Undergraduate Notes in Mathematics

Arkansas Tech University
Department of Mathematics

A Semester Course in Finite Mathematics for Business and Economics

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Contents

Preface

This book is a collection of lecture notes for a freshmen level course in mathematics designated for students in Business, Economics, Life Sciences and Social Sciences. The content is suitable for a one semester course. A college algebra background is required for this course.

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Mathematics of Finance

1. Simple Interest

Interest is a change of value of money. For example, when you deposit money into a savings account, the interest will increase your money based on the interest rate paid by your bank. In contrast, when you get a loan, the interest will increase the amount you owe based upon the interest rate charged by your bank. We can look at interest as the fee for using money.
There are two types of interest: Simple interest and compound interest. In this section we discuss the former one and postpone the discussion of the later to the next section.
Interest problems generally involve four quantities: principal(s), investment period length(s), interest rate(s), amount value(s).
The money invested in financial transactions will be referred to as the **principal** or the **present value**, denoted by P. The amount it has grown to will be called the **amount value** or the **future value** and will be denoted by A. The difference $I = A - P$ is the **amount of interest** earned during the period of investment. Interest expressed as a percent of the principal will be referred to as an **interest rate.**
The unit in which time of investment is measured is called the **measurement period.** The most common measurement period is one year but may be longer or shorter (could be days, months, years, decades, etc.)
Let r denote the annual fee per \$100 and P denote the principal. We call r the **simple interest rate.** Thus, if one deposits $P = \$500$ in a savings account that pays $r = 5\%$ interest then the interest earned for the two years period will be $500 \times 0.05 \times 2 = \$50$. In general, if I denotes the interest earned on an investment of P at the annual interest rate r for a period of t years then

$$I = Prt$$

and the amount of the investment at the end of t years is

$$A(t) = \text{principal} \ + \ \text{interest} = P(1 + rt).$$

Example 1.1
John borrows \$1,500 from a bank that charges 10% annual simple interest rate. He plans to make weekly payments for two years to repay the loan.
(a) Find the total interest paid for this loan.

(b) Find the total amount to be paid back.
(c) Find the weekly installment. Through out this document we shall round all amounts to the nearest cent.

Solution.
(a) The total interest paid is $I = Prt = 1,500 \times 0.10 \times 2 = \$300$.
(b) The total amount to be paid back is $A(2) = 1,500 + 300 = \$1,800$.
(c) The weekly installment is

$$\frac{1,800}{2 \times 52} = \$17.31 \blacksquare$$

Example 1.2
NBA bank is offering its customers an annual simple interest of 3% for deposits in savings accounts. How long will take for a deposit of \$1,500 to earn total interest of \$225?

Solution.
We are given $I = \$225,\ P = \$1,500$, and $r = 0.03$. We are asked to find t. From the formula $I = Prt$ we find

$$t = \frac{I}{Pr} = \frac{225}{1,500(0.03)} = 5 \text{ years} \blacksquare$$

Example 1.3
A new bank is offering its customers the option to double their investments in eight years with an annual simple interest rate r. Calculate r.

Solution.
We are given that

$$2P = P(1 + 8r).$$

Solving this equation for r, we find

$$\begin{aligned} P(1 + 8r) =& 2P \\ 1 + 8r =& 2 \\ 8r =& 1 \\ r =& \frac{1}{8} = 12.5\% \blacksquare \end{aligned}$$

Example 1.4
You deposit P dollars into a savings account that pays 12% annual simple interest. At the end of nine months, you have \$109 in your account. Calculate P.

Solution.
We are given: $r = 0.12,\ A = \$109,\ t = \frac{9}{12} = 0.75$. We are asked to find P. We have

$$P = \frac{A}{1+rt} = \frac{109}{1+0.12(0.75)} = \$100 \blacksquare$$

Example 1.5
Treasury bills (or T-bills) are financial instruments that the US government uses to finance public debt. Suppose you buy a 90-day T-bill with a maturity value of \$10,000 for \$9,800. Calculate the annual simple interest rate earned for this transaction. Round your answer to three decimal places. In all problems involving days, we assume a year has 360 days.

Solution.
We are given $P = \$9,800,\ A = \$10,000,\ t = 0.25$ and we are asked ti find r. We have

$$\begin{aligned}
9,800(1+0.25r) =& 10,000\\
1+0.25r =& \frac{50}{49}\\
0.25r =& \frac{50}{49} - 1 = \frac{1}{49}\\
r =& \frac{1}{49(0.25)} \approx 8.163\% \blacksquare
\end{aligned}$$

Example 1.6
The table below provides the commission that a brokerage firm charges for selling or buying stocks.

Transaction Amount	Commission
\$0 - \$2,499	\$29 + 1.6% of transaction amount
\$2,500 - \$9,999	\$49 + 0.8% of transaction amount
$\geq \$10,000$	\$99 + 0.3% of transaction amount

Suppose you purchase 45 shares of a stock at \$40.25 a share. After 180 days, you sell the stock for \$50.75 a share. Using the table above, find the annual simple interest rate earned by this investment. Round your answer to three decimal places.

Solution.
The amount of money needed for buying the stock is

$$45(40.25) = \$1,811.25.$$

The commission paid on this amount is

$$29 + 0.016(1,811.25) = \$57.98.$$

Thus, the total cost for this investment is

$$1,811.25 + 57.98 = \$1,869.23.$$

The return from selling the stocks is

$$50.75(45) = \$2,283.75.$$

The commission paid for this transaction is

$$29 + 0.016(2,283.75) = \$65.54.$$

The total return for this transaction is

$$2,283.75 - 65.54 = \$2,218.21.$$

Now, using the simple interest formula with $P = \$1,869.23,\ A = \$2,218.21,$ and $t = \frac{180}{360} = 0.5$, we find

$$\begin{aligned}
1,869.23(1 + 0.5r) =& 2,218.21\\
1 + 0.5r =& \frac{2,218.21}{1,869.23}\\
0.5r =& \frac{2,218.21}{1,869.23} - 1 = \frac{348.98}{1,869.23}\\
r =& \frac{348.98}{1,869.23(0.5)} \approx 37.339\% \blacksquare
\end{aligned}$$

Practice Problems

Problem 1.1
If \$4,500 is loaned for 6 months at a 5% annual simple interest rate, how much interest is earned?

Problem 1.2
A loan of \$3,630 was repaid at the end of 4 months. What size of repayment check (principal and interest) was written, if an 4% annual simple interest rate was charged?

Problem 1.3
A loan of \$2,000 was repaid at the end of 5 months with a check of \$2,135. What annual simple interest rate was charged?

Problem 1.4
A loan of \$7,260 was repaid at the end of t years. The interest on the loan was \$387.20 and the annual simple interest rate charged is 8%. Calculate t.

Problem 1.5
A 5-month loan of P dollars was retired by a check in the amount of \$3,097.50. The annual simple interest rate charged was 7.8%. Calculate the value of P.

Problem 1.6
A loan company charges 50¢a day for each \$1,000 borrowed. Suppose you borrow \$5,000 for 60 days, what amount will you repay?

Problem 1.7
A loan company charges 50¢a day for each \$1,000 borrowed. Suppose you borrow \$5,000 for 60 days, what annual simple interest rate will you repay the company?

Problem 1.8
A credit card company charges an 9% annual simple interest rate for overdue accounts. How much interest will be owed on an \$418 account that is 3 months overdue?

Problem 1.9
What annual interest rate is earned by a 26-week T-bill with a maturity of \$10,000 that sells for \$9,893.70?

Problem 1.10
What is the purchase price of a 33-day T-bill with a maturity of \$1,000 that earns an annual interest rate of 4.205%?

Problem 1.11
To complete the sale of a house, the seller accepts a 180-day note for \$10,000 at 7% simple interest. (Both interest and principal will be repaid at the end of 180 days). Wishing to be able to use the money sooner, the seller sells the note to a third party for \$10,124 after 60 days. What annual rate will the third party receive for the investment?

Problem 1.12
In two years from now Bob wants to take a long vacation. He estimates that he needs \$30,000 for this vacation. How much should he deposit now into an account that pays 5.5% annual simple interest rate so he will have the needed amount of money available for his vacation?

Problem 1.13
Suppose you make an initial deposit of \$1,000 into a savings account at a bank which offers a 3% yearly simple interest rate. If you make no withdrawals or deposits in the next 10 years, how much is the account worth?

Problem 1.14
The table below provides the commission that a brokerage firm charges for selling or buying stocks.

Transaction Amount	Commission
\$0 - \$2,999	\$25 + 1.8% of transaction amount
\$3,000 - \$10,000	\$37 + 1.4% of transaction amount
$> \$10,000$	\$107 + 0.7% of transaction amount

Suppose you purchase 200 shares of a stock at \$14.20 a share. After 39 weeks, you sell the stock for \$15.75 a share. Using the table above, find the annual simple interest rate earned by this investment. Round your answer to three decimal places.

Problem 1.15
A bond is an interest bearing security which promises to pay a stated amount (face value) of money at some future date (maturity date). Bonds coupons

are periodic payments of interest made by the issuer of the bond to the bondholder. The coupon rate gives the percentage of the face value to be paid periodically. A bond with a \$10,000 face value has a 3% coupon and a five-year maturity date. Calculate the total of the interest payments paid to the bondholder.

Problem 1.16
You have an account with \$500 that pays 5% annual simple interest. How long until your account doubles in value?

Problem 1.17
A total of \$8,000 is deposited in two simple interest accounts. In one account, the annual simple interest rate is 5%, and on the second account, the annual simple interest rate is 6%. How much should be invested in each account so that the total annual interest earned is \$450?

Problem 1.18
A sports foundation deposited a total of \$24,000 into two simple interest accounts. The annual simple interest rate on one account is 7%. The annual simple interest rate on the second account is 11%. How much is invested in each account if the total annual interest earned is 10% of the total investment?

Problem 1.19
A dentist invested a portion of \$15,000 in a 7% annual simple interest account and the remainder in a 6.5% annual simple interest government bond. The two investments earn \$1,020 in interest annually. How much was invested in each account?

Problem 1.20
You are expecting a tax refund of \$1,500 in 3 weeks. A tax preparer offers you an interest-free loan of \$1,500 for a fee of \$60 to be repaid by your refund check when it arrives in 3 weeks. Thinking of the fee as interest, what simple interest rate would you be paying on this loan?

2. Discrete and Continuous Compound Interest

Simple interest has the property that the interest earned is not invested to earn additional interest. In contrast, **compound interest** has the property that the interest earned at the end of one period is automatically invested in the next period to earn additional interest.
Compound rates are quoted annually. The **rate per compounding period** is found by dividing the annual nominal rate by the number n of compounding periods per year. For annual compound, $n = 1$. For semi-annual compound $n = 2$, for quarterly $n = 4$, for monthly $n = 12$, for weekly $n = 52$ and for daily $n = 360$. (Bankers rule)
Suppose that the annual interest r is compounded n times a year. Let P be the principal. Then at the end of the first period, the amount is given by

$$A_1 = P + \frac{r}{n}P = P\left(1 + \frac{r}{n}\right).$$

At the end of the second period, we have

$$A_2 = P\left(1 + \frac{r}{n}\right) + \frac{r}{n}P\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2.$$

Continuing this process, we find that the amount at the end of the first year is

$$A_n = P\left(1 + \frac{r}{n}\right)^n.$$

For t years of investment, the amount is

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Example 2.1
Find the amount after 3 years when \$1,000 is invested at 6%
(a) annually (b) semi-annually (c) quarterly (d) monthly.

Solution.
(a) For interest compounded annually, we have $n = 1$. Thus,

$$A(3) = 1,000(1 + 0.06)^3 = \$1,191.02.$$

(b) For interest compounded semi-annually, we have $n = 2$. Thus,

$$A(3) = 1,000\left(1 + \frac{0.06}{2}\right)^{2(3)} = \$1,194.05.$$

(c) For interest compounded quarterly, we have $n = 4$. Thus,

$$A(3) = 1,000\left(1 + \frac{0.06}{4}\right)^{4(3)} = \$1,195.62.$$

(d) For interest compounded monthly, we have $n = 12$. Thus,

$$A(3) = 1,000\left(1 + \frac{0.06}{12}\right)^{12(3)} = \$1,196.68 \blacksquare$$

Continuous Compound Interest
When the compound formula is used over smaller time periods the interest becomes slightly larger and larger. That is, frequent compounding earns a higher rate, though the increase is small. This suggests compounding more and more, or equivalently, finding the value of $A(t)$ in the long run. In Calculus, it can be shown that the expression

$$\left(1 + \frac{r}{n}\right)^n$$

approaches e^r as n increases without bound, where e (named after Euler) is a number whose value is $e = 2.71828\cdots$. The amount formula reduces to

$$A(t) = Pe^{rt}.$$

This formula is known as the **continuous compound formula.** and the annual rate is said to be **compounded continuously.**

Example 2.2
An amount of \$2,340 is deposited in a bank paying an annual interest rate of 3.1%, compounded continuously. Find the balance after 3 years.

Solution.
We are given $P = \$2,340,\ r = 0.031$, and $t = 3$. Hence, $A(3) = 2,340e^{0.031(3)} = \$2,568.06$ $\blacksquare$

Remark 2.1
Interest given by banks are known as **nominal rate** (e.g. "in name only"). When interest is compounded more frequently than once a year, the account effectively earns more than the nominal rate. Thus, we distinguish between

nominal rate and effective rate. The effective annual rate tells how much interest the investment actually earns. For an annual nominal rate compounded n times a year, the difference

$$r_e \left(1 + \frac{r}{n}\right)^n - 1$$

is known as the **effective interest rate** or the **annual percentage yield**. When the annual nominal rate is compounded continuously, the effective annual interest rate is

$$r_e = e^r - 1.$$

Example 2.3
Which is better: An account that pays 8% annual interest rate compounded quarterly or an account that pays 7.95% compounded continuously?

Solution.
The effective rate corresponding to the first option is

$$\left(1 + \frac{0.08}{4}\right)^4 - 1 = 8.24\%.$$

That of the second option is

$$e^{0.0795} - 1 = 8.27\%.$$

Thus, we see that 7.95% compounded continuously is better than 8% compounded quarterly ■

Example 2.4
Suppose that a savings account is compounded continuously at a rate of 8% per annum. After three years, the amount in the account is \$2,500. How much was deposited at the beginning, assuming that nothing was added or withdrawn during that period?

Solution.
We find

$$P = \frac{A}{e^{rt}} = \frac{2,500}{e^{0.08(3)}} = \$1,966.57 \blacksquare$$

Example 2.5
Suppose that today a principal of \$1,500 is deposited in a savings account. After 20 years, the amount increased to \$5,000. What was the nominal annual interest rate, assuming that nothing was added or withdrawn during that period if the interest was (a) compounded annually? (b) compounded continuously?

Solution.
(a) We have

$$\begin{aligned}
1,500(1+r)^{20} =& 5,000\\
(1+r)^{20} =& \frac{10}{3}\\
1+r =& \left(\frac{10}{3}\right)^{\frac{1}{20}}\\
r =& \left(\frac{10}{3}\right)^{\frac{1}{20}} - 1 = 6.20\%.
\end{aligned}$$

(b) We have

$$\begin{aligned}
1,500e^{20r} =& 5,000\\
e^{20r} =& \frac{10}{3}\\
20r =& \ln\left(\frac{10}{3}\right)\\
r =& \frac{1}{20}\ln\left(\frac{10}{3}\right) = 6.02\% \blacksquare
\end{aligned}$$

Practice Problems

Problem 2.1
What is the effective rate of interest corresponding to a nominal interest rate of 5% compounded quarterly?

Problem 2.2
Find the effective rate if \$1,000 is deposited at 5% annual interest rate compounded continuously.

Problem 2.3
Let \$1,000 be an initial deposit into a savings account with a annual nominal interest rate of 5%. What is the amount in the account after 1 year if the account (i) compounds annually, (ii) compounds quarterly, (iii) compounds monthly, (iv) compounds weekly, (v) compounds daily?

Problem 2.4
Suppose a principal of \$10,000 is compounded (a) annually, (b) quarterly, (c) monthly, (d) weekly, (e) daily, and (f) continuously at a nominal annual interest rate of 5%. Write out the corresponding effective interest rates.

Problem 2.5
How long will it take an investment of \$10,000 to grow to \$15,000 if it is invested at 9% compounded continuously?

Problem 2.6
How long will it take money to triple if it is invested at 5.5% compounded continuously?

Problem 2.7
What is the interest on a \$650 compound interest loan with an annual nominal interest of 10% compounded quarterly for 18 months?

Problem 2.8
How much money is needed now if one wants \$25,000 in 10 years and an 8% compounded weekly rate is found?

Problem 2.9
Given an investment of \$13,200, compound amount of \$22,680.06 invested for 8 years, what is the annual nominal interest rate if interest is compounded annually?

Problem 2.10
A necklace is appraised at \$6,300. If the value of the necklace has increased at an annual rate of 7%, how much was it worth 15 years ago?

Problem 2.11
Deposit \$100 into an account earning 4.5% interest compounded annually. How many years will it take to have a future value of \$200?

Problem 2.12
You are going to invest some money for one year. Bank A offers to give 6.2% interest compounded annually. Bank B offers to give 6.1% interest compounded monthly. Which is better?

Problem 2.13
You are going to invest some money for two years. Bank A offers to give 5% interest for the first year and 15% interest for the second year. Bank B offers to give 10% interest compounded yearly. Which is better?

Problem 2.14
How long will it take \$30,000 to accumulate to \$110,000 in an account that earns a 10% annual nominal rate compounded semiannually?

Problem 2.15
The Flagstar Bank in Michigan offered a 5-year certificate of deposit at 4.38% interest compounded quarterly. On the same day, Principal Bank offered a 5-year at 4.37% interest compounded monthly. Find the APY (effective rate) for each CD. Which bank offered a higher APY?

Problem 2.16
A company agrees to pay \$2.9 million in 5 years to settle a lawsuit. How much must it invest now in an account paying 8% compounded monthly to have that amount when it is due?

Problem 2.17
Scott borrowed \$5,200 from his friend Joe to buy computer equipment. He repaid the loan 10 months later with simple interest at 7%. Joe than invested the proceeds for 5 years in an account that pays 6.3% compounded quarterly. How much will he have at the end of 5 years?

Problem 2.18
You decide to invest \$16,000 in a money market fund that guarantees a 5.5% annual interest rate compounded monthly for 7 years. A one time fee of \$30 is charged to set up the account. In addition, there is an annual charge of 1.25% of the balance in the account at the end of each year.
(a) How much is in the account at the end of the first year?
(b) How much is in the account at the end of the seventh year?

Problem 2.19
Suppose a house that was worth \$68,000 in 1987 is worth \$104,000 in 2004. Assuming a constant rate of inflation from 1987 to 2004, what is the inflation rate?

Problem 2.20
What is the annual nominal rate compounded monthly for an investment that has an annual percentage yield of 5.9%?

3. Ordinay Annuity, Future Value and Sinking Fund

A sequence of equal payments made at equal periods of time is called an **annuity**. If the payments are made at the end of the periods, and if the frequency of payments is the same as the frequency of compounding, the annuity is called an **ordinary annuity**[1]

In what follows, the word annuity stands for ordinary annuity. The **future value** is the sum of all payments plus all interest earned. Let A denote the future value of an $n-$period annuity with periodic payment R and periodic rate r. Then we have the following table:

End of Period	Value of Payment at the end of the nth period
1	$R(1+r)^{n-1}$
2	$R(1+r)^{n-2}$
3	$R(1+r)^{n-3}$
$\vdots$	$\vdots$
$n-1$	$R(1+r)$
n	R

Thus, by the definition of A, we have

$$A = R + R(1+r) + \cdots + R(1+r)^{n-3} + R(1+r)^{n-2} + R(1+r)^{n-1}. \quad (1)$$

Multiply both sides of (1) by $(1+r)$ we obtian

$$(1+r)A = R(1+r)+R(1+r)^2+\cdots+R(1+r)^{n-2}+R(1+r)^{n-1}+R(1+r)^n. \quad (2)$$

Subtract (1) from (2) we obtain

$$rA = R(1+r)^n - R.$$

Hence,

$$A = R\left[\frac{(1+r)^n - 1}{r}\right]. \quad (3)$$

Example 3.1

Suppose a \$1,000 deposit is made at the end of each quarter and the money

[1]If the payments are made at the beginning of the periods, and if the frequency of payments is the same as the frequency of compounding, the annuity is called an **annuity due.**

in the account is compounded quarterly at 6.5% interest for 15 years.
(a) How much is in the account after the 15 year period?
(b) What is the total amount deposited?
(c) What is total amount of interest earned?

Solution.
(a) The future value of the annuity is

$$A = 1,000\left[\frac{\left(1+\frac{0.065}{4}\right)^{4(15)}-1}{\frac{0.065}{4}}\right] = \$100,336.68.$$

(b) The total amount deposited is $1,000(15)(4) = 60,000$.
(c) The total amount of interest earned is $100,336.68 - 60,000 = 40,336.68$ ■

Example 3.2
Larry has deposited \$150 per month into an ordinary annuity. After 14 years, the annuity is worth \$85,000. What annual rate compounded monthly has this annuity earned during the 14 year period?

Solution.
We are given $A = \$85,000, n = 14(12) = 168$, and $R = \$150$. We are asked to find r. We have

$$85,000 = 150\left[\frac{(1+r)^{168}-1}{r}\right]$$

or

$$\frac{85,000}{150} = \left[\frac{(1+r)^{168}-1}{r}\right].$$

Using a calculator, we graph the functions $y_1 = \frac{85,000}{150} = \frac{1700}{3}$ and $y_2 = \left[\frac{(1+r)^{168}-1}{r}\right]$ and find the point where they intersect. In our case, we find $r \approx 1.253\%$. An alternative is to use an equation solver if a calculator has this feature ■

Sinking Fund
By a **sinking fund** we mean an account that is established for accumulating funds to meet future obligations or debts. The **sinking fund payment** is defined to be the amount that must be deposited into an account periodically to have a given future amount. If payments are to be made in the form

of an ordinary annuity, then the sinking fund payment is found by solving equation (3) for R obtaining

$$R = A\left[\frac{r}{(1+r)^n - 1}\right]. \tag{4}$$

Example 3.3
How much must Vrege save each month in order to buy a new car for $12,000 in three years if the interest rate is 6% compounded monthly?

Solution.
We are given $A = \$12,000, n = 3(12) = 36$, and $r = \frac{0.06}{12} = 0.005$. We want to find R given by

$$R = 12,000\left[\frac{0.005}{(1.005)^{36} - 1}\right] = \$ \blacksquare$$

Example 3.4
The parents of a newborn baby set up an account to cover the cost of college. On the child's birthday each year, starting on the 1st and ending on the 18th, they deposit money into the account, which pays 7% compounded annually.
(a) How much should they deposit annually in order to have $100,000 available for college on the child's 18th birthday?
(b) How much interest is earned during the last year?

Solution.
(a) We are given $A = \$100,000, r = 0.07$, and $n = 18$. Thus,

$$R = 100,000\left[\frac{0.07}{1.07^{18} - 1}\right] = \$2,941.26.$$

(b) By the end of the 17th year, the amount in the account is

$$A = 2,941.26\left[\frac{1.07^{17} - 1}{0.07}\right] = \$90,709.10.$$

After the child's 18th birthday, the account grew from $90,709.10 to $100,000. Portion of this growth includes the paymet $2,941.26 and the rest is interest earned. Thus, the interest earned during the last year is

$$(100,000 - 90,709.10) - 2,941.26 = \$6,349.64 \blacksquare$$

Practice Problems

Problem 3.1
Find r and n for an ordinary annuity that consists of a quarterly payment of \$500 for 20 years with annual nominal interest of 8% compounded quarterly.

Problem 3.2
\$5,000 is deposited at the end of each year for the next 6years in a money market account paying 4.5% interest compounded annually. Find the future value of this annuity.

Problem 3.3
You need \$18,000 in three years to buy a car. How much per month should you put into a savings account that pays 6% compounded monthly to accumulate that amount?

Problem 3.4
You need to pay off a \$100,000 loan in 10 years. If the interest rate for a sinking fund is 8% compounded annually, what is the periodic payment you must make at the end of each year?

Problem 3.5
If you have \$500,000 when you retire and you plan to live for 40 years, and you can earn 8% compounded monthly, how much can you withdraw each month from your account?

Problem 3.6
If you can afford to pay \$250 at the end of each month for the next 5 years at 6% compounded monthly, how much do you have today to spend on an automobile purchase?

Problem 3.7
If your employer will match your retirement savings plan monthly contributions of \$225 (at the end of each month), and you can earn 9% compounded monthly for the next 25 years.
(a) How much will you have when you retire?
(b) How much did you contribute in total?
(c) What is the total gain for you?

Problem 3.8
If you can save \$200 at the end of each month for the next 35 years and you can earn 12% compounded monthly, how much will you have when you retire?

Problem 3.9
Steve deposits \$250 every 3 months into a savings account that pays interest at a rate of 8% compounded quarterly. How much money will be in the account at the end of 10 years?

Problem 3.10
Kelly wishes to buy a car that costs \$32,998. The car dealer tells her that they can finance the car at 6.25% per year compounded monthly for 5 years. She decides to secure the loan from the dealer. How much will her monthly payments be?

Problem 3.11
A person would like to have \$200,000 in an account for retirement 15 years from now. How much should be deposited quarterly in an account paying 6% per year compounded quarterly to obtain this amount?

Problem 3.12
A bond issue is approved for building a marina in a city. The city is required to make regular payments every 3 months into a sinking fund paying 5.4% compounded quarterly. At the end of 10 years, the bond obligation will be retired with a cost of \$5,000,000.
(a) What should each payment be?
(b) How much interest is earned during the 10th year?

Problem 3.13
Experts say the baby boom generation can't count on a company pension or Social Security to provide a comfortable retirement, as their parents did. It is recommended that they start to save early and regularly. Sarah, a baby boomer, has decided to deposit \$200 each month for 20 years in an account that pays interest of 7.2% compounded monthly.
a) How much will be in the account at the end of 20 years?
b) Sarah believes she needs to accumulate \$130,000 in the 20-year period to have enough for retirement. What interest rate would provide that amount?

Problem 3.14
You sell a land and then you will be paid a lump sum of \$60,000 in 7 years. Until then, the buyer pays 8% simple interest quarterly.
(a) Find the amount of each quarterly interest payment on the \$60,000.
(b) The buyer sets up a sinking fund so that enough money will be present to pay off the \$60,000. The buyer will make semiannual payments into the sinking fund; the account pays 6% compounded semi-annually. Find the amount of each payment into the fund.

Problem 3.15
Recently, Guaranty Income Life offered an annuity that pays 6.65% compounded monthly. If \$500 is deposited into this annuity every month, how much is in the account after 10 years? How much of this is interest?

Problem 3.16
Compu-bank, an online banking service, offered a money market account with an APY of 4.86%.
(a) If interest is compounded monthly, what is the equivalent annual nominal rate?
(b) If you wish to have \$10,000 in the account after 4 years, what equal deposit should you make each month?

Problem 3.17
A 45 year-old man puts \$2,500 in a retirement account at the end of each quarter until he reaches the age of 60, then makes no further deposits. If the account pays 6% interest compounded quarterly, how much will be in the account when the man retires at age 65?

Problem 3.18
A father opened a savings account for his daughter on the day she was born, depositing \$1,000. Each year on her birthday he deposits another \$1,000, making the last deposit on her 21st birthday. If the account pays 5.25% interest compounded annually,
(a) how much is in the account at the end of the day on his daughter's 21st birthday?
(b) How much interest has been earned?

Problem 3.19
A person makes annual payments of \$2,000 into an ordinary annuity account.

At the end of 4 years, the amount in the account is \$8417.33. What annual nominal compounding rate has this annuity earned?

Problem 3.20

Jane deposits \$2,000 annually into a Roth IRA that earns 6.85% compounded annually. (The interest earned on a Roth IRA is tax free.) Due to a change in employment, these deposits stop after 10 years.

Jane does not take the money out of this account when she changes jobs. Even though she does not add any more money to the account, the investment continues to earn interest until she retires 25 year after the last deposit was made.

(a) Calculate the amount of money in the Roth IRA at the time Jane changes job.

(b) Calculate the amount of money in the account at the time Jane retires.

4. Present Value of an Ordinay Annuity and Amortization

Consider an ordinary annuity that pays R at the end of each period for n periods with periodic rate r. We would like to find the present value of this annuity, that is, the time-0 value of this annuity. Let us denote the present value by the letter P. The present value of each payment is shown in the table below.

End of Period	Value of Payment at time 0
1	$R(1+r)^{-1}$
2	$R(1+r)^{-2}$
3	$R(1+r)^{-3}$
$\vdots$	$\vdots$
$n-1$	$R(1+r)^{-(n-1)}$
n	$R(1+r)^{-n}$

Thus, by the definition of P, we have

$$P = R(1+r)^{-1}+R(1+r)^{-2}+R(1+r)^{-3}+\cdots+R(1+r)^{-(n-1)}+R(1+r)^{-n}. \quad (5)$$

Multiply both sides of (5) by $(1+r)$ we obtian

$$(1+r)P = R+R(1+r)^{-1}+R(1+r)^{-2}+\cdots+R(1+r)^{-(n-2)}+R(1+r)^{-(n-1)}. \quad (6)$$

Subtract (5) from (6) we obtain

$$rP = R - R(1+r)^{-n}.$$

Hence,

$$P = R\left[\frac{1-(1+r)^{-n}}{r}\right]. \quad (7)$$

Solving for R, we find

$$R = P\left[\frac{r}{1-(1+r)^{-n}}\right]. \quad (8)$$

Example 4.1
What is the present value of an annuity that pays \$1,000 every six months for 3 years if money is worth 6% compounded semi-annually?

Solution.
We are given $R = \$1,000, r = \frac{0.06}{2} = 0.03$, and $n = 3(2) = 6$. We are asked to find P. We have

$$P = 1,000 \left[\frac{1 - (1.03)^{-6}}{0.03}\right] = \$5,417.19 \blacksquare$$

Example 4.2
A car costs \$12,000. After a down payment of \$2,000, the balance will be paid off in 36 equal monthly payments with interest of 6% per year on the unpaid balance. Find the amount of each payment.

Solution.
Interest will be charged for the loan of \$10,000. We are given $P = \$10,000, r = \frac{0.06}{12} = 0.005$, and $n = 36$. We are asked to find R. We have

$$R = 10,000 \left[\frac{0.005}{1 - (1.005)^{-36}}\right] = \$304.22 \blacksquare$$

Example 4.3
A person plans to make equal annual deposits into a retirement account for 25 years in order and then make 20 equal annual withdrawals of \$25,000 reducing the balance in the amount to zero. Assume the account offers a nominal annual rate of 6.5% compounded annually.
(a) How much must be deposited annually to accumulate sufficient funds to provide for these payments?
(b) How much total interest is earned during this entire 45-years process?

Solution.
(a) At time $t = 25$, we want to find the amount needed in the account so that \$25,000 can be withdrawn annually from the account for 20 years, reducing the balance in the account to 0.

$$P = 25,000 \left[\frac{1 - (1.065)^{-20}}{0.065}\right] = \$275,462.68.$$

Next, we want to find the annual deposit into an account that will accumulate to \$275,462.68 after 25 years. The annual deposit is

$$R = 275,462.68 \left[\frac{0.065}{(1.065)^{25} - 1}\right] = \$4,677.76.$$

(b) The total interest earned during the 45-year period is

Total interest = total withdrawals−total deposits = $20(25,000)-25(4,677.76) = \$383,056$ ■

Amortization

An important application of (8) is the question of retiring a debt by making periodic payments including compound interest. We refer to this process as **amortization**..

Example 4.4

You borrow \$5,000 to buy a car. You decide to retire this loan by making 36 equal monthly payments. The bank charges 12% compounded monthly.

(a) What is your monthly payment?

(b) How much interest did you pay for this loan?

Solution.

(a) The monthly payment is

$$R = 5,000\left[\frac{0.01}{1-(1.01)^{-36}}\right] = \$166.07 \text{ per month.}$$

(b) Total amount of interest paid for this loan is $36(166.07) - 5,000 = \$978.52$ ■

Amortization Schedule

When a loan is being repaid with the amortization method, each payment is partially a repayment of principal and partially a payment of interest. Determining the amount of each for a payment can be important (for income tax purposes, for example). An **amortization schedule** is a table which shows the division of each payment into principal and interest, together with the outstanding loan balance after each payment is made. We illustratte this concept in the next example.

Example 4.5

If you borrow \$500 that you agree to repay in six equal monthly payments at 1% interest per month on the unpaid balance, how much of each monthly payment is used for interest and how much is used to reduce the unpaid balance?

Solution.
We first find the monthly payment:

$$R = 500\left[\frac{0.01}{1-(1.01)^{-6}}\right] = \$86.27 \text{ per month.}$$

After the first payment, the interest due is $500(0.01) = \$5.00$ and the amount applied to the principal is $86.27 - 5.00 = 81.27$. Thus, the unpaid balance after the first payment is made is $500 - 81.27 = \$418.73$. We continue this process until the loan is retired. The amortization schedule is given next.

Period	R	Interest portion of R	Principal portion of R	Unpaid balance
0				\$500
1	86.27	5.00	81.27	418.73
2	86.27	4.19	82.08	336.65
3	86.27	3.37	82.90	253.75
4	86.27	2.54	83.73	170.02
5	86.27	1.70	84.57	85.45
6	86.30	0.85	85.45	0.0

Note that the last payment is adjusted so that we end with a zero balance ■

Example 4.6
A family purchase a home 10 years ago for \$80,000. The home was financed by paying 20% down for 30-year mortgage at 9%, on the unpaid balance. The net market of the house is now \$120,000.00 and the family wishes to sell the house. How much equity after making 120 monthly payments? Hint: Equity = current net market value − unpaid loan balance.

Solution.
The value of the loan is $80,0000.2(80,000) = \$64,000$. The monthly mortgage payment is

$$R = 64,000\left[\frac{\frac{0.09}{12}}{1-\left(1+\frac{0.09}{12}\right)^{-360}}\right] = \$514.96.$$

There are 240 payments remaining. The unpaid balance is just the present value of a 20-year annuity:

$$P = 514.96\left[\frac{1-\left(1+\frac{0.09}{12}\right)^{-240}}{\frac{0.09}{12}}\right] = \$57,235.$$

Thus, the equity of the house is $120,000 - 57,235 = \$62,765$. That is, if the family sells the house for $120,000 net, the family will have $62,765 after paying the unpaid loan balance $57,235 ■

Example 4.7
You have negotiated a price of $25,200 for a new truck. Now you must choose between 0% financing for 48 months or a $3,000 rebate. If you choose the rebate, you can obtain a loan for the balance at 4.5% compounded monthly for 48 months . Which option should you choose?

Solution.
Under 0% financing, we have $P = \$25,200; r = 0; n = 48$ so that

$$R_1 = \frac{25,200}{48} = \$525.$$

With the rebate option, we have $P = 25,200 - 3,000 = \$22,200; r = \frac{0.045}{12} = 0.00375; n = 48$. The monthly payment is

$$R_2 = 22,200\frac{0.00375}{1-(1.00375)^{-48}} = \$506.24.$$

By choosing the rebate you save $525 - 506.24 = \$18.76$ a month or $18.76(48) = \$900.48$ over the life of the loan ■

Practice Problems

Problem 4.1
Given: $n = 30; r = 0.04; R = \$200$. Find P.

Problem 4.2
Given: $P = \$40,000; n = 96; r = 0.0075$. Find R.

Problem 4.3
Given: $P = \$5,000; r = 0.01; R = \$200$. Find n.

Problem 4.4
Given: $P = \$9,000; R = \$600; n = 20$. Find i to three decimal places.

Problem 4.5
How much should you deposit in an account paying 8% compounded quarterly in order to receive quarterly payments of $1,000 for the next 4 years?

Problem 4.6
Suppose you have selected a new car to purchase for $19,500. If the car can be financed over a period of 4 years at an annual rate of 6.9% compounded monthly, how much will your monthly payments be? Construct an amortization table for the first 3 months.

Problem 4.7
Suppose your parents decide to give you $10,000 to be put in a college trust fund that will be paid in equally quarterly installments over a 5 year period. If you deposit the money into an account paying 1.5% per quarter, how much are the quarterly payments (Assume the account will have a zero balance at the end of period.)

Problem 4.8
Sharon has found the perfect car for her family (a new mini-van) at a price of $24,500. She will receive a $3,500 credit toward the purchase by trading in her old Gremlin, and will finance the balance at an annual rate of 4.8% compounded monthly.
a) How much are her payments if she pays monthly for 5 years?
b) How long would it take for her to pay off the car paying an extra $100 per month, beginning with the first month?

Problem 4.9
American General offers a 10-year ordinary annuity with a guaranteed rate of 6.65% compounded annually. How much should you pay for one of these annuities if you want to receive payments of \$5,000 annually over the 10-year period?

Problem 4.10
You want to purchase an automobile for \$27,300. The dealer offers you 0% financing for 60 months or a \$5,000 rebate. You can obtain 6.3% financial for 60 months at the local bank. Which option should you choose? Explain.

Problem 4.11
Construct the amortization schedule for a \$5,000 debt that is to be amortized in eight equal quarterly payments at 2.8% interest per quarter on the unpaid balance.

Problem 4.12
Construct the amortization schedule for a \$10,000 debt that is to be amortized in six equal quarterly payments at 2.6% interest per quarter on the unpaid balance.

Problem 4.13
A sailboat costs \$16,000. You pay 15% down and secure a loan for the remaining balance. How much are your monthly payments if 18% per year compounded monthly is charged over a period of 6 years?

Problem 4.14
Business partners, Bill and Bob, buy an apartment house for \$1,250,000 by making a down payment of \$125,000 and financing the rest with semiannual payments over the next 10 years. The interest rate on the debt is 8% per year compounded semiannually. How much is their semiannually payment?

Problem 4.15
Suppose you win a lottery that entitles you to receive \$500 per month for the next 20 years. If money is worth 6% compounded monthly, what is the present value of this annuity?

Problem 4.16
A home was purchased 15 years ago for \$75,000. The home was financed by

paying a 20% down payment and signing a 25-year mortgage at 9.0% compounded monthly on the unpaid balance. The market value is now \$100,000. The owner wishes to sell the house. How much equity (to the nearest dollar) does the owner have in the house after making 180 monthly payments?

Problem 4.17
Find the lump sum deposited today that will yield the same total amount as payments of \$10,000 at the end of each year for 15 years at 6% compounded annually.

Problem 4.18
Student borrowers now have more options to choose from when selecting repayment plans. The standard plan repays the loan in 10 years with equal monthly payments. The extended plan allows from 12 to 30 years to repay the loan. A student borrows \$35,000 at 7.43% compounded monthly:
(a) Find the monthly payment and total interest paid under the standard plan.
(b) Find the monthly payment and total interest paid under the extended plan for 20 years.

Problem 4.19
A couple purchased a home 20 years ago for \$65,000. The home was financed by paying 20% down and signing a 30 year mortgage at 8% on the unpaid balance. The net market value of the house is now \$130,000, and the couple wishes to sell the house. How much equity (to the nearest dollar) does the couple have in the house now after making 240 monthly payments?

Problem 4.20
A couple wishes to borrow \$125,000 in order to buy a house. They can pay a maximum of \$1,200 per month. If the loan is at 9.5% compounded monthly, how many months will it take to pay off the loan? (Round answer to the next higher month if not an integer.)

Matrices and Systems of Linear Equations

5. Solving Linear Systems Using Augmented Matrices

A **linear** equation in the variables x and y is an equation of the form $ax+by = c$. The graph of such an equation is a straight line in the $xy-$ coordinate system. A **linear system** in two equations in the variables x and y is a system of the form

$$\left\{\begin{array}{rr} ax + by = & c \\ a'x + b'y = & c'. \end{array}\right.$$

Graphically, a linear system consists of two straight lines. But two lines can either intersect or parallel. When they intersect, we say that the system is **consistent**. When they are parallel, we say that the system is **inconsistent.** A consistent system can be **dependent** (i.e., the two lines coincide) or **independent** (i.e., the two lines has exactly one point of intersection). A point of intersection of the two lines is called a **solution** of the system.
To **solve** a system is to find all the solutions. We discuss three methods for solving a system:

Method of Elimination
As the title indicates, we add the two equations together to eliminate one of the variables. Adding the equations means that we add the left sides of the two equations together, and we add the right sides together. This is legal because of the Addition Principle, which says that we can add the same amount to both sides of an equation.

Example 5.1
Solve the system of linear equations

$$\left\{\begin{array}{rr} 2x + 5y = & 1 \\ -3x + 2y = & 8. \end{array}\right.$$

Solution.
For this problem, to eliminate the x we have to multiply the first equation by 3 and the second equation by 2.

$$\left\{\begin{array}{rr} 6x + 15y = & 3 \\ -6x + 4y = & 16. \end{array}\right.$$

Adding results in the equation $19y = 19$ or $y = 1$. Substituting into one the given equations we find $x = -2$ ■

Example 5.2 *(Parallel Lines)*
Solve the system

$$\begin{cases} 6x - 2y = & 8 \\ 9x - 3y = & 6. \end{cases}$$

Solution.
Multiply the first equation by -3 and the second equation by 2 and add the resulting equations we find $0 = -12$ which is not a true statement. So the two lines do not intersect. In other words, the two lines are parallel. We also say that the system is **inconsistent** ■

Example 5.3 *(Overlapping Lines)*
Solve the system

$$\begin{cases} x - 2y = & 4 \\ 2x - 4y = & 8. \end{cases}$$

Solution.
Multiply the first equation by -2 and the resulting equation to the second equation we find $0 = 0$ which is always a true statement. So the two lines intersect infinitely many times. In other words, the two lines coincide. We also say that the system is a **dependent system**. ■

The Method of Substitution
With this method, we solve one of the equations for one variable in terms of the other, and then substitute that into the other equation. We illustrate this method by an example.

Example 5.4
Solve the system of linear equations

$$\begin{cases} x + 2y = & 3 \\ -3x + 4y = & 1. \end{cases}$$

Solution.
Solving the first equation for x we find $x = 3-2y$. Now we can use this result and substitute $x = 3-2y$ in for x in second equation and find $4y-3(3-2y) = 1$. Solving this equation we find $y = 1$. Hence, $x = 3 - 2y = 3 - 2 = 1$. The solution to the system is $(1, 1)$ ■

Solving Using Augmented Matrices

We define the **coefficient matrix** to our system to be the rectangular array

$$\begin{bmatrix} a & b \\ a' & b' \end{bmatrix}$$

We define the **augmented matrix** to be the rectangular array

$$\left[\begin{array}{cc|c} a & b & c \\ a' & b' & c' \end{array}\right]$$

Two linear systems are said to be **equivalent** if and only if they have the same solution set. One can build equivalent systems that are easier to solve than the original system by applying the following **elementary row operations** on the augemented matrix:

(I) Multiply a row by a non-zero number.

(II) Replace a row by the sum of this row and another row multiplied by a number.

(III) Interchange two rows.

We apply these operations to the augmented matrix to bring it to the form

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & c_1 \\ 0 & a_{22} & c_2 \end{array}\right]$$

We say that this matrix is **row equivalent** to the original augmented matrix and we use the symbol $\sim$ for that. Now, the system corresponding to the above matrix is

$$\left\{\begin{array}{ccccc} a_{11}x & + & a_{12}y & = c_1 \\ & & a_{22}y & = c_2. \end{array}\right.$$

This system is solved from bottom to top. We illustrate this process in the next examples.

Example 5.5

Solve using the augementex matrix methods.

$$\left\{\begin{array}{rr} 3x + 4y = & 1 \\ x - y = & 7. \end{array}\right.$$

Solution.

The augmented matrix is

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -1 & 7 \end{array}\right]$$

Interchanging the rows we find

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right]$$

Now add -3 times the first row to the second row, we find

$$\left[\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right]$$

The corresponding system is

$$\begin{cases} x \; - \; 2y & = 7 \\ \qquad 10y & = -20. \end{cases}$$

Solving this system from bottom to top, we find $y = -2$ and $x = 7 + 2y = 7 + 2(-2) = 3$ ■

Example 5.6
Solve using the augementex matrix methods.

$$\begin{cases} 6x - 2y = & 8 \\ 9x - 3y = & 6. \end{cases}$$

Solution.
The augmented matrix is

$$\left[\begin{array}{cc|c} 6 & -2 & 8 \\ 9 & -3 & 6 \end{array}\right]$$

Replace the first row by -9 times the first row and the second row by 6 times the second row to obtain

$$\left[\begin{array}{cc|c} -54 & 18 & -72 \\ 54 & -18 & 36 \end{array}\right]$$

Add the first row to the second row to obtain

$$\left[\begin{array}{cc|c} -54 & 18 & -72 \\ 0 & 0 & -36 \end{array}\right]$$

The corresponding system is

$$\begin{cases} -54x + 18y = & -72 \\ 0 \times x + 0 \times y = & -36. \end{cases}$$

The second equation is impossible. Hence, the system is inconsistent, i.e., has no solutions ■

Example 5.7
Solve using the augementex matrix methods.

$$\left\{\begin{array}{rl} x - 2y = & 4 \\ 2x - 4y = & 8. \end{array}\right.$$

Solution.
The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 2 & -4 & 8 \end{array}\right]$$

Add -2 times the first row to the second row, we find

$$\left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

The corresponding system is

$$\left\{\begin{array}{rl} x - 2y = & 4 \\ 0 \times x + 0 \times y = & 0. \end{array}\right.$$

Geometrically, the two lines coincide so that the system has infinitely many solutions. Letting $y = t$ we find $x = 4 + 2t$ so that the solution set is given by

$$\{(4 + 2t, t) : t \in \mathbb{R}\} \blacksquare$$

Practice Problems

Problem 5.1
Solve by the method of elimination.

$$\left\{\begin{array}{rr} -4x+2y= & 8 \\ 2x-y= & 0. \end{array}\right.$$

Problem 5.2
Solve by the method of elimination.

$$\left\{\begin{array}{rr} x+y= & 3 \\ 2x-y= & 0. \end{array}\right.$$

Problem 5.3
Solve by the method of elimination.

$$\left\{\begin{array}{rr} 2x-4y= & -10 \\ -x+2y= & 5. \end{array}\right.$$

Problem 5.4
Solve by the method of substitution

$$\left\{\begin{array}{rr} 2x-y= & 3 \\ x+2y= & 14. \end{array}\right.$$

Problem 5.5
Solve by the method of elimination.

$$\left\{\begin{array}{rr} 2x-3y= & -2 \\ -4x+6y= & 7. \end{array}\right.$$

Problem 5.6
Solve by the method of elimination.

$$\left\{\begin{array}{rr} -6x+10y= & -30 \\ 3x-5y= & 15. \end{array}\right.$$

Problem 5.7
A pharmacist has two solutions of alcohol. One is 25% alcohol and the other is 45% alcohol. He wants to mix these two solutions to obtain 36 ounces of a 30% alcohol solution. How many ounces of each solution should be mixed together?

Problem 5.8
You invested a total of \$38,000 in two municipal bonds that have a yield of 4% and 6% interest per year, respectively. The interest you earned from the bonds was \$1,930. How much did you invest in each bond?

Problem 5.9
Find the augmented matrix corresponding to the system

$$\begin{cases} -x + 2y = & -3 \\ 2x - y = & 4. \end{cases}$$

Problem 5.10
Find the linear system corresponding to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 6 & -3 & 12 \end{array}\right]$$

Problem 5.11
Solve using augmented matrix methods.

$$\begin{cases} x - 2y = & 1 \\ 2x - y = & 5. \end{cases}$$

Problem 5.12
Solve using augmented matrix methods.

$$\begin{cases} x + 2y = & 4 \\ 2x + 4y = & -8. \end{cases}$$

Problem 5.13
Solve using augmented matrix methods.

$$\begin{cases} 3x - 6y = & -9 \\ -2x + 4y = & 6. \end{cases}$$

Problem 5.14
A flight leaves New York at 8 PM and arrives in Paris at 9 AM (Paris time). This 13-hour difference includes the flight time plus the change in time zones. The return leaves Paris at 1 PM and arrives in New York at 3 PM (NY time). This 2-hour difference includes the flight time minus time zones, plus an extra hour due to the fact that flying westward is against the wind. Find the actual flight time eastward and the difference in time zones.

Problem 5.15
Solve using augmented matrix methods.

$$\begin{cases} 3x - y = & 7 \\ 2x + 3y = & 1. \end{cases}$$

Problem 5.16
Solve using augmented matrix methods.

$$\begin{cases} -2x + 4y = & 4 \\ 3x - 6y = & -6. \end{cases}$$

Problem 5.17
Solve using augmented matrix methods.

$$\begin{cases} 2x - 4y = & -4 \\ -3x + 6y = & 4. \end{cases}$$

Problem 5.18
At one store, 5 pairs of jeans and 2 sweatshirts costs $166, while 3 pairs of jeans and 4 sweatshirts costs $164. Find the cost of one sweatshirt.

Problem 5.19
Sonia invested a total of $12,000 into two accounts paying 7.5% and 6% simple interest. If her total return at the end of the first year was $840, how much did she invest in each account?

Problem 5.20
Pascal has $3.25 in change in his pocket, all in dimes and quarters. He has 22 coins in all. How many dimes does he have?

6. Gauss-Jordan Elimination

The previous section provided us with some experience with row operations on augmented matrices of linear systems of two equations in two unknowns. In this section, we consider a method for solving a linear system with any number of equations and any number of unknowns. This method is based on transforming the augmented matrix into a so-called reduced row-echelon form, a concept that we define next.

By a **leading entry** of a row in a matrix we mean the leftmost non-zero entry in the row. A rectangular matrix is said to be in **reduced row-echelon form** or simply in **reduced form** if it has the following three characterizations:

(1) All rows consisting entirely of zeros are at the bottom.

(2) The leading entry in each non-zero row is 1 and is located in a column to the right of the leading entry of the row above it.

(3) All entries in a column below a leading entry are zeros.

(4) Each leading 1 is the only nonzero entry in its column.

Example 6.1

Determine which matrices are in reduced form.

(a)

$$\begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution.

(a)The given matrix is not in reduced row-echelon form since the $(1,2)-$entry is not zero.

(b) The given matrix satisfies the characterization of a reduced row-echelon form $\blacksquare$

The importance of the reduced row-echelon matrices is indicated in the following theorem.

Theorem 6.1
Every nonzero matrix can be brought to reduced row-echelon form by a finite number of elementary row operations.

The process of reducing a matrix to a reduced row-echelon form is known as **Gauss-Jordan elimination**.

Example 6.2
Use Gauss-Jordan elimination to transform the following matrix into reduced form

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution.
The reduction of the given matrix to row-echelon form is as follows.

Step 1: $r_1 \leftrightarrow r_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 2: $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 + 2r_1$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 3: $r_2 \leftarrow \frac{1}{2}r_2$ and $r_3 \leftarrow \frac{1}{5}r_3$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 4: $r_3 \leftarrow r_3 - r_2$ and $r_4 \leftarrow r_4 + 3r_2$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Step 5: $r_3 \leftrightarrow r_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 6: $r_5 \leftarrow -\frac{1}{5}r_5$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 7: $r_1 \leftarrow r_1 - 4r_2$

$$\begin{bmatrix} 1 & 0 & -3 & 3 & 5 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 8: $r_1 \leftarrow r_1 - 3r_3$ and $r_2 \leftarrow r_2 + 3r_3$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \blacksquare$$

Example 6.3
Use Gauss-Jordan elimination to transform the following matrix into reduced form

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution.
By following the steps in the Gauss-Jordan algorithm we find

Step 1: $r_3 \leftarrow \frac{1}{3}r_3$

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{bmatrix}$$

Step 2: $r_1 \leftrightarrow r_3$

$$\left[\begin{array}{cccccc} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

Step 3: $r_2 \leftarrow r_2 - 3r_1$

$$\left[\begin{array}{cccccc} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

Step 4: $r_2 \leftarrow \frac{1}{2}r_2$

$$\left[\begin{array}{cccccc} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

Step 5: $r_3 \leftarrow r_3 - 3r_2$

$$\left[\begin{array}{cccccc} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Step 6: $r_1 \leftarrow r_1 + 3r_2$

$$\left[\begin{array}{cccccc} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Step 7: $r_1 \leftarrow r_1 - 5r_3$ and $r_2 \leftarrow r_2 - r_3$

$$\left[\begin{array}{cccccc} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \blacksquare$$

Gauss-Jordan elimination gives a systematic procedure for solving systems of linear equations; it is based on the idea of reducing the augmented matrix to reduced row-echelon form. Unknowns corresponding to leading entries in the reduced augmented matrix are called **dependent** or **leading variables**. If an unknown is not dependent then it is called **free** or **independent** variable. Now, the new system is equivalent to the original system and is easy to solve. If the augmented matrix is in reduced row-echelon form then to obtain the general solution one just has to move all independent variables to the right side of the equations and consider them as parameters. The dependent variables are given in terms of these parameters. We illustrate this process in the next examples.

Example 6.4
Solve the following linear system using Gauss-Jordan elimination.

$$\left\{\begin{array}{rcrcrcrcrcrcr}
x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0\\
2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1\\
 & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5\\
2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6
\end{array}\right.$$

Solution.
The augmented matrix for the system is

$$\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0\\
2 & 6 & -5 & -2 & 4 & -3 & -1\\
0 & 0 & 5 & 10 & 0 & 15 & 5\\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{array}\right]$$

Using the Gaussian-Jordan algorithm we bring the augmented matrix to reduced form as follows:

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_4 \leftarrow r_4 - 2r_1$

$$\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0\\
0 & 0 & -1 & -2 & 0 & -3 & -1\\
0 & 0 & 5 & 10 & 0 & 15 & 5\\
0 & 0 & 4 & 8 & 0 & 18 & 6
\end{array}\right]$$

Step 2: $r_2 \leftarrow -r_2$

$$\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0\\
0 & 0 & 1 & 2 & 0 & 3 & 1\\
0 & 0 & 5 & 10 & 0 & 15 & 5\\
0 & 0 & 4 & 8 & 0 & 18 & 6
\end{array}\right]$$

Step 3: $r_3 \leftarrow r_3 - 5r_2$ and $r_4 \leftarrow r_4 - 4r_2$

$$\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0\\
0 & 0 & 1 & 2 & 0 & 3 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 6 & 2
\end{array}\right]$$

Step 4: $r_3 \leftrightarrow r_4$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 5: $r_3 \leftarrow \frac{1}{6}r_3$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 6: $r_2 \leftarrow r_2 - 3r_3$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 7: $r_1 \leftarrow r_1 + 2r_2$

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The free variables are x_2, x_4, and x_5. Let $x_2 = s, x_4 = t$, and $x_5 = w$. Solving the system starting from the bottom we find $x_1 = -3s-4t-2w, x_2 = s, x_3 = -2t, x_4 = t, x_5 = w, x_6 = \frac{1}{3}$ ■

Example 6.5
Solve the following linear system using Gauss-Jordan elimination.

$$\left\{\begin{array}{cccccccccc} x_1 & + & 2x_2 & + & & & x_4 & & = & 6 \\ & & & & x_3 & + & 6x_4 & & = & 7 \\ & & & & & & & x_5 & = & 1. \end{array}\right.$$

Solution.
The augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 6 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

already in reduced form. The free variables are x_2 and x_4. So let $x_2 = s$ and $x_4 = t$. Solving the system starting from the bottom we find $x_1 = -2s - t + 6, x_3 = 7 - 6t$, and $x_5 = 1$ ■

Example 6.6
Find the general solution of the system whose augmented matrix is given by

$$\left[\begin{array}{cc|c} 1 & 2 & -7 \\ -1 & -1 & 1 \\ 2 & 1 & 5 \end{array}\right]$$

Solution.
We first transform the system to reduced form as follows.

Step 1: $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 - 2r_1$

$$\left[\begin{array}{cc|c} 1 & 2 & -7 \\ 0 & 1 & -6 \\ 0 & -3 & 19 \end{array}\right]$$

Step 2: $r_3 \leftarrow r_3 + 3r_2$

$$\left[\begin{array}{cc|c} 1 & 2 & -7 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{array}\right]$$

Step 3: $r_1 \leftarrow r_1 - 2r_2$

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{array}\right]$$

The corresponding system is given by

$$\left\{\begin{array}{rcrcr} x_1 & & & = & 5 \\ & & x_2 & = & -6 \\ & & 0 & = & 1 \end{array}\right.$$

Because of the last equation the system is inconsistent ■

Practice Problems

Problem 6.1
Which of the following matrices are not in reduced row-ehelon form and why?
(a)

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 6.2
Use Gauss-Jordan elimination to transform the matrix into reduced matrix.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$

Problem 6.3
Use Gauss-Jordan elimination to transform the matrix into reduced matrix.

$$B = \begin{bmatrix} 3 & 1 & 0 & 1 & -9 \\ 0 & -2 & 12 & -8 & -6 \\ 2 & -3 & 22 & -14 & -17 \end{bmatrix}$$

Problem 6.4
Use Gauss-Jordan elimination to convert the following matrix into reduced row-echelon form.

$$\begin{bmatrix} -2 & 1 & 1 & 15 \\ 6 & -1 & -2 & -36 \\ 1 & -1 & -1 & -11 \\ -5 & -5 & -5 & -14 \end{bmatrix}$$

Problem 6.5
Use Gauss-Jordan elimination to convert the following matrix into reduced row-echelon form.

$$\left[\begin{array}{ccccc} 3 & 1 & 7 & 2 & 13 \\ 2 & -4 & 14 & -1 & -10 \\ 5 & 11 & -7 & 8 & 59 \\ 2 & 5 & -4 & -3 & 39 \end{array}\right].$$

Problem 6.6
Solve the linear system using Gauss-Jordan elimination.

$$\left\{\begin{array}{ccccccccc} x_1 & - & 3x_2 & + & x_3 & - & x_4 & = & 2 \\ & & x_2 & + & 2x_3 & - & x_4 & = & 3 \\ & & & & x_3 & + & x_4 & = & 1. \end{array}\right.$$

Problem 6.7
Find the general solution of the system whose augmented matrix is given by

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 7 & -3 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Problem 6.8
Determine the value(s) of h such that the following matrix is the augmented matrix of a consistent linear system

$$\left[\begin{array}{cc|c} 1 & 4 & 2 \\ -3 & h & -1 \end{array}\right]$$

Problem 6.9
Solve the linear system using Gauss-Jordan elimination.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

Problem 6.10
Solve the linear system whose augmented matrix is reduced to the following reduced row-echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{array}\right]$$

Problem 6.11
Solve the following system using Gauss-Jordan elimination.

$$\left\{\begin{array}{rcrcrcrcr} 3x_1 & + & x_2 & + & 7x_3 & + & 2x_4 & = & 13 \\ 2x_1 & - & 4x_2 & + & 14x_3 & - & x_4 & = & -10 \\ 5x_1 & + & 11x_2 & - & 7x_3 & + & 8x_4 & = & 59 \\ 2x_1 & + & 5x_2 & - & 4x_3 & - & 3x_4 & = & 39 \end{array}\right.$$

Problem 6.12
Solve the following system.

$$\left\{\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & x_3 & = & -1 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & & & - & 2x_3 & = & 5 \end{array}\right.$$

Problem 6.13
Solve the following system using elementary row operations on the augmented matrix:

$$\left\{\begin{array}{rcrcrcr} 5x_1 & - & 5x_2 & - & 15x_3 & = & 40 \\ 4x_1 & - & 2x_2 & - & 6x_3 & = & 19 \\ 3x_1 & - & 6x_2 & - & 17x_3 & = & 41 \end{array}\right.$$

Problem 6.14
Solve using Gauss-Jordan elimination.

$$\left\{\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 5 \\ 4x_1 & + & 5x_2 & - & 3x_3 & + & 6x_4 & = & 9 \\ -2x_1 & + & 5x_2 & - & 2x_3 & + & 6x_4 & = & 4 \\ 4x_1 & + & 11x_2 & - & 4x_3 & + & 8x_4 & = & 2. \end{array}\right.$$

Problem 6.15
Solve using Gauss-Jordan elimination.

$$\left\{\begin{array}{rcrcrcr} 2x_1 & - & 5x_2 & + & 3x_3 & = & -4 \\ x_1 & - & 2x_2 & - & 3x_3 & = & 3 \\ -3x_1 & + & 4x_2 & + & 2x_3 & = & -4. \end{array}\right.$$

Problem 6.16
Solve using Gauss-Jordan elimination.

$$\left\{\begin{array}{rcrcrcrcrcr} 2x_1 & + & 4x_2 & + & 2x_3 & + & 4x_4 & + & 2x_5 & = & 4 \\ 2x_1 & + & 4x_2 & + & 3x_3 & + & 3x_4 & + & 3x_5 & = & 4 \\ 3x_1 & + & 6x_2 & + & 6x_3 & + & 3x_4 & + & 6x_5 & = & 6 \\ & & & & x_3 & - & x_4 & - & x_5 & = & 4. \end{array}\right.$$

Problem 6.17
A restaurant owner orders a replacement set of knives, forks, and spoons. The box arrives containing 40 utensils and weighting 141.3 oz (ignoring the weight of the box). A knife, fork, and spoon weigh 3.9 oz, 3.6 oz, and 3.0 oz, respectively.How many solutions are there for the number of knives, forks and spoons in the box?

Problem 6.18
Katherine invests \$10,000 received from her grandmother in three ways. With one part, she buys U.S saving bonds at an interest rate of 2.5% per year. She uses the second part, which amounts to twice the first, to buy mutual funds that offer a return of 6% per year. She puts the rest of the money into a money market account paying 4.5% annual interest. The first year her investments bring a return of \$470. How much did she invest in each way?

Problem 6.19
A Company that rents small moving trucks wants to purchase 25 trucks with a combined capacity of 28,000 cubic ft. Three different types of trucks are available: a 10-foot truck with a capacity of 350 cubic feet, a 14-foot truck with a capacity of 700 cubic feet, and a 24-foot truck with a capacity of 1,400 cubic feet. How many truck of each type of truck should the company purchase?

Problem 6.20
A corporation wants to lease a fleet of 12 airplanes with a combined carrying capacity of 220 passengers. The three available types of planes carry 10, 15, and 20 passengers. The monthly cost of leasing each of these types of planes is \$8,000, \$14, 000, and \$16,000.
(a) How many of each type of plane should be leased?
(b) Which of these solutions would minimize the monthly leasing cost?

7. The Algebra of Matrices

Matrices are essential in the study of linear systems. In tis section, we introdice this concept and examine four operations on matrices- addition, subtraction, scalar multiplication, and multiplication-and give their basic properties.
A **matrix A of size** $m \times n$ is a rectangular array of the form

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{array}\right]$$

where the a_{ij}'s are the **entries** of the matrix, m is the number of rows, n is the number of columns. The **zero matrix 0** is the matrix whose entries are all 0. The $n \times n$ **identity matrix** I_n is a square matrix whose main diagonal consists of $1's$ and the off diagonal entries are all 0. A matrix A can be represented with the following compact notation $A = [a_{ij}]$. The **ith row** of the matrix A is

$$[a_{i1}, a_{i2}, ..., a_{in}]$$

and the **jth column** is

$$\left[\begin{array}{c} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{array}\right]$$

In what follows we discuss the basic arithmetic of matrices.

Two matrices are said to be **equal** if they have the same size and their corresponding entries are all equal. If the matrix A is not equal to the matrix B we write $A \neq B$.

Example 7.1
Find x_1, x_2 and x_3 such that

$$\left[\begin{array}{ccc} x_1 + x_2 + 2x_3 & 0 & 1 \\ 2 & 3 & 2x_1 + 4x_2 - 3x_3 \\ 4 & 3x_1 + 6x_2 - 5x_3 & 5 \end{array}\right] = \left[\begin{array}{ccc} 9 & 0 & 1 \\ 2 & 3 & 1 \\ 4 & 0 & 5 \end{array}\right]$$

Solution.
Because corresponding entries must be equal, this gives the following linear system

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 - 5x_3 = 0 \end{cases}$$

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

The reduction of this matrix to row-echelon form is

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Step 2: $r_2 \leftrightarrow r_3$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 3 & -11 & -27 \\ 0 & 2 & -7 & -17 \end{bmatrix}$$

Step 3: $r_2 \leftarrow r_2 - r_3$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 2 & -7 & -17 \end{bmatrix}$$

Step 4: $r_3 \leftarrow r_3 - 2r_2$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The corresponding system is

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ \qquad x_2 - 4x_3 = -10 \\ \qquad\qquad x_3 = 3 \end{cases}$$

Using backward substitution we find: $x_1 = 1, x_2 = 2, x_3 = 3$ ■

Next, we introduce the operations of addition and subtraction of two matrices. If A and B are two matrices of the same size, then the **sum** (respectively **difference**) $A + B$ (respectively $A - B$) is the matrix obtained by adding (respectively subtracting) together the corresponding entries in the two matrices. Matrices of different sizes cannot be added or subtracted.

Example 7.2
Consider the matrices

$$A = \left[\begin{array}{cc} 2 & 1 \\ 3 & 4 \end{array}\right], B = \left[\begin{array}{cc} 2 & 1 \\ 3 & 5 \end{array}\right], C = \left[\begin{array}{ccc} 2 & 1 & 0 \\ 3 & 4 & 0 \end{array}\right]$$

Compute, if possible, $A + B$, $A - B$ and $B + C$.

Solution.
We have

$$A + B = \left[\begin{array}{cc} 4 & 2 \\ 6 & 10 \end{array}\right]$$

and

$$A - B = \left[\begin{array}{cc} 0 & 0 \\ 0 & -1 \end{array}\right]$$

$B + C$ are undefined since B and C are of different size ■

If A is a matrix and c is a constant, then the product cA is the matrix obtained by multiplying each entry of A by c.

Example 7.3
Consider the matrices

$$A = \left[\begin{array}{ccc} 2 & 3 & 4 \\ 1 & 2 & 1 \end{array}\right], B = \left[\begin{array}{ccc} 0 & 2 & 7 \\ 1 & -3 & 5 \end{array}\right]$$

Compute $A - 3B$.

Solution.
Using the above definitions we have

$$A - 3B = \left[\begin{array}{ccc} 2 & -3 & -17 \\ -2 & 11 & -14 \end{array}\right] ■$$

Finally, we introduce the product of two matrices. Let $A = (a_{ij})$ be a matrix of size $m \times n$ and $B = (b_{ij})$ be a matrix of size $n \times p$. Then the **product** matrix is a matrix of size $m \times p$ and entries

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj},$$

that is, c_{ij} is obtained by multiplying componentwise the entries of the i^{th} row of A by the entries of the j^{th} column of B. It is very important to keep in mind that the number of columns of the first matrix must be equal to the number of rows of the second matrix; otherwise the product is undefined.

Example 7.4
Let

$$A = \left[\begin{array}{cc} 1 & 2 \\ 3 & 2 \end{array}\right], B = \left[\begin{array}{cc} 2 & -1 \\ -3 & 4 \end{array}\right]$$

Show that $AB \neq BA$. Hence, matrix multiplication is not commutative.

Solution.
Using the definition of matrix multiplication we find

$$AB = \left[\begin{array}{cc} -4 & 7 \\ 0 & 5 \end{array}\right], BA = \left[\begin{array}{cc} -1 & 2 \\ 9 & 2 \end{array}\right]$$

Hence, $AB \neq BA$ ■

Example 7.5
Consider the linear system

$$\left\{\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9. \end{array}\right.$$

(a) Find the coefficient and augmented matrices of the linear system.
(b) Find the matrix notation.

Solution.
(a) The coefficient matrix of this system is

$$\left[\begin{array}{ccc} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{array}\right]$$

and the augmented matrix is

$$\left[\begin{array}{rrrr} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

(b) We can write the given system in matrix form as

$$\left[\begin{array}{rrr} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] = \left[\begin{array}{r} 0 \\ 8 \\ -9 \end{array}\right] \blacksquare$$

Matrix multiplication shares many properties of the product of real numbers which are listed in the following theorem

Theorem 7.1
Let A be a matrix of size $m \times n$. Then
(a) $A(BC) = (AB)C$, where B is of size $n \times p$, C of size $p \times q$.
(b) $A(B + C) = AB + AC$, where B and C are of size $n \times p$.
(c) $(B + C)A = BA + CA$, where B and C are of size $l \times m$.
(d) $c(AB) = (cA)B = A(cB)$, where c denotes a scalar.

Practice Problems

Problem 7.1
Solve the following matrix equation for a, b, c, and d

$$\left[\begin{array}{cc} a-b & b+c \\ 3d+c & 2a-4d \end{array}\right] = \left[\begin{array}{cc} 8 & 1 \\ 7 & 6 \end{array}\right]$$

Problem 7.2
Solve the following matrix equation.

$$\left[\begin{array}{cc} 3 & 2 \\ -1 & 1 \end{array}\right] + \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ -1 & 2 \end{array}\right]$$

Problem 7.3
Compute the matrix

$$3\left[\begin{array}{cc} 2 & -1 \\ 1 & 0 \end{array}\right] - 2\left[\begin{array}{cc} 1 & -1 \\ 2 & 3 \end{array}\right]$$

Problem 7.4
Find w, x, y, and z.

$$\left[\begin{array}{ccc} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & z \end{array}\right] = \left[\begin{array}{ccc} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{array}\right]$$

Problem 7.5
Determine two numbers s and t so that

$$A = \left[\begin{array}{ccc} 2 & s & t \\ 2s & 0 & s+t \\ 3 & 3 & t \end{array}\right] = \left[\begin{array}{ccc} 2 & 2s & 3 \\ s & 0 & 3 \\ t & s+t & t \end{array}\right]$$

Problem 7.6
Let A be the matrix

$$A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$$

Show that

$$A = a\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] + b\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] + c\left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right] + d\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]$$

Problem 7.7
Compute
$$\begin{bmatrix} 1 & 9 & -2 \\ 3 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 8 & -4 & 3 \\ -7 & 1 & 6 \end{bmatrix}$$

Problem 7.8
Consider the matrices
$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, C = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$$
(a) Show that $A(BC) = (AB)C$.
(b) Show that $A(B+C) = AB + AC$.

Problem 7.9
Write the linear system whose augmented matrix is given by
$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Problem 7.10
Consider the linear system
$$\left\{ \begin{array}{rcrcrcrcr} 2x_1 & + & 3x_2 & - & 4x_3 & + & x_4 & = & 5 \\ -2x_1 & & & + & x_3 & & & = & 7 \\ 3x_1 & + & 2x_2 & & & - & 4x_3 & = & 3 \end{array} \right.$$
(a) Find the coefficient and augmented matrices of the linear system.
(b) Find the matrix notation.

Problem 7.11
Find k such that
$$\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0.$$

Problem 7.12
Express the matrix notation as a system of linear equations.
$$\begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 7 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}.$$

Problem 7.13
Calculate
$$\begin{bmatrix} 5 & 4 \\ -3 & 7 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 8 \\ 6 & 0 \\ -5 & 3 \end{bmatrix}$$

Problem 7.14
Find a, b, c, d, e, and f so that
$$\begin{bmatrix} a+b & 3b & 4c \\ d & 7e & 8 \end{bmatrix} + \begin{bmatrix} -7 & 2b & 6 \\ -3d & -6 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 25 & 6 \\ -8 & 1 & 6 \end{bmatrix}$$

Problem 7.15
Find the product
$$\begin{bmatrix} -3 & 4 & 2 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} -6 & 4 \\ 2 & 3 \\ 3 & -2 \end{bmatrix}$$

Problem 7.16
Sal's Shoes and Fred's Footwear both have outlets in California and Arizona. Sal's sells shoes for \$80, sandals for \$40, and boots for \$120. Fred's prices are \$60, \$30, and \$150 for shoes, sandals and boots, respectively. Half of all sales in California stores are shoes, 1/4 are sandals, and 1/4 are boots. In Arizona the fractions are 1/5 shoes, 1/5 are sandals, and 3/5 are boots.
(a) Write a 2×3 matrix called P representing prices for the two stores and three types of footwear.
(b) Write a 2×3 matrix called F representing fraction of each type of footwear sold in each state.
(c) Only one of the two products PF and FP is meaningful. Determine which one it is, calculate the product, and describe what the entries represent.

Problem 7.17
A contractor builds three kinds of houses, models A, B, and C, with a choice of two styles, Spanish and contemporary. Matrix P shows the number of each kind of house planned for a new 100-home subdivision. The amounts for each of the exterior materials depend primarily on the style of the house. These amounts are shown in matrix Q. (concrete is in cubic yards, lumber in units of 1000 board feet, brick in 1000s, and shingles in units of 100 ft^2.) Matrix R gives the cost in dollars for each kind of material.

(a) What is the total cost of these materials for each model?
(b) How much of each of four kinds of material must be ordered?
(c) What is the total cost for exterior materials?

$$\begin{array}{cc} & \begin{array}{cc} \textit{Spanish} & \textit{Contemporary} \end{array} \\ \begin{array}{c} \textit{Model A} \\ \textit{Model B} \\ \textit{Model C} \end{array} & \begin{bmatrix} 0 & 30 \\ 10 & 20 \\ 20 & 20 \end{bmatrix} = P \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} \textit{Concrete} & \textit{Lumber} & \textit{Brick} & \textit{Shingles} \end{array} \\ \begin{array}{c} \textit{Spanish} \\ \textit{Contemporary} \end{array} & \begin{bmatrix} 10 & 2 & 0 & 2 \\ 50 & 1 & 20 & 2 \end{bmatrix} = Q \end{array}$$

$$\begin{array}{cc} & \textit{Cost per unit} \\ \begin{array}{c} \textit{Concrete} \\ \textit{Lumber} \\ \textit{Brick} \\ \textit{Shingles} \end{array} & \begin{bmatrix} 20 \\ 180 \\ 60 \\ 25 \end{bmatrix} = R \end{array}$$

8. Inverse Matrices and their Applications to Linear Systems

Most problems in practice reduces to a system with matrix notation $Ax = b$. Thus, in order to get x we must somehow be able to eliminate the coefficient matrix A. One is tempted to try to divide by A. Unfortunately such an operation has not been defined for matrices. In this section we introduce a special type of square matrices and formulate the matrix analogue of numerical division. Recall that the $n \times n$ identity square matrix is the matrix I_n whose main diagonal entries are 1 and off diagonal entries are 0.
A square matrix A of size n is called **invertible** or **non-singular** if there exists a square matrix A^{-1} of the same size such that $AA^{-1} = A^{-1}A = I_n$. In this case A^{-1} is called the **inverse** of A. A square matrix that is not invertible is called **singular.**

Example 8.1
Show that the matrix
$$B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$
is the inverse of the matrix
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution.
Using matrix multiplication one checks that $AB = BA = I_2$ ■

Example 8.2
Show that the matrix
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
is singular.

Solution.
Let $B = (b_{ij})$ be a 2×2 matrix. If $BA = I_2$ then the $(2,2)$-th entry of BA is zero while the $(2,2)-$entry of I_2 is 1, which is impossible. Thus, A is singular ■

How do you go about finding A^{-1}. We describe an algorithm derives in linear algebra for finding the inverse of a matrix, if it exists. The algorithm goes as follows: We construct the matrix $[A|I_n]$. If A is invertible then we should be able to apply elemetary row operations on the mentioned matrix and obtain an equivalent matrix of the form $[I_n|B]$. The matrix B is the inverse of A. We illustrate this algorithm in the next example.

Example 8.3
Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution.
We first construct the matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

Applying the above algorithm to obtain

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_3 \leftarrow r_3 - r_1$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

Step 2: $r_3 \leftarrow r_3 + 2r_2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

Step 3: $r_1 \leftarrow r_1 - 2r_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

Step 4: $r_2 \leftarrow r_2 - 3r_3$ and $r_1 \leftarrow r_1 + 9r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & -3 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

Step 5: $r_3 \leftarrow -r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & -5 & -2 & -1 \end{array}\right]$$

It follows that

$$A^{-1} = \left[\begin{array}{ccc} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{array}\right] \blacksquare$$

How can we tell when a square matrix A is singular? i.e., when does the algorithm of finding A^{-1} fail? The answer is provided by the following theorem

Theorem 8.1
An $n \times n$ matrix A is singular if and only if A is row equivalent to a matrix B that has a row of zeros.

Example 8.4
Show that the matrix below is singular.

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & -2 & -1 \end{array}\right]$$

Solution.
Adding -2 times of the first row to the second row, we obtain the equivalent matrix

$$\left[\begin{array}{ccc} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 5 & -2 & -1 \end{array}\right]$$

Since this matrix has a row consisting of zeros, this matrix is singular by the previous theorem $\blacksquare$

Matrix inverses can be used to solve systems of linear equations as suggested by the following theorem.

Theorem 8.2
If A is an $n \times n$ invertible matrix and b is a column matrix then the equation $Ax = b$ has a unique solution $x = A^{-1}b$.

Example 8.5
Solve the following system by using the previous theorem

$$\left\{ \begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 5x_2 & + & 3x_3 & = & 3 \\ x_1 & & & + & 8x_3 & = & 17 \end{array} \right.$$

Solution.
Writing the system in matrix notation, we find

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

By Example 8.3, the coefficient matrix is invertible so that we can write

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \end{aligned}$$

That is, $x_1 = -1, \; x_2 = -1, \; x_3 = 2$ ∎

Practice Problems

Problem 8.1
Show that the matrix is singular.

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Problem 8.2
Find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$$

Problem 8.3
If $D = (d_{ii})$ is a diagonal matrix of size n such that $d_{11}d_{22}\cdots d_{nn} \neq 0$ then D is invertible with inverse $D^{-1} = \left(\frac{1}{d_{ii}}\right)$. Find the inverse of the matrix

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Problem 8.4
Use the inverse of the coefficient matrix to solve the linear system

$$\left\{ \begin{array}{rr} 2x_1 + 5x_2 = & 15 \\ x_1 + 4x_2 = & 9. \end{array} \right.$$

Problem 8.5
Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

Problem 8.6
Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

Problem 8.7
Find the inverse of the matrix

$$A = \left[\begin{array}{rrr} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right]$$

Problem 8.8
Use the inverse of the coefficient matrix to solve the linear system:

$$\left\{\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & x_3 & = & 3 \\ -x_1 & + & x_2 & & & = & -3 \\ x_1 & & & + & x_3 & = & 2. \end{array}\right.$$

Problem 8.9
Find the inverse of the matrix

$$A = \left[\begin{array}{rrr} 1 & 4 & -1 \\ 2 & 3 & -2 \\ -1 & 2 & 3 \end{array}\right]$$

Problem 8.10
Find the inverse of the matrix, if it exists.

$$A = \left[\begin{array}{rrr} 4 & 2 & 2 \\ -1 & -3 & 4 \\ 3 & -1 & 6 \end{array}\right]$$

Problem 8.11
The management of a rent a car company plans to expand its fleet of rental cars for the next quarter by purchasing compact and full-size cars. The average cost of a compact is \$10,000, and the average cost of a full-size car is \$24,000.
(a) If a total of 800 cars is to be purchased with a budget of \$12 million, how many cars of each size will be acquired?
(b) If the predicated demand calls for a total purchase of 1,000 cars with a budget of \$14 million, how many cars of each type will be acquired?

Problem 8.12
Use the inverse of the coefficient matrix to solve the linear system:

$$\left\{\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 3x_3 & = & 2 \\ 2x_1 & + & x_2 & & 2x_3 & = & 2 \\ -2x_1 & - & 2x_2 & + & x_3 & = & 3. \end{array}\right.$$

Problem 8.13

Use the inverse of the coefficient matrix to solve the linear system:

$$\left\{\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 2x_3 & = & 1 \\ 2x_1 & + & x_2 & & & = & 2 \\ x_1 & + & 2x_2 & + & 2x_3 & = & 3. \end{array}\right.$$

Problem 8.14

Use the inverse of the coefficient matrix to solve the linear system:

$$\left\{\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 2x_1 & - & x_2 & & 2x_3 & = & 2 \\ 3x_1 & + & x_2 & + & 3x_3 & = & 4. \end{array}\right.$$

Problem 8.15

Labor and material costs for manufacturing two guitar models are given in the table below: Suppose that in a given week \$1,800 is used for labor and \$1,200 used for materials. How many of each model should be produced to use exactly each of these allocations?

Guitar model	Labor cost	Material cost
A	\$30	\$20
B	\$40	\$30

Linear Programming

9. Solving Systems of Linear Inequalities

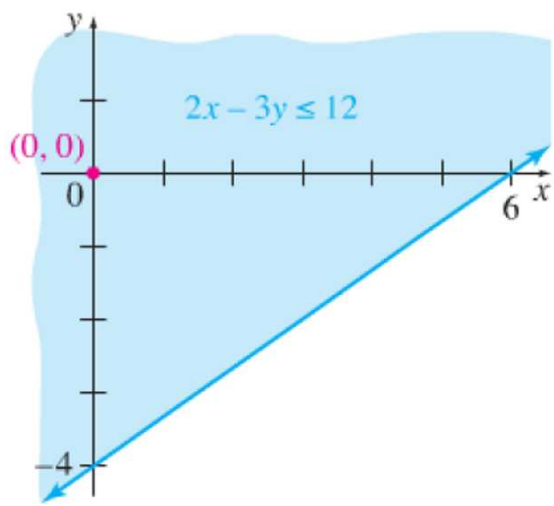
A **linear inequality** in the variables x and y is any of the inequalities

$$ax + by \leq c; ax + by \geq c; ax + by < c; ax + by > c.$$

Linear inequalities are solved graphically. To solve a linear inequality, we start by graphing the linear equation $ax + by = c$. This is a solid line in the case of $\geq$ or $\leq$ and dashed line otherwise. This line divides the cartesian plane into two halves. One of them is the solution to the inequality. To determine the right region, we select a test point in one of them not on the line. If the coordinates of the point satisfy the inequality then that's the solution region. Otherwise, the other half is the answer.

Example 9.1
Solve $2x - 3y \leq 12$.

Solution.
The solid line $2x - 3y = 12$ divides the $xy-$plane into two halves. The test point $(0, 0)$ satisfies $2(0) - 3(0) \leq 12$ so that the solution region is the one above the line as shown below ■

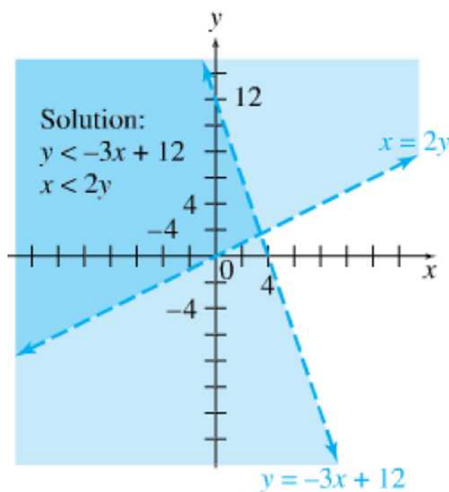


Next, we wish to solve systems of linear inequalities. That is, to find all ordered pairs (x, y) that simultaneously satisfy all the inequalities in the system. The graph is called the **feasible region** or the **solution region.** To find such a region, we find the solution region of each inequality in the system and then take the intersection of all the regions.

Example 9.2
Solve

$$\begin{cases} 3x + y & < 12 \\ x - 2y & < 0. \end{cases}$$

Solution.
Solving each inequality separately and then finding the intersetion of the regions we find

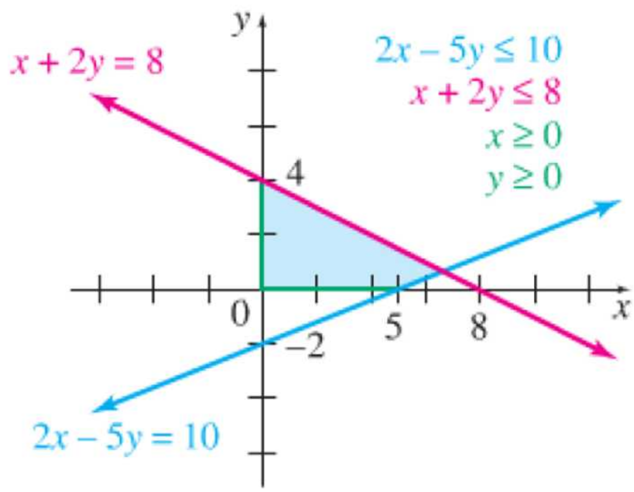


A feasible region is **bounded** if it can be enclosed within a circle. Otherwise the feasible region is said to be **undounded**. A **corner point** of a feasible region is a point in the feasible region where the boundary lines cross.

Example 9.3
Graph the feasible region of the system

$$\begin{cases} 2x - 5y & \leq 10 \\ x + 2y & \leq 8 \\ x, y & \geq 0. \end{cases}$$

Solution.
The feasible region is shown below. Note that the feasible region is bounded. The point $(\frac{20}{3}, \frac{2}{3})$ is a corner point ■

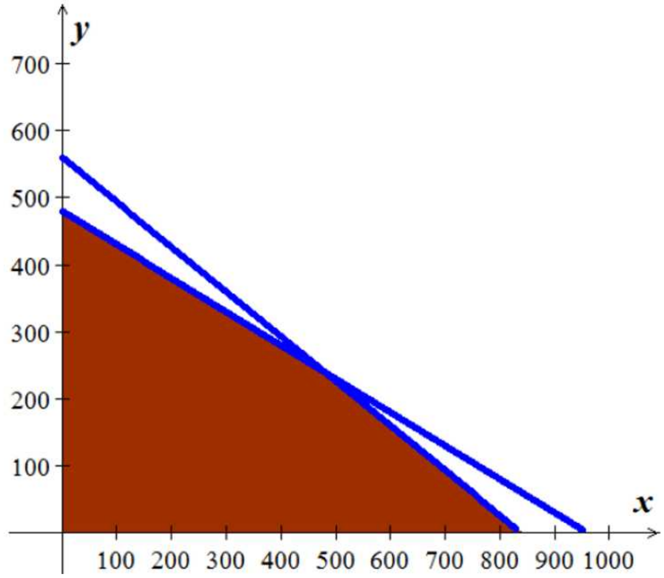


Example 9.4
A manufacturing plant makes two types of inflatable boats, a two-person boat and a four-person boat. Each two-person boat requires 0.9 labor-hour in the cutting department and 0.8 labor-hour in the assembly department. Each four-person boat requires 1.8 labor-hours in the cutting department and 1.2 labor-hour in the assembly department. The maximum labor-hours available each month in the cutting and assembly departments are 864 and 672, respectively.
(a) If x two-person boat and y four-person boats are manufactured each month, write a system of linear inequalities that reflect the conditions indicated.
(b) Find the set of feasible solutions graphically.

Solution.
(a) The system is

$$\begin{cases} 0.9x + 1.8y & \leq 864 \\ 0.8x + 1.2y & \leq 672 \\ \quad x, y & \geq 0. \end{cases}$$

(b) The feasible region is given below ■



Practice Problems

Problem 9.1
Determine if the given ordered pair is a solution to the given system of inequalities.

$$\begin{cases} 2x - 3y & \leq 9 \\ 2x - y & \geq 6. \end{cases}$$

(a) $(1, -2)$ (b) $(-1, 2)$ (c) $(3, -1)$.

Problem 9.2
Graph the feasible region of the system

$$\begin{cases} 2x + y & \leq 12 \\ x + 2y & \leq 12 \\ x, y & \geq 0. \end{cases}$$

Problem 9.3
Graph the feasible region of the system

$$\begin{cases} 3x - 2y & \geq 6 \\ x + y & \leq -5 \\ y & \geq 4. \end{cases}$$

Problem 9.4
Graph the feasible region of the system

$$\begin{cases} x + y & \geq 6 \\ 2x - y & \geq 0. \end{cases}$$

Problem 9.5
Graph the feasible region of the system

$$\begin{cases} 5x + y & \geq 20 \\ x + y & \geq 12 \\ x + 3y & \geq 18 \\ x, y & \geq 0. \end{cases}$$

Problem 9.6
Solve the following system of linear inequalities graphically, and find the corner points.

$$\left\{\begin{array}{ll} 2x+y & \leq 22 \\ x+y & \leq 13 \\ 2x+5y & \leq 50 \\ x,y & \geq 0. \end{array}\right.$$

Problem 9.7
Is the feasible region of the following system bounded or unbounded?

$$\left\{\begin{array}{ll} x-y & \leq 0 \\ x+y & \geq 3. \end{array}\right.$$

Problem 9.8
Is the feasible region of the following system bounded or unbounded?

$$\left\{\begin{array}{ll} x+y & \leq 6 \\ 2x+y & \leq 8 \\ x,y & \geq 0. \end{array}\right.$$

Problem 9.9
Solve the following system of linear inequalities graphically.

$$\left\{\begin{array}{ll} -x+y & \geq 6 \\ x+y & \geq 6 \\ y & \leq -4. \end{array}\right.$$

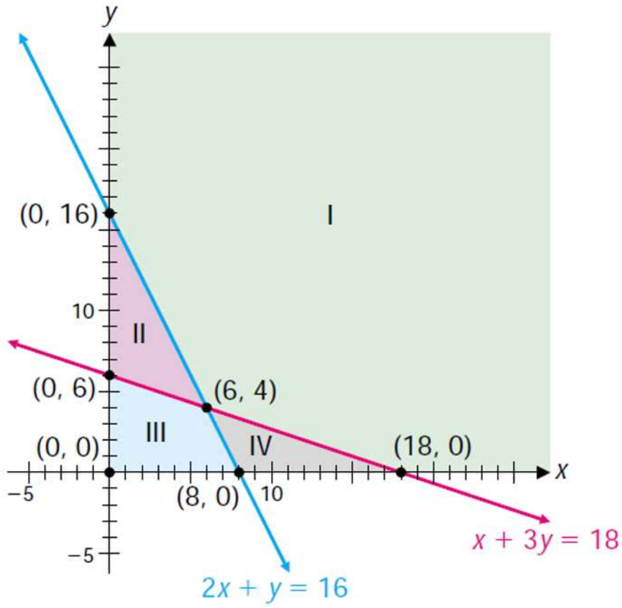
Problem 9.10
Solve the following system of linear inequalities graphically.

$$\left\{\begin{array}{ll} x-y & < -5 \\ x-y & > 5. \end{array}\right.$$

Problem 9.11
Solve the following system of linear inequalities graphically.

$$\left\{\begin{array}{ll} 2x+y & \geq 50 \\ x+y & \geq 40 \\ x,y & \geq 0. \end{array}\right.$$

Problem 9.12
Match the solution region of each system of linear inequalities with one of the four regions shown in the figure.



(a)
$$\begin{cases} x+3y & \leq 18 \\ 2x+y & \geq 16 \\ x,y & \geq 0. \end{cases}$$

(b)
$$\begin{cases} x+3y & \leq 18 \\ 2x+y & \leq 16 \\ x,y & \geq 0. \end{cases}$$

(c)
$$\begin{cases} x+3y & \geq 18 \\ 2x+y & \geq 16 \\ x,y & \geq 0. \end{cases}$$

(d)
$$\begin{cases} x+3y & \geq 18 \\ 2x+y & \leq 16 \\ x,y & \geq 0. \end{cases}$$

Problem 9.13
Find the corner points of the feasible region corresponding to the system

$$\left\{\begin{array}{ll} 2x+4y & \geq 8 \\ 3x+y & \leq 7 \\ x\geq 0, y & \leq 4. \end{array}\right.$$

Problem 9.14
A farmer can buy two types of plant food, mix A and mix B. Each cubic yard of mix A contains 20 pounds of phosphoric acid, 30 pounds of nitrogen, and 5 pounds of potash. Each cubic yard of mix B contains 10 pounds of phosphoric acid, 30 pounds of nitrogen, and 10 pounds of potash. The minimum requirements are 460 pounds of phosphoric acid, 960 pounds of nitrogen, and 220 pounds of potash. If x is the number of cubic yards of mix A used and y is the number of cubic yards of mix B used, write a system of inequalities that indicates appropriate restraints on x and y. Find the set of feasible solutions graphically for the amount of mix A and mix B that can be used.

Problem 9.15
Happy Ice Cream Cone Company makes cake cones and sugar cones, both of which must be processed in the mixing department and the baking department. Manufacturing one batch of cake cones requires 1 hour in the mixing department and 2 hours in the baking department, and producing one batch of sugar cones requires 2 hours in the mixing department and 1 hour in the baking department. Each department is operated for at most 12 hours per day.
(a) Write a system of inequalities that expresses these restrictions. (b) Graph the feasible region.
(c) Using the graph from part (b), can 3 batches of cake cones and 2 batches of sugar cones be manufactured in one day.
(d) can 4 batches of cake cones and 6 batches of sugar cones be manufactured in one day

10. Geometric Method for Solving Linear Programming Problems

Linear programming problems are often seen in economics and business applications. **Linear programming** is a technique for the optimization of a **linear objective function** such as the profit or cost associated with producing some product. This function is subject to linear equalities and linear inequalities referred to as **constraints** (such as restrictions on the resources necessary to produce the product). A linear programming algorithm finds a point in the feasible solution where the objective function has the smallest (or largest) value if such point exists. In this section we solve linear programming problems using the geometric method (also known as the graphical method or the method of corners). In Section 11, we discuss an algebraic method known as the simplex method.
The graphical method involves the following steps:
(I) Translate into mathematics a problem which will be stated in everyday conversational language. Determine the variables involved (known as **decision variables**), state the objective function to be maximized or minimized and write the constraint inequalities/equalities.
(II) Graph the feasible region represented by the constraints.
(III) Evaluate the objective function at the corner points. If the problem has a solution (known as an **optimal solution**) then it will occur at a corner point.
How can we know in advance that a linear programming has an optimal solution? The answer is provided by the following theorem.

Theorem 10.1
(i) Maxima and minima always exist for bounded feasible regions.
(ii) If the feasible region is unbounded and the coefficients of the objective function are positive then the minimum value of the objective function exists but the maximum value does not.
(iii) If the feasible region is empty then both maximum value and minimum value of the objective function do not exist.

We illustrate the geometric approach in the following examples.

Example 10.1
Maximize: $z = 3x + 4y$

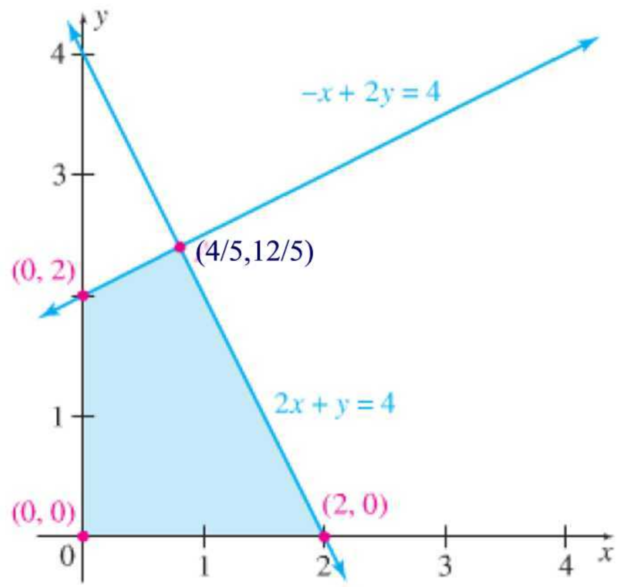
Subject to

$$\begin{cases} 2x + y & \leq 4 \\ -x + 3y & \leq 4 \\ x, y & \geq 0. \end{cases}$$

Solution.
The feasible region is bounded so that a maximum exists. Evaluating the function at the corner points, we find

$$\begin{aligned} z(0,0) =& 0 \\ z(2,0) =& 6 \\ z(0,2) =& 8 \\ z\left(\frac{4}{5}, \frac{12}{5}\right) =& 12. \end{aligned}$$

Thus, the largest value of z is 12 ■



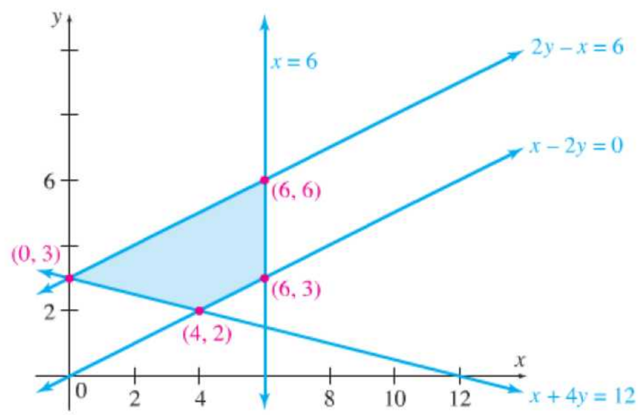
Example 10.2
Maximize and Minimize: $z = x + 10y$
Subject to

$$\begin{cases} x + 4y & \geq 12 \\ x - 2y & \leq 0 \\ -x + 2y & \leq 6 \\ x \leq 6, y & \geq 0. \end{cases}$$

Solution.
The feasible region is bounded so that a maximum and a minimum exist. Evaluating the function at the corner points, we find

$$
\begin{aligned}
z(0,3) =& 30\\
z(4,2) =& 24\\
z(6,3) =& 36\\
z(6,6) =& 66.
\end{aligned}
$$

The minimum value of z is 24 at the corner point $(4,2)$. The maximum value of z is 66 at the corner point $(6,6)$. ■



Example 10.3
Mr. Trenga plans to start a new business called River Explorers, which will rent canoes and kayaks to people to travel 10 miles down the Clarion River in Cook Forest State Park. He has \$45,000 to purchase new boats. He can buy the canoes for \$600 each and the kayaks for \$750 each. His facility can hold up to 65 boats. The canoes will rent for \$25 a day and the kayaks will rent for \$30 a day. How many canoes and how many kayaks should he buy to earn the most revenue?

Solution.
Let x represent the number of canoes and y the number of kayaks. The mathematical model for this problem for the given linear programming problem is as follows

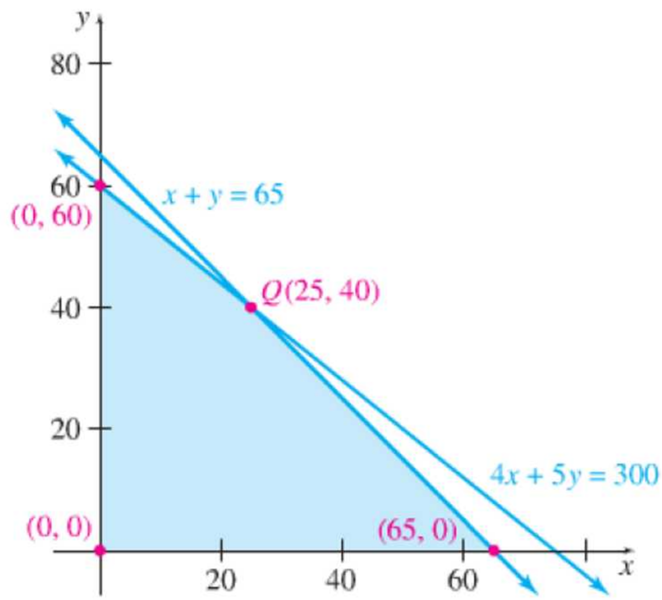
Maximize: $z = 25x + 30y$
Subject to

$$\begin{cases} x + y & \leq 65 \\ 600x + 750y & \leq 45,000 \\ x, y & \geq 0. \end{cases}$$

The feasible region is bounded so that a maximum exists. Evaluating the function at the corner points, we find

$$\begin{aligned} z(0,0) =& 0 \\ z(0,60) =& 1800 \\ z(65,0) =& 1625 \\ z(25,40) =& 1825. \end{aligned}$$

The objective function, which represents revenue, is maximized when x = 25 and y = 40. He should buy 25 canoes and 40 kayaks.



Practice Problems

Problem 10.1
Maximize and Minimize: $z = 4x + 2y$
Subject to

$$\begin{cases} 2x + y & \leq 20 \\ 10x + y & \geq 36 \\ 2x + 5y & \geq 36 \\ x, y & \geq 0. \end{cases}$$

Problem 10.2
A manufacturing company makes two types of water skis, a trick ski and a slalom ski. The trick ski requires 9 labor-hours for fabricating and 1 labor-hour for finishing. The slalom ski requires 5 labor-hours for fabricating and 1 labor-hour for finishing. The maximum labor-hours available per day for fabricating and finishing are 135 and 20 respectively. If x is the number of trick skis and y is the number of slalom skis produced per day, write a system of linear inequalities that indicates appropriate restraints on x and y. Find the set of feasible solutions graphically for the number of each type of ski that can be produced.

Problem 10.3
In a small town in South Carolina, zoning rules require that the window space (in square feet) in a house be at least one-sixth of the space used up by solid walls. The cost to build windows is \$10 per ft^2 , while the cost to build solid walls is \$20 per ft^2 . The total amount available for building walls and windows is no more than \$12,000. The estimated monthly cost to heat the house is \$0.32 for each square foot of windows and \$0.20 for each square foot of solid walls. Find the maximum total area (windows plus walls) if no more than \$160 per month is available to pay for heat.

Problem 10.4
An anthropology article presents a hypothetical situation that could be described by a linear programming model. Suppose a population gathers plants and animals for survival. They need at least 360 units of energy, 300 units of protein, and 8 hides during some time period. One unit of plants provides 30 units of energy, 10 units of protein, and no hides. One animal provides 20 units of energy, 25 units of protein, and 1 hide.

Problem 10.5
A dietician is planning a snack package of fruit and nuts. Each ounce of fruit will supply zero units of protein, 2 units of carbohydrates, and 1 unit of fat, and will contain 20 calories. Each ounce of nuts will supply 3 units of protein, 1 unit of carbohydrates, and 2 units of fat, and will contain 30 calories. Every package must provide at least 6 units of protein, at least 10 units of carbohydrates, and no more than 9 units of fat. Find the number of ounces of fruit and number of ounces of nuts that will meet the requirement with the least number of calories. What is the least number of calories?

Problem 10.6
A certain predator requires at least 10 units of protein and 8 units of fat per day. One prey of species I provides 5 units of protein and 2 units of fat; one prey of species II provides 3 units of protein and 4 units of fat. Capturing and digesting each species-II prey requires 3 units of energy, and capturing and digesting each species-I prey requires 2 units of energy. How many of each prey would meet the predator's daily food requirements with the least expenditure of energy?

Problem 10.7
A pension fund manager decides to invest a total of at most \$39 million in U.S. treasury bonds paying 4% annual interest and in mutual funds paying 8% annual interest. He plans to invest at least \$5 million in bonds and at least \$10 million in mutual funds. Bonds have an initial fee of \$100 per million dollars, while the fee for mutual funds is \$200 per million. The fund manager is allowed to spend no more than \$5000 on fees. How much should be invested in each to maximize annual interest? What is the maximum annual interest?

Problem 10.8
A small country can grow only two crops for export, coffee and cocoa. The country has 500,000 hectares of land available for the crops. Long-term contracts require that at least 100,000 hectares be devoted to coffee and at least 200,000 hectares to cocoa. Cocoa must be processed locally, and production bottlenecks limit cocoa to 270,000 hectares. Coffee requires two workers per hectare, with cocoa requiring five. No more than 1,750,000 people are available for working with these corps. Coffee produces a profit of \$220 per hectares and cocoa a profit of 4550 per hectare. How many hectares

should the country devote to each crop in order to maximize profit? Find the maximum profit.

Problem 10.9
The manufacturing process requires that oil refineries must manufacture at least 2 gal of gasoline for every gallon of fuel oil. To meet the winter demand for fuel oil, at least 3 million gal a day must be produced. The demand for gasoline is no more than 6.4 million gal per day. It takes 0.25 hour to ship each million gal of gasoline and 1 hour to ship each million gal of fuel oil out of the warehouse. No more than 4.65 hours are available for shipping. If the refinery sells gasoline for $2.50 per gal and fuel oil for $2 per gal, how many of each should be produced to maximize revenue? Find the maximum revenue.

Problem 10.10
The Muro Manufacturing Company makes two kinds of plasma screen TV sets. It produces the Flexscan set that sells for $350 profit and the Panoramic I that sells for $500 profit. On the assembly line, the Flexscan requires 5 hours, and the Panoramic I takes 7 hours. The cabinet shop spends 1 hour on the cabinet for the Flexscan and 2 hours on the cabinet for the Panoramic I. Both sets require 4 hours for testing and packing. On a particular production run, the Muro Company has available 3600 work-hour on the assembly line, 900 work-hour in the cabinet shop, and 2600 work-hours in the testing and packing department. How many sets of each type should it produce to make a maximum profit? What is the maximum profit?

Problem 10.11
A company produces small engines for several manufacturers. The company receives orders from two assembly plants for their Top-flight engine. Plant I needs at least 45 engines, and plant II needs at least 32 engines. The company can send at most 90 engines to these two assembly plants. It costs $30 per engine to ship to plant I and $40 per engine to ship to plant II. Plant I gives the company $20 in rebates toward its products for each engine they buy, while plant II gives similar $15 rebates. The company estimates that they need at least $1,200 in rebates to cover products they plan to buy from the two plants. How many engines should be shipped to each plant to minimize shipping costs? What is the minimum cost?

Problem 10.12

A chicken farmer can buy a special food mix A at 20¢per pound and a special food mix B at 40¢per pound. Each pound of mix A contains 3,000 units of nutrient N_1 and 1,000 units of nutrient N_2, and Each pound of mix B contains 4,000 units of nutrient N_1 and 4,000 units of nutrient N_2. If the minimum daily requirements for the chickens collectively are 36,000 units of nutrient N_1 and 20,000 units of nutrient N_2, how many pounds of each food mix should be used each day to minimize daily food costs while meeting (or exceeding) the minimum daily nutrient requirements? What is the minimum daily cost? Construct a mathematical model and solve using the geometric method.

Problem 10.13

A manufacturing plant makes two types of inflatable boats, a two-person boat and a four-person boat. Each two-person boat requires 0.9 labor-hour in the cutting department and 0.8 labor-hour in the assembly department. Each four-person boat requires 1.8 labor-hour in the cutting department and 1.2 labor-hour in the assembly department. The maximum labor-hour available each month in the cutting and assembly departments are 864 and 672, respectively. The company makes a profit of \$25 on each two-person boat and \$40 on each four-person boat. (a) Identify the decision variables.
(b) Write the objective function P.
(c) Write the problem constraints and the nonnegative constraints.
(d) Determine how many boats should be manufactured each month to maximize the profit. What is the maximum profit?

Problem 10.14

Certain animals in a rescue shelter must have at least 30 g of protein and at least 20 g of fat per feeding period. These nutrients come from food A, which costs 18 cents per unit and supplies 2 g of protein and 4g of fat; and food B, which costs 12 cents per unit and supplies 6 g of protein and 2 g of fat. Food B is bought under a long-term contract requiring that at least 2 units of B be used per serving. How much of each food must be bought to produce the minimum cost per serving?

Problem 10.15

A 4-H member raises only goats and pigs. She wants to raise no more than 16 animals, including no more than 10 goats. She spends \$25 to raise a goat and \$75 to raise a pig, and she has \$900 available for the project. The 4-H member wishes to maximize her profits. Each goat produces \$12 in profit and each pig \$40 in profit.

11. Simplex Method for Solving Linear Programming Problems

In this section, we discuss an algebraic method for solving linear programming problems. This method is called the **simplex method**.. Before we start discussing the simplex method, we point out that every linear program can be converted into **standard form.**

Maximize: $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
Subject to:

$$\left\{\begin{array}{ccc} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \\ x_1, x_2, \cdots, x_n & \geq & 0 \\ b_1, b_2, \cdots, b_m & \geq & 0 \end{array}\right.$$

where the objective is maximized, the constraints are equalities and the variables are all nonnegative. This is done as follows:
• If the problem is min z, convert it to max $-z$.
• If a constraint is $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$, convert it into an equality constraint by adding a nonnegative **slack variable** s_i. The resulting constraint is $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + s_i = b_i$ where $s_i \geq 0$.
• If a constraint is $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i$, convert it into an equality constraint by subtracting a nonnegative **surplus variable** s_i. The resulting constraint is $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - s_i = b_i$,, where $s_i \geq 0$.

Example 11.1
Transform the following linear program into standard form.

Minimize: $z = -2x_1 + 3x_2$
Subject to:

$$\left\{\begin{array}{ccccccc} x_1 & - & 3x_2 & + & 2x_3 & \leq & 3 \\ -x_1 & + & 2x_2 & & & \geq & 2 \\ x_1, & & x_2, & & x_3 & \geq & 0. \end{array}\right.$$

Solution.
The linear problem in standard form is
Maximize: $-z = 2x_1 - 3x_2$
Subject to:

$$\begin{cases} x_1 & - & 3x_2 & + & 2x_3 & + & s_1 & & & = & 3 \\ -x_1 & + & 2x_2 & & & & & - & s_2 & = & 2 \\ x_1, & & x_2, & & x_3, & & s_1, & & s_2 & \geq & 0 \blacksquare \end{cases}$$

For simplicity, in this book we solve hand only the case where the constraints are of the form $\leq$. We will explain the steps of the simplex method while we progress through an example.

Example 11.2
Maximize: $z = x_1 + x_2$
Subject to:

$$\begin{cases} 2x_1 & + & x_2 & \leq & 4 \\ x_1 & + & 2x_2 & \leq & 3 \\ x_1, & & x_2, & \geq & 0. \end{cases}$$

Solution.
First, we convert the problem into standard form by adding slack variables $s_1 \geq 0$ and $s_2 \geq 0$.
Maximize: $z = x_1 + x_2$
Subject to:

$$\begin{cases} 2x_1 & + & x_2 & + & s_1 & & & = & 4 \\ x_1 & + & 2x_2 & & & + & s_2 & = & 3 \\ x_1, & & x_2, & & s_1, & & s_2, & \geq & 0. \end{cases}$$

Putting the objective equation written in the form $z - x_1 - x_2 = 0$ together with the constraints, we get the following system of linear equations.

$$\begin{cases} z & - & x_1 & - & x_2 & & & & & = & 0 & Row0 \\ & & 2x_1 & + & x_2 & + & s_1 & & & = & 4 & Row1 \\ & & x_1 & + & 2x_2 & & & + & s_2 & = & 3. & Row2 \end{cases}$$

Our goal is to maximize z, while satisfying these equations and, in addition, $x_1, x_2, s_1, s_2 \geq 0$. The variables (other than the special variable z) which appear in only one equation are called the **basic variables.** All other variables are called **nonbasic variables**. Thus, in our case, s_1, s_2 are the basic variables whereas x_1 and x_2 are the nonbasic variables. A **basic solution** is obtained from the system of equations by setting the nonbasic variables to zero. This yields

$$x_1 = x_2 = 0, s_1 = 4, s_2 = 3, z = 0.$$

A basic solution where all values are non-negative is referred to as a **basic feasible solution.**
Now, is the basic solution optimal or we can increase z? By looking at Row 0 above, we see that we can increase z by increasing x_1 or x_2. This is because these variables have a negative coefficient in Row 0. If all coefficients in Row 0 had been nonnegative, we could have concluded that the current basic solution is optimum, since there would be no way to increase z (remember that all variables x_i and s_i must remain ≥ 0). We have just discovered the first rule of the simplex method.

Rule 1 If all variables have a nonnegative coefficient in Row 0, the current basic solution is optimal. Otherwise, pick a variable x_j with a negative coefficient in Row 0.

The variable chosen by Rule 1 is called the **entering variable.** Here let us choose, say, x_1 as our entering variable. It really does not matter which variable we choose as long as it has a negative coefficient in Row 0. The idea is to **pivot** in order to make the nonbasic variable x_1 become a basic variable. In the process, some basic variables will become nonbasic (the **leaving variable**). This change of basis is done using the Gauss-Jordan procedure. What is needed next is to choose the pivot element? It will be found using Rule 2 of the simplex method. In order to better understand the rationale behind this second rule, let us try both possible pivots and see why only one is acceptable.
First, try the pivot element in Row 1 which is $2x_1$. This yields

$$\left\{\begin{array}{lllllllll} z & & - & \frac{1}{2}x_2 & + & \frac{1}{2}s_1 & & = & 2 \quad Row0 \\ & x_1 & + & \frac{1}{2}x_2 & + & \frac{1}{2}s_1 & & = & 2 \quad Row1 \\ & & & \frac{3}{2}x_2 & - & \frac{1}{2}s_1 & + \; s_2 & = & 1 \quad Row2 \end{array}\right.$$

with basic solution $x_2 = s_1 = 0, x_1 = 2, s_2 = 1, z = 2$.
Now, try the pivot element in Row 2,i.e., x_1. This yields,

$$\begin{cases} z \quad + x_2 \quad + s_2 = 3 & Row0 \\ \quad - 3x_2 + s_1 - 2s_2 = -2 & Row1 \\ x_1 + 2x_2 \quad + s_2 = 3 & Row2 \end{cases}$$

with basic solution $x_2 = s_2 = 0, x_1 = 3, s_1 = -2, z = 3$.
Which pivot should we choose? The first one, of course, since the second yields an **infeasible basic solution**! Indeed, remember that we must keep all variables ≥ 0. With the second pivot, we get $s_1 = -2$ which is infeasible. How could we have known this ahead of time, before actually performing the pivots? The answer is, by comparing the ratios

$$\frac{\text{Right Hand Side}}{\text{Entering Variable Coefficient}}$$

in Rows 1 and 2 of the original system. Here we get $\frac{4}{2}$ in Row 1 and $\frac{3}{1}$ in Row 2. If you pivot in a row with minimum ratio, you will end up with a feasible basic solution (i.e. you will not introduce negative entries in the Right Hand Side), whereas if you pivot in a row with a ratio which is not minimum you will always end up with an infeasible basic solution. *A negative pivot element* would not be good either, for the same reason. We have just discovered the second rule of the simplex method.

Rule 2 For each Row $i, i \geq 1$, where there is a strictly positive entering variable coefficient, compute the ratio of the Right Hand Side to the entering variable coefficient. Choose the pivot row as being the one with MINIMUM ratio.

Once you have indicated the pivot element by Rule 2, you perform a Gauss-Jordan pivot. This gives you a new basic solution. Is it an optimal solution? This question is addressed by Rule 1, so we have closed the loop. The simplex method iterates between Rules 1, 2 and pivoting until Rule 1 guarantees that the current basic solution is optimal. That's all there is to the simplex method.
Now back to our problem, after our first pivot, we obtained the following

system of equations.

$$\begin{cases} z & & - & \frac{1}{2}x_2 & + & \frac{1}{2}s_1 & & & = & 2 & Row0 \\ & x_1 & + & \frac{1}{2}x_2 & + & \frac{1}{2}s_1 & & & = & 2 & Row1 \\ & & & \frac{3}{2}x_2 & - & \frac{1}{2}s_1 & + & s_2 & = & 1 & Row2 \end{cases}$$

with basic solution $x_2 = s_1 = 0, x_1 = 2, s_2 = 1, z = 2$.
Is this solution optimal? No, Rule 1 tells us to choose x_2 as entering variable. Where should we pivot? Rule 2 tells us to pivot in Row 2, since the ratios are $\frac{2}{\frac{1}{2}} = 4$ for Row 1, and $\frac{1}{\frac{3}{2}} = \frac{2}{3}$ for Row 2, and the minimum occurs in Row 2. So we pivot on $\frac{3}{2}x_2$ in the above system of equations. This yields

$$\begin{cases} z & & & + & \frac{1}{3}s_1 & + & \frac{1}{3}s_2 & = & \frac{7}{3} & Row0 \\ & x_1 & & + & \frac{2}{3}s_1 & - & \frac{1}{3}s_2 & = & \frac{5}{3} & Row1 \\ & & x_2 & - & \frac{1}{3}s_1 & + & \frac{2}{3}s_2 & = & \frac{2}{3} & Row2 \end{cases}$$

with basic solution $s_1 = s_2 = 0, x_1 = \frac{5}{3}, x_2 = \frac{2}{3}, z = \frac{7}{3}$. Now Rule 1 tells us that this basic solution is optimal, since there are no more negative entries in Row 0.
All the above computations can be represented very compactly in **simplex tableau** form.

x_1	x_2	s_1	s_2	z	RHS	Basic solution
-1	-1	0	0	1	0	basic $s_1 = 4, s_2 = 3$
2	1	1	0	0	4	nonbasic $x_1 = x_2 = 0$
1	2	0	1	0	3	$z = 0$
0	$-\frac{1}{2}$	$\frac{1}{2}$	0	1	2	basic $x_1 = 2, s_2 = 1$
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	2	nonbasic $x_2 = s_1 = 0$
0	$\frac{3}{2}$	$-\frac{1}{2}$	1	0	1	$z = 2$
0	0	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{7}{3}$	basic $x_1 = \frac{5}{3}, x_2 = \frac{2}{3}$
1	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{5}{3}$	nonbasic $s_1 = s_2 = 0$
0	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{2}{3}$	$z = \frac{7}{3}$

Example 11.3
Maximize: $z = 3x_1 + 2x_2 + x_3$
Subject to:

$$\begin{cases} 2x_1 & + & x_2 & + & x_3 & \leq & 150 \\ x_1 & + & 2x_2 & + & 8x_3 & \leq & 200 \\ 2x_1 & + & 3x_2 & + & x_3 & \leq & 320 \\ x_1, & & x_2, & & x_3 & \geq & 0. \end{cases}$$

Solution.
First, we convert the problem into standard form by adding slack variables $s_1 \geq 0, s_2 \geq 0$ and $s_3 \geq 0$.
Maximize: $z = 3x_1 + x_2 + x_3$
Subject to:

$$\begin{cases} 2x_1 + x_2 + x_3 + s_1 = 150 \\ x_1 + 2x_2 + 8x_3 + s_2 = 200 \\ 2x_1 + 3x_2 + x_3 + s_3 = 320 \\ x_1, \quad x_2, \quad x_3 \quad s_1, \quad s_2, \quad s_3 \geq 0. \end{cases}$$

The initial linear system is

$$\begin{cases} z - 3x_1 - 2x_2 - x_3 = 0 \\ 2x_1 + x_2 + x_3 + s_1 = 150 \\ x_1 + 2x_2 + 8x_3 + s_2 = 200 \\ 2x_1 + 3x_2 + x_3 + s_3 = 320 \end{cases}$$

Using simplex tableaux, we find

x_1	x_2	x_3	s_1	s_2	s_3	z	RHS	Basic solution
-3	-2	-1	0	0	0	1	0	basic $s_1 = 150, s_2 = 200, s_3 = 320$
2	1	1	1	0	0	0	150	nonbasic $x_1 = x_2 = x_3 = 0$
1	2	8	0	1	0	0	200	
2	3	1	0	0	1	0	320	$z = 0$
0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	0	0	1	225	basic $x_1 = 75, s_2 = 125, s_3 = 170$
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	75	nonbasic $x_2 = x_3 = s_1 = 0$
0	$\frac{3}{2}$	$\frac{15}{2}$	$-\frac{1}{2}$	1	0	0	125	
0	2	0	$-\frac{1}{2}$	0	1	0	170	$z = 225$
0	0	3	$\frac{4}{3}$	$\frac{1}{3}$	0	1	$\frac{800}{3}$	basic $x_1 = \frac{100}{3}, x_2 = \frac{250}{3}, s_3 = \frac{10}{3}$
1	0	-2	$\frac{2}{3}$	$-\frac{1}{3}$	0	0	$\frac{100}{3}$	nonbasic $x_3 = s_1 = s_2 = 0$
0	1	5	$-\frac{1}{3}$	$\frac{2}{3}$	0	0	$\frac{250}{3}$	
0	0	-10	$-\frac{1}{3}$	$-\frac{4}{3}$	1	0	$\frac{10}{3}$	$z = \frac{800}{3}$ ■

Practice Problems

Problem 11.1
Solve by the simplex method.

Maximize: $z = 50x_1 + 80x_2$
Subject to:

$$\left\{\begin{array}{rcrcl} x_1 & + & 2x_2 & \leq & 32 \\ 3x_1 & + & 4x_2 & \leq & 84 \\ x_1, & & x_2, & \geq & 0. \end{array}\right.$$

Problem 11.2
Solve by the simplex method.

Maximize: $z = 2x_1 + 3x_2$
Subject to:

$$\left\{\begin{array}{rcrcl} -3x_1 & + & 4x_2 & \leq & 12 \\ & & x_2 & \leq & 2 \\ x_1, & & x_2, & \geq & 0. \end{array}\right.$$

Problem 11.3
Solve by the simplex method.

Maximize: $z = 2x_1 + x_2$
Subject to:

$$\left\{\begin{array}{rcrcl} 5x_1 & + & x_2 & \leq & 9 \\ x_1 & + & x_2 & \leq & 5 \\ x_1, & & x_2, & \geq & 0. \end{array}\right.$$

Problem 11.4
Solve by the simplex method.
Maximize: $z = x_1 + 2x_2 - x_3$
Subject to:

$$\left\{\begin{array}{rcrcrcl} 2x_1 & + & x_2 & + & x_3 & \leq & 14 \\ 4x_1 & + & 2x_2 & + & 3x_3 & \leq & 28 \\ 2x_1 & + & 5x_2 & + & 5x_3 & \leq & 30 \\ x_1, & & x_2, & & x_3 & \geq & 0. \end{array}\right.$$

Problem 11.5
Solve by the simplex method.
Maximize: $z = 6x_1 + 5x_2 + 4x_3$
Subject to:

$$\begin{cases} 2x_1 & + & x_2 & + & x_3 & \leq & 180 \\ x_1 & + & 3x_2 & + & 2x_3 & \leq & 300 \\ 2x_1 & + & x_2 & + & 2x_3 & \leq & 240 \\ x_1, & & x_2, & & x_3 & \geq & 0. \end{cases}$$

Problem 11.6
Carrie is working to raise money for the homeless by sending information letters and making follow-up calls to local labor organizations and church groups. She discovers that each church group requires 2 hours of letter writing and 1 hour of follow-up, while for each labor union she needs 2 hours of letter writing and 3 hours of follow-up. Carrie can raise \$100 from each church group and \$200 from each union local, and she has a maximum of 16 hours of letter-writing time and a maximum of 12 hours of follow-up time available per month. Determine the most profitable mixture of groups she should contact and the most money she can raise in a month.

Problem 11.7
The Texas Poker Company assembles three different poker sets. Each Royal Flush poker set contains 1000 poker chips, 4 decks of cards, 10 dice, and 2 dealer buttons. Each Deluxe Diamond poker set contains 600 poker chips, 2 decks of cards, 5 dice, and one dealer button. The full House poker set contains 300 poker chips, 2 decks of cards, 5 dice, and one dealer button. The Texas Poker Company has 2,800,000 poker chips, 10,000 decks of cards, 25,000 dice, and 6000 dealer buttons in stock. They earn a profit of \$38 for each Royal Flush poker set, \$22 for each Deluxe Diamond poker set, and \$12 for each Full House poker set. How many of each type of poker set should they assemble to maximize profit? What is the maximum profit?

Problem 11.8
The Muro Manufacturing Company makes two kinds of plasma screen TV sets. It produces the Flexscan set that sells for \$350 profit and the Panoramic I that sells for \$500 profit. On the assembly line, the Flexscan requires 5 hours, and the Panoramic I takes 7 hours. The cabinet shop spends 1 hour on the cabinet for the Flexscan and 2 hours on the cabinet for the Panoramic I.

Both sets require 4 hours for testing and packing. On a particular production run, the Muro Company has available 3600 work-hours on the assembly line, 900 work-hours in the cabinet shop, and 2600 work-hours in the testing and packing department.
(a) How many sets of each type should it produce to make a maximum profit? What is the maximum profit?
(b) Find the values of any nonzero slack variables and describe what they tell you about unused time.

Problem 11.9
A baker has 150 units of flour, 90 of sugar, and 150 of raisins. A loaf of raisin bread requires 1 unit of flour, 1 of sugar, and 2 of raisins, while a raisin cake needs 5, 2, and 1 units, respectively.
(a) If raisin bread sells for \$1.75 a loaf and raisin cake for \$4.00 each, how many of each should baked so that gross income is maximized?
(b) What is the maximum gross income?
(c) Does it require all of the available units of flour, sugar, and raisins to produce the number the maximum profit? If not, how much of each ingredient is left over? Compare any leftover to the value of the relevant slack variable.

Problem 11.10
A farmer owns a 100 acre farm and plans to plant at most three crops. The seed for crops A, B, and C costs \$24, \$40, and \$30 per acre, respectively. A maximum of \$3,600 can be spent on seed. Crops A, B, and C require 1, 2, and 2 workdays per acre, respectively, and three are a maximum of 160 workdays available. If the farmer can make a profit of \$140 per acre on crop A, \$200 per acre on crop B, and \$160 per acre on crop C, how many acres of each crop that should be planted to maximize the profit?

Problem 11.11
A candy company makes three types of candy, solid, fruit, and cream filled, and packages these candies in three different assortments. A box of assortment I contains 4 solid, 4 fruit, and 12 cream and sells for \$9.40. A box of assortment II contains 12 solid, 4 fruit, and 4 cream and sells for \$7.60. A box of assortment III contains 8 solid, 8 fruit, and 8 cream and sells for \$11.00. The manufacturing costs per piece of candy are \$0.20 for solid, \$0.25 for fruit, and \$0.30 for cream. The company can manufacture up to 4800 solid, 4000 fruit, and 5600 cream candies weekly. How many boxes of each

type should the company produce in order to maximize profit? What is their maximum profit?

Problem 11.12
A small company manufactures three different electronic components for computers. Component A requires 2 hours of fabrication and 1 hour of assembly; component B requires 3 hours of fabrication and 1 hour of assembly; and component C requires 2 hours of fabrication and 2 hours of assembly. The company has up to 1,000 labor-hours of fabrication time and 800 labor-hours of assembly time available per week. The profit on each component, A, B, and C is \$7, \$8, and \$10, respectively. How many components of each week in order to maximize its profit (assuming that all components that it manufactures can be sold)? What is the maximum profit?

Problem 11.13
An investor has at most \$100,000 to invest in government bonds, mutual funds, and money market funds. The average yields for government bonds, mutual funds, and money market funds are 8%, 13%,, and 15%, respectively. The investor's policy requires that the total amount invested in mutual and money market funds not exceed the amount invested in government bonds. How much should be invested in each type of investment in order to maximize the return? What is the maximum return?

Problem 11.14
A department store chain up to \$20,000 to spend on television advertising for a sale. All ads will be placed with one television station, where 30-second as costs \$1,000 on daytime TV and is viewed by 14,000 potential customers, \$2,000 on prime-time TV and is viewed by 24,000 potential customer, and \$1,500 on late-night TV and is viewed by 18,000 potential customers. The television station will not accept a total of more than 15 ads in all three time periods. How many ads should be placed in each time period in order to maximize the number of potential customers who will see the ads? How many potential customers will see the ads?

Problem 11.15
A political scientist has received a grant to find a research project involving voting trends. The budget of the grant includes \$3,200 for conducting door-to-door interviews the day before an election. Undergraduate students,

graduate students, and faculty members will be hired to conduct the interviews. Each undergraduate student will conduct 18 interviews and be paid \$100. Each graduate student will conduct 25 interviews and be paid \$150. Each faculty members will conduct 30 interviews and be paid \$200. Due to limited transportation facilities, no more than 20 interviews can be hired. How many undergraduate students, graduate students, and faculty members should be hired in order to maximize the number of interviews that will be conducted? What is the maximum number of interviews?

12. The Dual Problem: Minimization with $\geq$ Constraints

In this section, we consider solving the linear problem

$$\text{Minimize: } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
Subject to:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \geq & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq & b_m \\ x_1, x_2, \cdots, x_n & \geq & 0 \\ b_1, b_2, \cdots, b_m & \geq & 0. \end{cases}$$

Associated with each minimization problem with $\geq$ constraints a maximization problem with $\leq$ constraints, called the **dual problem**. The formation of thid dual problem consists of the following steps:

- Use the coefficients and constants in the problem constraints and the objective function to form a matrix A with the coefficients of the objective function in the last row.
- Interchange the rows and columns of the matrix A to form the matrix A^T, the **transpose** of A.
- Use the rows of A^T to form a maximization problem with $\leq$ constraints.

Example 12.1
Form the dual problem: Minimize: $z = 16x_1 + 45x_2$
Subject to:

$$\begin{cases} 2x_1 & + & 5x_2 & \geq & 50 \\ x_1 & + & 3x_2 & \geq & 27 \\ x_1, & & x_2, & \geq & 0. \end{cases}$$

Solution.
The coefficient matrix is

$$A = \begin{bmatrix} 2 & 5 & 50 \\ 1 & 3 & 27 \\ 16 & 45 & 1 \end{bmatrix}$$

The transpose matrix is

$$A^T = \begin{bmatrix} 2 & 1 & 16 \\ 5 & 3 & 45 \\ 50 & 27 & 1 \end{bmatrix}$$

The dual problem is Maximize: $z = 50y_1 + 27y_2$
Subject to:

$$\left\{ \begin{array}{rcrcr} 2y_1 & + & y_2 & \leq & 16 \\ 5y_1 & + & 3y_2 & \leq & 45 \\ y_1, & & y_2, & \geq & 0 \end{array} \right. \blacksquare$$

The following result establishes the relationship between the solution of a minimization problem and the solution of its dual problem:

Theorem 12.1
A minimization problem has a solution if and only if its dual problem has a solution. If a solution exists, both problems yield the same optimal value. However, the optimal solutions are different.

It follows from the above theorem that one finds the solution to the minimization problem by solving the dual problem using the simplex method. We illustrate the process of solving a minimization problem in the next two examples.

Example 12.2
Minimize: $z = 16x_1 + 9x_2 + 21x_3$
Subject to:

$$\left\{ \begin{array}{rcrcrcr} x_1 & + & x_2 & + & 3x_3 & \geq & 12 \\ 2x_1 & + & x_2 & + & x_3 & \geq & 16 \\ x_1, & & x_2, & & x_3 & \geq & 0. \end{array} \right.$$

Solution.
We have

$$A = \begin{bmatrix} 1 & 1 & 3 & 12 \\ 2 & 1 & 1 & 16 \\ 16 & 9 & 21 & 1 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & 2 & 16 \\ 1 & 1 & 9 \\ 3 & 1 & 21 \\ 12 & 16 & 1 \end{bmatrix}$$

Thus, the dual problem is

Maximize: $Z = 12y_1 + 16y_2$
Subject to:

$$\begin{cases} y_1 & + & 2y_2 & \leq & 16 \\ y_1 & + & y_2 & \leq & 9 \\ 3y_1 & + & y_2 & \leq & 21 \\ y_1, & & y_2, & \geq & 0. \end{cases}$$

Convert the problem into standard form by adding slack variables $x_1 \geq 0, x_2 \geq 0$ and $x_3 \geq 0$ we find
Maximize: $Z = 12y_1 + 16y_2$
Subject to:

$$\begin{cases} y_1 & + & 2y_2 & + & x_1 & & & & & = & 16 \\ y_1 & + & y_2 & & & + & x_2 & & & = & 9 \\ 3y_1 & + & y_2 & & & & & + & x_3 & = & 21 \\ y_1, & & y_2, & & x_1, & & x_2, & & x_3, & \geq & 0. \end{cases}$$

Note that the slack variables are the same as the variables of the minimization problem. This choice will be clear at the end of the example. Now, applying the simplex method we find

y_1	y_2	x_1	x_2	x_3	Z	RHS	Basic solution
-12	-16	0	0	0	1	0	basic $x_1 = 16, x_2 = 9, x_3 = 21$
1	2	1	0	0	0	16	nonbasic $y_1 = y_2 = 0$
1	1	0	1	0	0	9	
3	1	0	0	1	0	21	$Z = 0$
-4	0	8	0	0	1	128	basic $y_2 = 8, x_2 = 1, x_3 = 17$
0.5	1	0.5	0	0	0	8	nonbasic $y_1 = x_1 = 0$
0.5	0	-0.5	1	0	0	1	
2.5	0	-0.5	0	1	0	17	$Z = 128$
0	0	4	8	0	1	136	basic $y_1 = 2, y_2 = 7, x_3 = 12$
0	1	1	-1	0	0	7	nonbasic $x_1 = x_2 = 0$
1	0	-1	2	0	0	2	
0	0	2	-5	1	0	12	$Z = 136$

Now, comes the confusing part. Using the geometric method of Section 10, one finds the optimal solution $x_1 = 4, x_2 = 8, x_3 = 0$ with optimal value $z = 136$. But this can be found from the first row. This justifies our use of x_1, x_2, and x_3 as slack variables ■

Example 12.3
Minimize: $z = 2x_1 + 3x_2$
Subject to:

$$\begin{cases} x_1 & - & 2x_2 & \geq & 2 \\ -x_1 & + & x_2 & \geq & 1 \\ x_1, & & x_2, & \geq & 0. \end{cases}$$

Solution.
We have

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 2 & 1 & 1 \end{bmatrix}$$

Thus, the dual problem is

Maximize: $Z = 2y_1 + y_2$
Subject to:

$$\begin{cases} y_1 & - & y_2 & \leq & 2 \\ -2y_1 & + & y_2 & \leq & 3 \\ y_1, & & y_2, & \geq & 0. \end{cases}$$

Convert the problem into standard form by adding slack variables $x_1 \geq 0$ and $x_2 \geq 0$ we find
Maximize: $Z = 2y_1 + y_2$
Subject to:

$$\begin{cases} y_1 & - & y_2 & + & x_1 & & & = & 2 \\ -2y_1 & + & y_2 & & & + & x_2 & = & 3 \\ y_1, & & y_2, & & x_1, & & x_2, & \geq & 0. \end{cases}$$

Using the simlplex method, we find

y_1	y_2	x_1	x_2	Z	RHS	Basic solution
-2	-1	0	0	1	0	basic $x_1 = 2, x_2 = 3$
1	-1	1	0	0	2	nonbasic $y_1 = y_2 = 0$
-2	1	0	1	0	3	$Z = 0$
0	-3	2	0	1	4	basic $x_2 = 7.5, y_1 = 2$
1	-1	1	0	0	2	nonbasic $y_2 = x_1 = 0$
0	-1	2	1	0	7	$Z = 0$

The -3 in the first row indicates that the second column is the pivot column. Since no positive elements appear in the pivot column below the first row, we are unable to select a pivot row. We stop the operation and conclude that this maximization problem has no optimal solution. Hence, by Theorem 12.1, the original minimization problem has no solution ■

Practice Problems

Problem 12.1
Minimize: $z = 3x_1 + 2x_2$
Subject to:

$$\left\{\begin{array}{rcrcl} 2x_1 & + & x_2 & \geq & 6 \\ x_1 & + & x_2 & \geq & 4 \\ x_1, & & x_2, & \geq & 0. \end{array}\right.$$

Problem 12.2
Minimize: $z = 2x_1 + 10x_2 + 8x_3$
Subject to:

$$\left\{\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & \geq & 6 \\ & + & x_2 & + & 2x_3 & \geq & 8 \\ -x_1 & + & 2x_2 & + & 2x_3 & \geq & 4 \\ x_1, & & x_2, & & x_3 & \geq & 0. \end{array}\right.$$

Problem 12.3
Minimize: $z = 20,000x_1 + 25,000x_2$
Subject to:

$$\left\{\begin{array}{rcrcr} 400x_1 & + & 300x_2 & \geq & 25,000 \\ 300x_1 & + & 400x_2 & \geq & 27,000 \\ 200x_1 & + & 500x_2 & \geq & 30,000 \\ x_1, & & x_2, & \geq & 0. \end{array}\right.$$

Problem 12.4
Minimize: $z = 7x_1 + 5x_2$
Subject to:

$$\left\{\begin{array}{rcrcl} x_1 & + & x_2 & \geq & 16 \\ 5x_1 & + & 2x_2 & \geq & 50 \\ x_1, \geq 0 & & x_2 & \geq & 5. \end{array}\right.$$

Problem 12.5
Minimize: $z = 21x_1 + 14x_2$
Subject to:

$$\left\{\begin{array}{rcrcl} 2x_1 & + & 2x_2 & \geq & 12 \\ 3x_1 & + & x_2 & \geq & 6 \\ x_1 & + & 3x_2 & \geq & 9 \\ x_1, & & x_2, & \geq & 0. \end{array}\right.$$

Problem 12.6
Minimize: $z = 7x_1 + 5x_2 + 4x_3 + 3x_4$ Subject to:

$$\begin{cases} x_1 + x_2 & \leq 700 \\ x_3 + x_4 & \leq 900 \\ x_1 + x_3 & \geq 500 \\ x_2 + x_4 & \geq 1,000 \\ x_1, \quad x_2, \quad x_3, \quad x_4 & \geq 0. \end{cases}$$

Problem 12.7
Minimize: $z = 250x_1 + 350x_2 + 290x_3 + 320x_4$ Subject to:

$$\begin{cases} x_1 + x_2 & \leq 1,000 \\ x_3 + x_4 & \leq 2,000 \\ x_1 + x_3 & \geq 1,200 \\ x_2 + x_4 & \geq 1,600 \\ x_1, \quad x_2, \quad x_3, \quad x_4 & \geq 0. \end{cases}$$

Problem 12.8
Minimize: $z = 30x_1 + 36x_2 + 39x_3$ Subject to:

$$\begin{cases} 20x_1 + 10x_2 + 20x_3 & \geq 480 \\ 10x_1 + 10x_2 + 20x_3 & \geq 320 \\ 10x_1 + 15x_2 + 5x_3 & \geq 225 \\ x_1, \quad x_2, \quad x_3, & \geq 0. \end{cases}$$

Problem 12.9
Minimize: $z = 20x_1 + 24x_2 + 18x_3$ Subject to:

$$\begin{cases} 20x_1 + 10x_2 + 10x_3 & \geq 300 \\ 10x_1 + 10x_2 + 10x_3 & \geq 200 \\ 10x_1 + 15x_2 + 10x_3 & \geq 240 \\ x_1, \quad x_2, \quad x_3, & \geq 0. \end{cases}$$

Problem 12.10
A computer manufacturing company has two assembly plants, plant A and plant B, and two distribution outlets, outlet I and outlet II. Plant A can assemble at most 700 computers a month, and plant B can assemble at most 900 computers a month. Outlet I must have at least 500 computers a month, and outlet II must have at least 1,000 computers a month. Transportation

costs for shipping one computer from each plant to each outlet are as follows: \$7 from plant A to outlet I, \$5 from plant A to outlet II, \$4 from plant B to outlet I, \$3 from plant B to outlet II. Find a shipping schedule that will minimize the total cost of shipping the computers from the assembly plants to the distribution outlets. What is this minimum cost?

Problem 12.11
Acme Micros markets computers with single-sided and double-sided drives. The disk driers are supplied by two other companies, Associated Electronics and Digital Drives. Associated Electronics charges \$250 for a single-sided disk drive and \$350 for a double-sided disk drive. Digital Drives charges \$290 for a single-sided disk drive and \$320 for a double-sided disk drive. Associated Electronics can supply at most 1,000 disk drives in any combination of single-sided and double-sided drives. The combined monthly total supplied by Digital Drives cannot exceed 2,000 disk drives. Acme Micros needs at least 1,200 single-sided drives and at least 1,600 double-sided drives each month. How many disk drives of each type should Acme Micros order from each supplier in order to meet its monthly demand and minimize the purchase cost? What is the minimum purchase cost?

Problem 12.12
A farmer can buy three types of plant food, mix A, mix B, and mix C. Each cubic yard of mix A contains 20 pounds of phosphoric acid, 10 pounds of nitrogen, and 10 pound of potash. Each cubic yard of mix B contains 10 pounds of phosphoric acid, 10 pounds of nitrogen, and 15 pound of potash. Each cubic yard of mix C contains 20 pounds of phosphoric acid, 20 pounds of nitrogen, and 5 pound of potash. The minimum monthly requirements are 480 pounds of phosphoric acid, 320 pounds of nitrogen, and 225 pound of potash. If mix A costs \$30 per cubic yard, mix B costs \$36 per cubic yard, and mix C \$39 per cubic yard, how many cubic yards of each mix should the farmer blend to meet the minimum monthly requirements at a minimal cost? What is the minimum cost?

Problem 12.13
A dietician in a hospital is to arrange a special dietusing three foods, L, M, and N. Each ounce of food L contains 20 units of calcium, 10 units of iron, 10 units of vitamin A, and 20 units of cholesterol. Each ounce of food M contains 10 units of calcium, 10 units of iron, 15 units of vitamin A, and

24 units of cholesterol. Each ounce of food N contains 10 units of calcium, 10 units of iron, 10 units of vitamin A, and 18 units of cholesterol. If the minimum daily requirements are 300 units of calcium, 200 units of iron, and 240 units of vitamin A, how many ounces of each food should be used to meet the minimum requirements and at the same time minimize cholesterol intake? What is the minimum cholesterol intake?

Counting Principles, Permuations, and Combinations

13. Sets

Set is the most basic term in mathematics and computer science. Hardly any discussion in either subject can proceed without *set* or some synonym such as *class* or *collection*. In this section we introduce the concept of sets and its various operations and then study the properties of these operations.

Throughout this book, we assume that the reader is familiar with the following number systems:

• The set of all positive integers

$$\mathbb{N} = \{1, 2, 3, \cdots\}.$$

• The set of all integers

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}.$$

• The set of all rational numbers

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \ with \ b \neq 0\}.$$

• The set $\mathbb{R}$ of all real numbers.
• The set of all complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

where $i = \sqrt{-1}$.

We define a **set** A as a collection of well-defined objects (called **elements** or **members** of A) such that for any given object x either one (but not both) of the following holds:

- x belongs to A and we write $x \in A$.
- x does not belong to A, and in this case we write $x \notin A$.

Example 13.1
Which of the following is a well-defined set.

(a) The collection of good books.
(b) The collection of left-handed individuals in Russellville.

Solution.
(a) The collection of good books is not a well-defined set since the answer to the question "Is *My life* a good book" may be subject to dispute.
(b) This collection is a well-defined subject since a person is either left-handed or right-handed ■

We denote sets by capital letters $A, B, C, \cdots$ and elements by lowercase letters $a, b, c, \cdots$
There are two different ways to represent a set. The first one is to list, without repetition, the elements of the set. The other way is to describe a property that characterizes the elements of the set.
We define the **empty** set, denoted by $\emptyset$, to be the set with no elements.

Example 13.2
List the elements of the following sets.
(a) $\{x|x \text{ is a real number such that } x^2 = 1\}$.
(b) $\{x|x \text{ is an integer such that } x^2 - 3 = 0\}$.

Solution.
(a) $\{-1, 1\}$.
(b) $\emptyset$ ■

Example 13.3
Use a property to give a description of each of the following sets.
(a) $\{a, e, i, o, u\}$.
(b) $\{1, 3, 5, 7, 9\}$.

Solution.
(a) $\{x|x \textit{ is a vowel}\}$.
(b) $\{n \in \mathbb{N}^*|n \textit{ is odd and less than } 10\}$ ■

The number of elements of a set is called the **cardinality** of the set. We write $|A|$ to denote the cardinality of the set A. If A has a finite cardinality we say that A is a **finite** set. Otherwise, it is called **infinite**. For infinite set, we write $|A| = \infty$. For example, $|\mathbb{N}| = \infty$.

Example 13.4
What is the cardinality of each of the following sets.
(a) $\emptyset$.
(b) $\{\emptyset\}$.
(c) $\{a, \{a\}, \{a, \{a\}\}\}$.

Solution.
(a) $|\emptyset| = 0$
(b) $|\{\emptyset\}| = 1$
(c) $|\{a, \{a\}, \{a, \{a\}\}\}| = 3$ ■

Let A and B be two sets. We say that A is a **subset** of B, denoted by $A \subseteq B$, if and only if every element of A is also an element of B. If there exists an element of A which is not in B then we write $A \not\subseteq B$. For any set A we have $\emptyset \subseteq A \subseteq A$. That is, every set has at least two subsets: the empty set and the set itself.

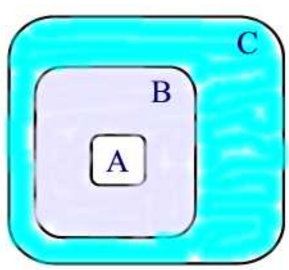
Example 13.5
Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, and $C = \{4, 6\}$. Determine which of these sets are subsets of which other(s) of these sets.

Solution.
$B \subseteq A$ and $C \subseteq A$ ■

If sets A and B are represented as regions in the plane, relationships between A and B can be represented by pictures, called **Venn diagrams.**

Example 13.6
Represent $A \subseteq B$ using a Venn diagram.

Solution.



Two sets A and B are said to be **equal** if and only if $A \subseteq B$ and $B \subseteq A$. We write $A = B$. Thus, to show that $A = B$ it suffices to show the double inclusions mentioned in the definition. For non-equal sets we write $A \neq B$.

Example 13.7
Determine whether each of the following pairs of sets are equal.
(a) $\{1, 3, 5\}$ and $\{5, 3, 1\}$.
(b) $\{\{1\}\}$ and $\{1, \{1\}\}$.

Solution.
(a) $\{1, 3, 5\} = \{5, 3, 1\}$.
(b) $\{\{1\}\} \neq \{1, \{1\}\}$ since $1 \notin \{\{1\}\}$ ■

Let A and B be two sets. We say that A is a **proper** subset of B, denoted by $A \subset B$, if $A \subseteq B$ and $A \neq B$. Thus, to show that A is a proper subset of B we must show that every element of A is an element of B and there is an element of B which is not in A.

Example 13.8
Order the sets of numbers: $\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{N}$ using $\subset$

Solution.
$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ ■

Example 13.9
Determine whether each of the following statements is true or false.
(a) $x \in \{x\}$ (b) $\{x\} \subseteq \{x\}$ (c) $\{x\} \in \{x\}$
(d) $\{x\} \in \{\{x\}\}$ (e) $\emptyset \subseteq \{x\}$ (f) $\emptyset \in \{x\}$

Solution.
(a) True (b) True (c) False (d) True (e) True (f) False ■

If U is a given set whose subsets are under consideration, then we call U a **universal set.** Let U be a universal set and A, B be two subsets of U. The **absolute complement** of A (See Figure 13.1(I)) is the set

$$A^c = \{x \in U | x \notin A\}.$$

The **relative complement** of A with respect to B (See Figure 13.1(II)) is the set

$$B - A = \{x \in U | x \in B \ and \ x \notin A\}.$$

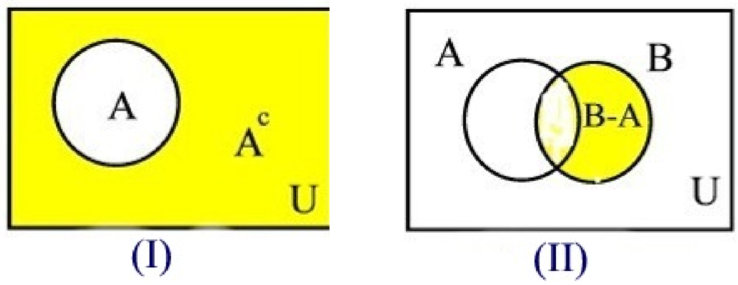


Figure 13.1

Example 13.10
Let $U = \mathbb{R}$. Consider the sets $A = \{x \in \mathbb{R} | x < -1 \ or \ x > 1\}$ and $B = \{x \in \mathbb{R} | x \leq 0\}$. Find

(a) A^c.
(b) $B - A$.

Solution.
(a) $A^c = [-1, 1]$.
(b) $B - A = [-1, 0]$ ■

Example 13.11
Write the set of irrational numbers using complements.

Solution.
The set of irrational numbers is just the set $\mathbb{R} - \mathbb{Q}$ ■

Given two sets A and B. The **union** of A and B is the set

$$A \cup B = \{x | x \in A \ or \ x \in B\}$$

where the 'or' is inclusive.(See Figure 13.2)

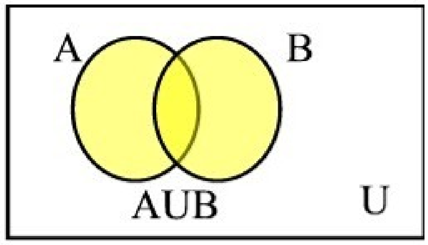


Figure 13.2

The above definition can be extended to more than two sets. More precisely, if $A_1, A_2, \cdots$, are sets then

$$\cup_{n=1}^{\infty} A_n = \{x | x \in A_i \ for \ some \ i \in \mathbb{N}\}.$$

The **intersection** of A and B is the set (See Figure 13.3)

$$A \cap B = \{x | x \in A \ and \ x \in B\}.$$

If $A \cap B = \emptyset$ we say that A and B are **disjoint** sets.

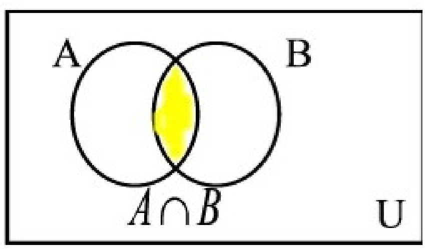


Figure 13.3

Given the sets $A_1, A_2, \cdots$, we define

$$\cap_{n=1}^{\infty} A_n = \{x | x \in A_i \ for \ all \ i \in \mathbb{N}\}.$$

Remark 13.1
Nore that the Venn diagrams of $A \cap B$ and $A \cup B$ show that $A \cap B = B \cap A, A \cup B = B \cup A, A \subset A \cup B, B \subset A \cup B, A \cap B \subset A$, and $A \cap B \subset B$.

Example 13.12
Let $A = \{a, b, c\}, B = \{b, c, d\}$, and $C = \{b, c, e\}$.

(a) Find $A \cup (B \cap C), (A \cup B) \cap C$, and $(A \cup B) \cap (A \cup C)$. Which of these sets are equal?
(b) Find $A \cap (B \cup C), (A \cap B) \cup C$, and $(A \cap B) \cup (A \cap C)$. Which of these sets are equal?
(c) Find $A - (B - C)$ and $(A - B) - C$. Are these sets equal?

Solution.
(a) $A \cup (B \cap C) = A, (A \cup B) \cap C = \{b, c\}, (A \cup B) \cap (A \cup C) = \{b, c\} = (A \cup B) \cap C$.
(b) $A \cap (B \cup C) = \{b, c\}, (A \cap B) \cup C = C, (A \cap B) \cup (A \cap C) = \{b, c\} = (A \cap B) \cup C$.
(c) $A - (B - C) = A$ and $(A - B) - C = \{a\} \neq A - (B - C)$ ■

Practice Problems

Problem 13.1
Which of the following sets are equal?
(a) $\{a, b, c, d\}$
(b) $\{d, e, a, c\}$
(c) $\{d, b, a, c\}$
(d) $\{a, a, d, e, c, e\}$

Problem 13.2
Let $A = \{c, d, f, g\}$, $B = \{f, j\}$, and $C = \{d, g\}$. Answer each of the following questions. Give reasons for your answers.
(a) Is $B \subseteq A$?
(b) Is $C \subseteq A$?
(c) Is $C \subseteq C$?
(d) Is C is a proper subset of A?

Problem 13.3
(a) Is $3 \in \{1, 2, 3\}$?
(b) Is $1 \subseteq \{1\}$?
(c) Is $\{2\} \in \{1, 2\}$?
(d) Is $\{3\} \in \{1, \{2\}, \{3\}\}$?
(e) Is $1 \in \{1\}$?
(f) Is $\{2\} \subseteq \{1, \{2\}, \{3\}\}$?
(g) Is $\{1\} \subseteq \{1, 2\}$?
(h) Is $1 \in \{\{1\}, 2\}$?
(i) Is $\{1\} \subseteq \{1, \{2\}\}$?
(j) Is $\{1\} \subseteq \{1\}$?

Problem 13.4
Let $A = \{b, c, d, f, g\}$ and $B = \{a, b, c\}$. Find each of the following:
(a) $A \cup B$.
(b) $A \cap B$.
(c) $A - B$.
(d) $B - A$.

Problem 13.5
Indicate which of the following relationships are true and which are false:
(a) $\mathbb{Z}^+ \subseteq \mathbb{Q}$.

(b) $\mathbb{R}^- \subset \mathbb{Q}$.
(c) $\mathbb{Q} \subset \mathbb{Z}$.
(d) $\mathbb{Z}^+ \cup \mathbb{Z}^- = \mathbb{Z}$.
(e) $\mathbb{Q} \cap \mathbb{R} = \mathbb{Q}$.
(f) $\mathbb{Q} \cup \mathbb{Z} = \mathbb{Z}$.
(g) $\mathbb{Z}^+ \cap \mathbb{R} = \mathbb{Z}^+$
(h) $\mathbb{Z} \cup \mathbb{Q} = \mathbb{Q}$.

Problem 13.6
Find sets A, B, and C such that $A \cap C = B \cap C$ but $A \neq B$.

Problem 13.7
Give: $U = \{1,2,3,4,5,6,7,8,9\}, A = \{2,4,6,8\}, B = \{2,3,4,5,6\}, C = \{1,2,3,8,9\}$. Find
(a) $A \cap B$.
(b) $A \cup B$.
(c) B^c.

Problem 13.8
Give: $U = \{1,2,3,4,5,6,7,8,9\}, A = \{2,4,6,8\}, B = \{2,3,4,5,6\}, C = \{1,2,3,8,9\}$. Find
(a) $A^c \cap C$.
(b) $B \cap (B \cup C)$.
(c) $A^c \cap (B^c \cup C)$.
(d) $(A \cap B^c) \cup C^c$.

Problem 13.9
Give: $U = \{1,2,3,4,5,6,7,8,9\}, A = \{1,2,3,4\}, B = \{1,3,5,7\}, C = \{2,4\}$. Find
(a) $A \cup B$
(b) $A \cap B$
(c) $B \cap C$
(d) B^c.

Problem 13.10
A small town has two radio stations, an AM station and an FM station. A survey of 100 residents of the town produced the following results: In the last 30 days, 65 people have listened to the AM station, 45 have listened to the FM station, and 30 have listened to both stations. Draw a Venn diagram for this problem.

Problem 13.11
A department store classifies credit applicants by gender, marital status, and employment status. Let the universal set be the set of all applicants, M be the set of male applicants, S be the set of single applicants, and E be the set of employed applicants. Describe each set in words.
(a) $M \cap E$ (b) $M^c \cup S^c$ (c) $M^c \cap S^c$.

Problem 13.12
A researcher collecting data on 100 households finds that • 21 have a DVD player;
• 56 have a videocassette recorder (VCR); and
• 12 have both.
(a) How many do not have a VCR?
(b) How many have neither a DVD player nor a VCR?
(c) How many have a DVD player but not a VCR?

Problem 13.13
A survey of 77 freshman business students at a large university produced the following results

25 of the students read Business Week;
19 read the Wall Street Journal;
27 do not read Fortune

Moreover, • 11 read Business Week but not Wall Street Journal;
• 11 read the Wall Street Journal and Fortune;
• 13 read Business Week and Fortune;
• 9 read all three.
(a) How many students read none of the publications?
(b) How many read only Fortune?
(c) How many read Business Week and the Wall Street Journal, but not Fortune?

Problem 13.14
In a survey of 500 businesses it was found that 250 had copiers and 300 had fax machines. It was also determined that 100 businesses had both copiers and fax machines.
(a) How many had either a copier or a fax machine?

(b) How many had neither a copier nor a fax machine?
(c) How many had a copier, but no fax machine?
(d) How many had a fax machine, but no copier?
(e) How many had no fax machines?

Problem 13.15
In a class of 35 students, 19 are married and 20 are blondes. Given that there are 7 students that are both married and blonde, answer the following questions.
(a) How many are married, but not blonde?
(b) How many are blonde but not married?
(c) How many are blonde or married?
(d) How many are neither blonde nor married?
(e) How many are not blonde?

14. Counting Principles

For a set X, $n(X)$ denotes the number of elements of X. It is easy to see that for any two sets A and B we have the following result known as the **addition principle of counting**

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Indeed, $n(A)$ gives the number of elements in A including those that are common to A and B. The same holds for $n(B)$. Hence, $n(A)+n(B)$ includes twice the number of common elements. Hence, to get an accurate count of the elements of $A \cup B$, it is necessary to subtract $n(A \cap B)$ from $n(A)+n(B)$. Note that if A and B are disjoint then $n(A \cap B) = 0$ and consequently $n(A \cup B) = n(A) + n(B)$.

Example 14.1
A total of 35 programmers interviewed for a job; 25 knew FORTRAN, 28 knew PASCAL, and 2 knew neither languages. How many knew both languages?

Solution.
Let A be the group of programmers that knew FORTRAN, B those who knew PASCAL. Then $A \cap B$ is the group of programmers who knew both languages. By the Inclusion-Exclusion Principle we have

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

That is,

$$33 = 25 + 28 - |A \cap B|.$$

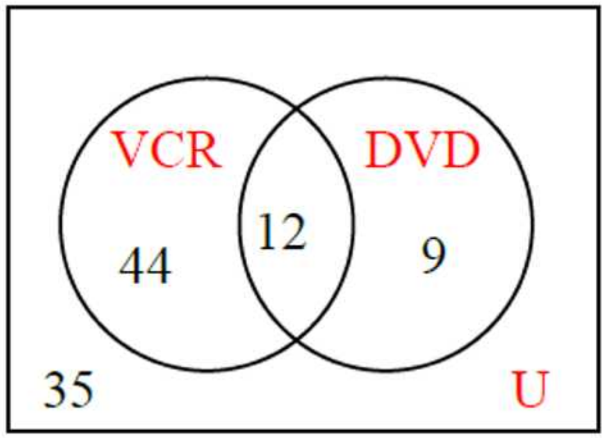
Solving for $n(A \cap B)$ we find $n(A \cap B) = 20$ ■

Example 14.2
A researcher collecting data on 100 households finds that

21 have a DVD player;
56 have a videocassette recorder (VCR); and
12 have both

(a) How many do not have a VCR?
(b) How many have neither a DVD player nor a VCR?
(c) How many have a DVD player but not a VCR?

Solution.
The Venn diagram of this problem is shown below.



(a) $35 + 9 = 44$ households.
(b) 35 households.
(c) 9 households ■

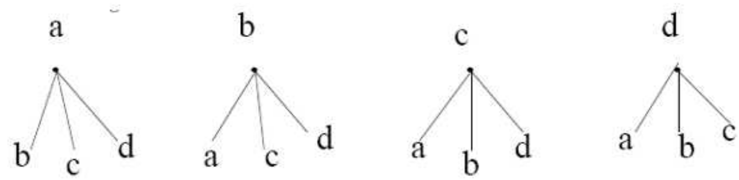
Another important rule of counting is the **multiplication rule**. It states that if a decision consists of k steps, where the first step can be made in n_1 different ways, the second step in n_2 ways, $\cdots$, the kth step in n_k ways, then the decision itself can be made in $n_1 n_2 \cdots n_k$ ways.

Example 14.3
(a) How many possible outcomes are there if 2 distinguishable dice are rolled?
(b) Suppose that a state's license plates consist of 3 letters followed by four digits. How many different plates can be manufactured? (no repetitions)

Solution.
(a) By the multiplication rule there are $6 \times 6 = 36$ possible outcomes.
(b) By the multiplication rule there are $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7$ possible license plates. ■

Example 14.4
Let $\Sigma = \{a, b, c, d\}$ be an alphabet with 4 letters. Let Σ^2 be the set of all words of length 2 with letters from Σ. Find the number of all words of length 2 where the letters are not repeated. First use the product rule. List the words by means of a **tree diagram**.

Solution.
By the multiplication rule there are $4 \times 3 = 12$ different words. Constructing a tree diagram



we find that the words are

$$\{ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc\} \blacksquare$$

Example 14.5
In designing a study of the effectiveness of migraine medicines, 3 factors were considered:

(i)	Medicine (A,B,C,D, Placebo)
(ii)	Dosage Level (Low, Medium, High)
(iii)	Dosage Frequency (1,2,3,4 times/day)

In how many possible ways can a migraine patient be given medicine?

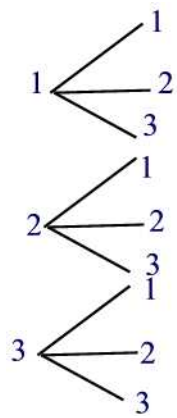
Solution.
The choice here consists of three stages, that is, $k = 3$. The first stage, can be made in $n_1 = 5$ different ways, the second in $n_2 = 3$ different ways, and the third in $n_3 = 4$ ways. Hence, the number of possible ways a migraine patient can be given medecine is $n_1 \cdot n_2 \cdot n_3 = 5 \cdot 3 \cdot 4 = 60$ different ways $\blacksquare$

Example 14.6
(a) In how many ways can 4 cards be drawn, with replacement, from a deck of 52 cards?
(b) In how many ways can 4 cards be drawn, without replacement, from a deck of 52 cards?

Solution.
(a) By the multiplication rule there are 52^4 possible combinations.
(b) Again by the multiplication rule there are $52 \times 51 \times 50 \times 49$ possible ways $\blacksquare$

Example 14.7
A lottery allows you to select a two-digit number. Each digit may be either 1,2 or 3. Use a tree diagram to show all the possible outcomes and tell how many different numbers can be selected.

Solution.



The different numbers are $\{11, 12, 13, 21, 22, 23, 31, 32, 33\}$ ■

Practice Problems

Problem 14.1
How many license-plates with 3 letters followed by 3 digits exist?

Problem 14.2
How many numbers in the range 1000 - 9999 have no repeated digits?

Problem 14.3
A certain combination lock can be set to open to any 3-letter sequence.
(a) How many sequences are possible?
(b) How many sequences are possible if no letter is repeated?

Problem 14.4
Each question on a multiple-choice test has 5 choices. If there are 5 such question on a test, how many different response sheets are possible if only 1 choice is marked for each question?

Problem 14.5
Morse code uses a sequence of dots and dashes to represent letters and words. How many sequences are possible with at most 3 symbols?

Problem 14.6
A teacher has 5 different books that he wishes to arrange side by side. How many different arrangements are possible?

Problem 14.7
A company offers its employees health plans from three different companies R, S, and T. Each company offers two levels of coverage, A and B, with one level requirement additional employee contributions. What are the combined choices, and how many choices are?

Problem 14.8
An entertainment guide recommends 6 restaurants and 3 plays that appeal to a couple.
(a) If the couple goes to dinner or to a play, how many selections are possible?
(b) If the couple goes to dinner and then to a play, how many combined selections are possible?

Problem 14.9
A coin is tossed with possible outcomes heads, H, or tails, T. Then a single die is tossed with possible outcomes 1, 2, 3, 4, 5, or 6. How many combined outcomes are there?

Problem 14.10
Suppose there are 5 chicken dishes and 8 beef dishes. How many selections does a customer have ?

Problem 14.11
Angie will draw one card from a standard deck of playing cards. How many ways can she choose
(a) a king or a queen?
(b) a king or a black card?
(c) an even number or a spade?
(d) a heart, a diamond, or a club?

Problem 14.12
The state of California makes license plates with 3 letters followed by 3 numbers, and also plates with 3 numbers followed by 3 letters. How many different license plates are possible in California?

Problem 14.13
A family consists of a mother, a father, 3 girl children and 5 boy children. How many ways can the family choose
(a) one girl to wash the dishes and one boy to dry the dishes?
(b) a boy, a girl, and a parent to go grocery shopping?
(c) one child and one parent to sweep the garage?
(d) a father or a girl to take out the trash?
(e) a female or a child to wash the car?

Problem 14.14
Max is a waiter in a restaurant. In one week, 35 people ordered shrimp and 45 people ordered steak. Max noted that 15 of these people ordered a platter that had both shrimp and steak on it. Use the addition principle of counting to count the number of people served in that week who ordered either shrimp or steak.

Problem 14.15
A group of 75 people includes 32 who play tennis, 37 play golf, and 8 who play both tennis and golf. How many people in the group play neither sport?

15. Permutations and Combinations

The multiplication principle can be used to develop two conting techniques that are useful in probability theory. More specifically, these techniques count the possible number of arrangements without the need to list them.

Permutations

Consider the following problem: In how many ways can 8 horses finish in a race (assuming there are no ties)? We can look at this problem as a decision consisting of 8 steps. The first step is the possibility of a horse to finish first in the race, the second step the horse finishes second, ... , the 8th step the horse finishes 8th in the race. Thus, by the Fundamental Principle of counting there are

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40,320 \text{ ways.}$$

This problem exhibits an example of an ordered arrangement, that is, the order the objects are arranged is important. Such ordered arrangement is called a **permutation**. Products such as $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ can be written in a shorthand notation called factoriel. That is, $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8!$ (read "8 factoriel"). In general, we define **n factoriel** by

$$n! = \begin{cases} n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0 \end{cases}$$

where n is a positive integer or zero.

Example 15.1

Evaluate the following expressions:

(a) $6!$ (b) $\frac{10!}{7!}$.

Solution.

(a) $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$
(b) $\frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 \cdot 8 = 720$ ■

Using factoriels we see that the number of permutations of n objects is $n!$.

Example 15.2

There are $6!$ permutations of the 6 letters of the word "square." In how many of them is r the second letter?

Solution.
Let r be the second letter. Then there are 5 ways to fill the first spot, 4 ways to fill the third, 3 to fill the fourth, and so on. There are 5! such permutations ■

Example 15.3
Five different books are on a shelf. In how many different ways could you arrange them?

Solution.
The five books can be arranged in $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$ ways ■

Counting Permutations
We next consider the permutations of a set of objects taken from a larger set. Suppose we have n items. How many ordered arrangements of r items can we form from these n items? The number of permutations is denoted by $P(n, r)$. The n refers to the number of different items and the r refers to the number of them appearing in each arrangement. This is equivalent to finding how many different ordered arrangements of people we can get on r chairs if we have n people to choose from. We proceed as follows.
The first chair can be filled by any of the n people; the second by any of the remaining $(n-1)$ people and so on. The rth chair can be filled by $(n-r+1)$ people. Hence we easily see that

$$P(n, r) = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}.$$

Example 15.4
How many ways can gold, silver, and bronze medals be awarded for a race run by 8 people?

Solution.
Using the permuation formula we find $P(8, 3) = \frac{8!}{(8-3)!} = 336$ ways ■

Example 15.5
How many five-digit zip codes can be made where all digits are unique? The possible digits are the numbers 0 through 9.

Solution.
$P(10,5) = \frac{10!}{(10-5)!} = 30,240$ zip codes ■

Combinations
As mentioned above, in a permutation the order of the set of objects or people is taken into account. However, there are many problems in which we want to know the number of ways in which r objects can be selected from n distinct objects in arbitrary order. For example, when selecting a two-person committee from a club of 10 members the order in the committee is irrelevant. That is choosing Mr A and Ms B in a committee is the same as choosing Ms B and Mr A. A **combination** is a group of items in which the order does not make a difference.

Counting Combinations
Let $C(n,r)$ denote the number of ways in which r objects can be selected from a set of n distinct objects. Since the number of groups of r elements out of n elements is $C(n,r)$ and each group can be arranged in $r!$ ways, $P(n,r) = r!C(n,r)$. It follows that

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}.$$

Example 15.6
How many ways can two slices of pizza be chosen from a plate containing one slice each of pepperoni, sausage, mushroom, and cheese pizza.

Solution.
In choosing the slices of pizza, order is not important. This arrangement is a combination. Thus, we need to find $C(4,2) = \frac{4!}{2!(4-2)!} = 6$. So, there are six ways to choose two slices of pizza from the plate ■

Example 15.7
How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of 3 faculty members from the mathematics department and 4 from the computer science department, if there are 9 faculty members of the math department and 11 of the CS department?

Solution.
There are $C(9,3) \cdot C(11,4) = \frac{9!}{3!(9-3)!} \cdot \frac{11!}{4!(11-4)!} = 27,720$ ways ■

Practice Problems

Problem 15.1
Compute each of the following expressions.

(a) $(2!)(3!)(4!)$
(b) $(4 \times 3)!$
(c) $4 \cdot 3!$
(d) $4! - 3!$
(e) $\frac{8!}{5!}$
(f) $\frac{8!}{0!}$

Problem 15.2
Compute each of the following.

(a) $P(7, 2)$ (b) $P(8, 8)$ (c) $P(25, 2)$

Problem 15.3
How many four-letter code words can be formed using a standard 26-letter alphabet
(a) if repetition is allowed?
(b) if repetition is not allowed?

Problem 15.4
Certain automobile license plates consist of a sequence of three letters followed by three digits.

(a) If no repetitions of letters are permitted, how many possible license plates are there?
(b) If no letters and no digits are repeated, how many license plates are possible?

Problem 15.5
A combination lock has 40 numbers on it.

(a) How many different three-number combinations can be made?
(b) How many different combinations are there if the numbers must be all different?

Problem 15.6

(a) Miss Murphy wants to seat 12 of her students in a row for a class picture. How many different seating arrangements are there?

(b) Seven of Miss Murphy's students are girls and 5 are boys. In how many different ways can she seat the 7 girls together on the left, and then the 5 boys together on the right?

Problem 15.7

Using the digits 1, 3, 5, 7, and 9, with no repetitions of the digits, how many

(a) one-digit numbers can be made?
(b) two-digit numbers can be made?
(c) three-digit numbers can be made?
(d) four-digit numbers can be made?

Problem 15.8

There are five members of the Math Club. In how many ways can the positions of officers, a president and a treasurer, be chosen?

Problem 15.9

(a) A baseball team has nine players. Find the number of ways the manager can arrange the batting order.

(b) Find the number of ways of choosing three initials from the alphabet if none of the letters can be repeated.

Problem 15.10

Compute each of the following: (a) C(7,2) (b) C(8,8) (c) C(25,2)

Problem 15.11

The Library of Science Book Club offers three books from a list of 42. If you circle three choices from a list of 42 numbers on a postcard, how many possible choices are there?

Problem 15.12

At the beginning of the second quarter of a mathematics class for elementary school teachers, each of the class's 25 students shook hands with each of the other students exactly once. How many handshakes took place?

Problem 15.13
There are five members of the math club. In how many ways can the two-person Social Committee be chosen?

Problem 15.14
A consumer group plans to select 2 televisions from a shipment of 8 to check the picture quality. In how many ways can they choose 2 televisions?

Problem 15.15
The Chess Club has six members. In how many ways
(a) can all six members line up for a picture?
(b) can they choose a president and a secretary?
(c) can they choose three members to attend a regional tournament with no regard to order?

Problem 15.16
A school has 30 teachers. In how many ways can the school choose 3 people to attend a national meeting?

Problem 15.17
How many different 12-person juries can be chosen from a pool of 20 juries?

Probability

16. Sample Spaces, Events, and Probability

In this and the coming sections we discuss the fundamental concepts of probability at a level at which no previous exposure to the topic is assumed.
Probability has been used in many applications ranging from medicine to business and so the study of probability is considered an essential component of any mathematics curriculum.
So what is probability? Before answering this question we start with some basic definitions.

An **experiment** is any situation whose outcome cannot be predicted with certainty. Examples of an experiment include rolling a die, flipping a coin, and choosing a card from a deck of playing cards.
By an **outcome** or **simple event** we mean any result of the experiment. For example, the experiment of rolling a die yields six outcomes, namely, the outcomes 1,2,3,4,5, and 6.
The **sample space** S of an experiment is the set of all possible outcomes for the experiment. For example, if you roll a die one time then the experiment is the roll of the die. A sample space for this experiment could be $S = \{1, 2, 3, 4, 5, 6\}$ where each digit represents a face of the die.
An **event** is a subset of the sample space. For example, the event of rolling an odd number with a die consists of three simple events $\{1, 3, 5\}$. An event with more than one outcome is called a **compound event**.

Example 16.1
Consider the random experiment of tossing a coin three times.
(a) Find the sample space of this experiment.
(b) Find the outcomes of the event of obtaining more than one head.

Solution.
(a) The sample space is composed of eight simple events:

$$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}.$$

(b) The event of obtaining more than one head is the set

$$\{THH, HTH, HHT, HHH\} \blacksquare$$

Example 16.2
Consider the random experiment of rolling a die.
(a) Find the sample space of this experiment.
(b) Find the event of rolling the die an even number.

Solution.
(a) The sample space is composed of six simple events:

$$S = \{1, 2, 3, 4, 5, 6\}.$$

(b) The event of rolling the die an even number is the set $\{2, 4, 6\}$ ■

Example 16.3
An experiment consists of the following two stages: (1) first a fair die is rolled and the number of dots recorded, (2) if the number of dots appearing is even, then a fair coin is tossed and its face recorded, and if the number of dots appearing is odd, then the die is tossed again, and the number of dots recorded. Find the sample space of this experiment.

Solution.
The sample space for this experiment is the set of 24 ordered pairs

$$\begin{gathered}\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,H),(2,T),\\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,H),(4,T),\\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,H),(6,T)\}\end{gathered}$$ ■

Probability is the measure of occurrence of an event. If the event is impossible to occur then its probability is 0. If the occurrence is certain then the probability is 1. The closer to 1 the probability is, the more likely the event is. The probability of occurrence of an event E (called its **success**) will be denoted by $P(E)$. If an event has no outcomes, that is as a subset of S if $E = \emptyset$ then $P(\emptyset) = 0$. On the other hand, if $E = S$ (i.e., the event is **certain**) then $P(S) = 1$.

Example 16.4
A hand of 5 cards is dealt from a deck. Let E be the event that the hand contains 5 aces. List the elements of E.

Solution.
Recall that a standard deck of 52 playing cards can be described as follows:

hearts (red)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
clubs (black)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
diamonds (red)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
spades (black)	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King

Cards labeled Jack, Queen, or King are called **face cards.**
Since there are only 4 aces in the deck, event E cannot occur. Hence E is an impossible event and $E = \emptyset$ so that $P(E) = 0$ ■

When the outcome of an experiment is just as likely to occur as another, as in the example of tossing a coin, the outcomes are said to be **equally likely**. Various probability concepts exist nowadays. The **classical** probability concept applies *only* when all possible outcomes are equally likely, in which case we use the formula

$$P(E) = \frac{\text{number of outcomes favorable to event}}{\text{total number of outcomes}} = \frac{n(E)}{n(S)},$$

where $n(E)$ denotes the number of elements in E.
Since for any event E we have $\emptyset \subseteq E \subseteq S$, $0 \le n(E) \le n(S)$ so that $0 \le \frac{n(E)}{n(S)} \le 1$. It follows that $0 \le P(E) \le 1$.

Example 16.5
Which of the following numbers cannot be the probability of some event?
(a) 0.71 (b)-0.5 (c) 150% (d) $\frac{4}{3}$.

Solution.
Only (a) can represent the probability of an event. The reason that -0.5 is not because it is a negative number. As for 150% this is a number greater than 1 and so cannot be a probability of an event. The same is true for $\frac{4}{3}$ ■

Example 16.6
What is the probability of drawing an ace from a well-shuffled deck of 52 playing cards?

Solution.
Since there are four aces in a deck of 52 playing cards, the probabiltiy of getting an ace is $\frac{4}{52} = \frac{1}{13}$ ■

Example 16.7
What is the probability of rolling a 3 or a 4 with a fair die?

Solution.
Since the event of having a 3 or a 4 has two simple events $\{3, 4\}$, the probability of rolling a 3 or a 4 is $\frac{2}{6} = \frac{1}{3}$ ■

It is important to keep in mind that the above definition of probability applies only to a sample space that has equally likely outcomes. Applying the definition to a space with outcomes that are not equally likely leads to incorrect conclusions. For example, the sample space for spinning the spinner in Figure 16.1 is given by S={Red,Blue}, but the outcome Blue is more likely to occur than is the outcome Red. Indeed, $P(Blue) = \frac{3}{4}$ whereas $P(Red) = \frac{1}{4}$.

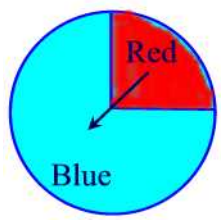


Figure 16.1

A widely used probability concept is the **experimental** or **emperical** probability which uses the relative frequency of an event and is given by the formula:

$$P(E) = \text{Relative frequency} = \frac{f}{n},$$

where f is the frequency of the event and n is the size of the sample space.

Example 16.8
Personality types are broadly defined according to four main preferences. Do married couples choose similar or different personality types in their mates? The following data give an indication:

Similarities and Differences in a Random Sample of 545 Married Couples

Number of Similar Preferences	Number of Married Couples
All four	108
Three	59
Two	146
One	141
None	91

Suppose that a married couple is selected at random. Use the data to estimate the probability (to two decimal places) that they will have no personality preferences in common.

Solution.
The probability they will have no personality preferences in common is $P(0) = \frac{91}{545} = 0.17$ ■

Next, we define the probability of nonoccurrence of an event E (called its **failure**) to be the number $P(E^c)$. Not surprisingly, the probabilities of an event E and its complement E^c are related. The probability of the event E^c is easily found from the identity

$$\frac{\text{number of outcomes in A}}{\text{total number of outcomes}} + \frac{\text{number of outcomes not in A}}{\text{total number of outcomes}} = 1,$$

so that $P(E) + P(E^c) = 1$.

Example 16.9
The probability that a college student without a flu shot will get the flu is 0.45. What is the probability that a college student without the flu shot will not get the flu?

Solution.
Let E denote the event with outcomes those students without a flu shot who will get the flu. Then $P(E) = 0.45$. The probability that a student without the flu shot will not get the flu is then $P(E^c) = 1 - P(E) = 1 - 0.45 = 0.55$ ■

Finding Probabilities Using Combinations and Permutations
Combinations can be used in finding probabilities as illustrated in the next example.

Example 16.10
Given a class of 12 girls and 10 boys.
(a) In how many ways can a committee of five consisting of 3 girls and 2 boys be chosen?
(b) What is the probability that a committee of five, chosen at random from the class, consists of three girls and two boys?
(c) How many of the possible committees of five have no boys?(i.e. consists only of girls)
(d) What is the probability that a committee of five, chosen at random from the class, consists only of girls?

Solution.
(a) First note that the order of the children in the committee does not matter. From 12 girls we can choose $C(12,3)$ different groups of three girls. From the 10 boys we can choose $C(10,2)$ different groups. Thus, by the Fundamental Principle of Counting the total number of committee is

$$C(12,3)\cdot C(10,2)=\frac{12!}{3!9!}\cdot\frac{10!}{2!8!}=\frac{12\cdot 11\cdot 10}{3\cdot 2\cdot 1}\cdot\frac{10\cdot 9}{2\cdot 1}=9900$$

(b) The total number of committees of 5 is $C(22,5)=26,334$. Using part (a), we find the probability that a committee of five will consist of 3 girls and 2 boys to be

$$\frac{C(12,3)\cdot C(10,2)}{C(22,5)}=\frac{9900}{26,334}\approx 0.3759.$$

(c) The number of ways to choose 5 girls from the 12 girls in the class is

$$C(10,0)\cdot C(12,5)=C(12,5)=\frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{5\cdot 4\cdot 3\cdot 2\cdot 1}=792$$

(d) The probability that a committee of five consists only of girls is

$$\frac{C(12,5)}{C(22,5)}=\frac{792}{26,334}\approx 0.03 \blacksquare$$

Practice Problems

Problem 16.1
An experiment consists of flipping a fair coin twice and recording each flip. Determine its sample space.

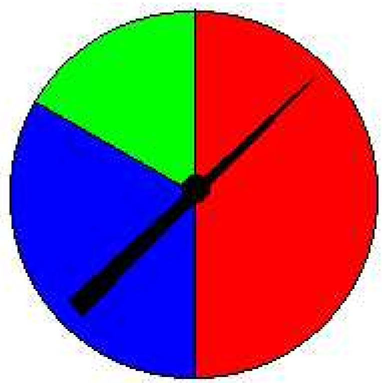
Problem 16.2
Three coins are thrown. List the outcomes which belong to each of the following events.
(a) exactly two tails (b) at least two tails (c) at most two tails.

Problem 16.3
For each of the following events A, B, C, list and count the number of outcomes it contains and hence calculate the probability of A, B or C occurring.
(a) A = "throwing 3 or higher with one die",
(b) B = "throwing exactly two heads with three coins",
(c) C = "throwing a total score of 14 with two dice".

Problem 16.4
An experiment consists of throwing two four-faced dice.
(a) Write down the sample space of this experiment.
(b) If E is the event total score is at least 4 list the outcomes belonging to E^c.
(c) If each die is fair find the probability that the total score is at least 6 when the two dice are thrown. What is the probability that the total score is less than 6?
(d) What is the probability that a double: (i.e. $\{(1,1),(2,2),(3,3),(4,4)\}$) will not be thrown?
(e) What is the probability that a double is not thrown nor is the score greater than 6?

Problem 16.5
A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. One article is chosen at random. Find the probability that:
(a) it has no defects,
(b) it has no major defects,
(c) it is either good or has major defects.

Problem 16.6
Consider the experiment of spinning the pointer on the game spinner pictured below. There are three possible outcomes, that is, when the pointer stops it must point to one of the three colors. (We rule out the possibility of landing on the border between two colors.)
(a) What is the probability that the spinner is pointing to the red area?
(b) What is the probability that the spinner is pointing to the blue area?
(c) What is the probability that the spinner is pointing to the green area?



Problem 16.7
Consider the experiment of flipping a coin three times. If we denote a head by H and a tail by T, we can list the 8 possible ordered outcomes as (H,H,H), (H,H,T) each of which occurs with probability of 1/8. Finish listing the remaining members of the sample space. Calculate the probability of the following events:
(a) All three flips are heads.
(b) Exactly two flips are heads.
(c) The first flip is tail.
(d) At least one flip is head.

Problem 16.8
Suppose an experimet consists of drawing one slip of paper from a jar containing 12 slips of paper, each with a different month of the year written on it. Find each of the following:
(a) The sample space S of the experiment.
(b) The event A consisting of the outcomes having a month begining with J.
(c) The event B consisting of outcomes having the name of a month that has

exactly four letters.
(d) The event C consisting of outcomes having a month that begins with M or N.

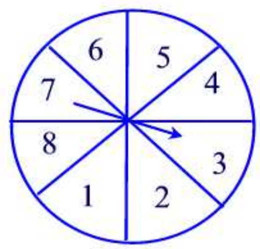
Problem 16.9
Let $S = \{1, 2, 3, \cdots, 25\}$. If a number is chosen at **random**, that is, with the same chance of being drawn as all other numbers in the set, calculate each of the following probabilities:
(a) The even A that an even number is drawn.
(b) The event B that a number less than 10 and greater than 20 is drawn.
(c) The event C that a number less than 26 is drawn.
(d) The event D that a prime number is drawn.
(e) The event E that a number both even and prime is drawn.

Problem 16.10
Consider the experiment of drawing a single card from a standard deck of cards and determine which of the following are sample spaces with equally likely outcomes: (a) {face card, not face card}
(b) {club, diamond, heart, spade}
(c) {black, red}
(d) {king, queen, jack, ace, even card, odd card}

Problem 16.11
An experiment consists of selecting the last digit of a telephone number. Assume that each of the 10 digits is equally likely to appear as a last digit. List each of the following:
(a) The sample space
(b) The event consisting of outcomes that the digit is less than 5
(c) The event consisting of outcomes that the digit is odd
(d) The event consisting of outcomes that the digit is not 2
(e) Find the probability of each of the events in (b) - (d)

Problem 16.12
Each letter of the alphabet is written on a seperate piece of paper and placed in a box and then one piece is drawn at random.
(a) What is the probability that the selected piece of paper has a vowel written on it?
(b) What is the probability that it has a consonant written on it?

Problem 16.13
The following spinner is spun:



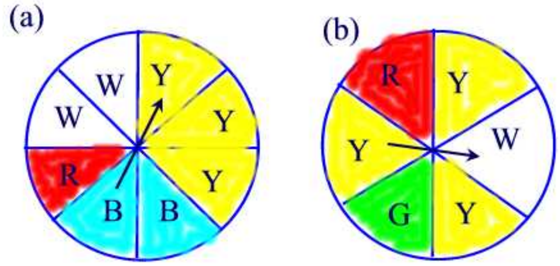
Find the probabilities of obtaining each of the following:
(a) P(factor of 35)
(b) P(multiple of 3)
(c) P(even number)
(d) P(11)
(e) P(composite number)
(f) P(neither prime nor composite)

Problem 16.14
An experiment consists of tossing four coins. List each of the following.
(a) The sample space
(b) The event of a head on the first coin
(c) The event of three heads

Problem 16.15
Identify which of the following events are certain, impossible, or possible.
(a) You throw a 2 on a die
(b) A student in this class is less than 2 years old
(c) Next week has only 5 days

Problem 16.16
Two dice are thrown. If each face is equally likely to turn up, find the following probabilities.
(a) The sum is even
(b) The sum is not 10
(c) The sum is a prime
(d) The sum is less than 9
(e) The sum is not less than 9

Problem 16.17
What is the probability of getting yellow on each of the following spinners?



Problem 16.18
A department store's records show that 782 of 920 women who entered the store on a Saturday afternoon made at least one purchase. Estimate the probability that a woman who enters the store on a Saturday afternoon will make at least one purchase.

Problem 16.19
Suppose that a set of 10 rock samples includes 3 that contain gold nuggets. If you were to pick up a sample at random, what is the probability that it includes a gold nugget?

Problem 16.20
When do creative people get their best ideas? A magazine did a survey of 414 inventors (who hold U.S. patents) and obtained the following information:

Time of Day When Best Ideas Occur

Time	Number of Inventors
6 A.M. - 12 noon	46
12 noon - 6 P.M.	188
6 P.M. - 12 midnight	63
12 midnight - 6 A.M.	117

Assuming that the time interval includes the left limit and all the times up to but not including the right limit, estimate the probability (to two decimal places) that an inventor has a best idea during the time interval 6 A.M. - 12 noon.

Problem 16.21
John and Beth are hoping to be selected from their class of 30 as president and vice-president of the Social Committee. If the three-person committee (president, vice-president, and secretary) is selected at random, what is the probability that John and Beth would be president and vice president?

Problem 16.22
There are 10 boys and 13 girls in Mr. Benson's fourth-grade class and 12 boys and 11 girls in Mr. Johnson fourth-grade class. A picnic committee of six people is selected at random from the total group of students in both classes.
(a) What is the probability that all the committee members are girls?
(b) What is the probability that the committee has three girls and three boys?

Problem 16.23
A school dance committee of 4 people is selected at random from a group of 6 ninth graders, 11 eighth graders, and 10 seventh graders.
(a) What is the probability that the committee has all seventh graders?
(b) What is the probability that the committee has no seventh graders?

Problem 16.24
In an effort to promote school spirit, Georgetown High School created ID numbers with just the letters G, H, and S. If each letter is used exactly three times,
(a) how many nine-letter ID numbers can be generated?
(b) what is the probability that a random ID number starts with GHS?

Problem 16.25
The license plates in the state of Utah consist of three letters followed by three single-digit numbers.
(a) If Edward's initials are EAM, what is the probability that his license plate will have his initials on it (in any order)?
(b) What is the probability that his license plate will have his initials in the correct order?

17. Probability of Unions and Intersections; Odds

The **union** of two events A and B is the event $A \cup B$ whose outcomes are either in A or in B. The **intersection** of two events A and B is the event $A \cap B$ whose outcomes are outcomes of both events A and B. Two events A and B are said to be **mutually exclusive** if they have no outcomes in common. In this case $A \cap B = \emptyset$. Thus, $P(A \cap B) = P(\emptyset) = 0$.

Example 17.1
Consider the sample space of rolling a die. Let A be the event of rolling an even number, B the event of rolling an odd number, and C the event of rolling a 2. Find
(a) $A \cup B, A \cup C$, and $B \cup C$.
(b) $A \cap B, A \cap C$, and $B \cap C$.
(c) Which events are mutually exclusive?

Solution.
(a) We have

$$\begin{aligned} A \cup B &= \{1,2,3,4,5,6\} \\ A \cup C &= \{2,4,6\} \\ B \cup C &= \{1,2,3,5\} \end{aligned}$$

(b)

$$\begin{aligned} A \cap B &= \emptyset \\ A \cap C &= \{2\} \\ B \cap C &= \emptyset \end{aligned}$$

(c) A and B are mutually exclusive as well as B and C ■

Example 17.2
Let A be the event of drawing a King from a well-shuffled standard deck of playing cards and B the event of drawing a "ten" card. Are A and B mutually exclusive?

Solution.
Since $A = \{\text{king of diamonds, king of hearts, king of clubs, king of spades}\}$ and $B = \{\text{ten of diamonds, ten of hearts, ten of clubs, ten of spades}\}$, A and B are mutually exclusive since there are no cards common to both events ■

The next result provides a relationship between the probabilities of the events A, B, and $A \cup B$.

Theorem 17.1
If A and B are two events of a sample space S then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive then $P(A \cap B) = 0$ and and the above equality becomes

$$P(A \cup B) = P(A) + P(B).$$

Example 17.3
Let $P(A) = 0.9$ and $P(B) = 0.6$. Find the minimum possible value for $P(A \cap B)$.

Solution.
Since $P(A) + P(B) = 1.5$ and $0 \leq P(A \cup B) \leq 1$, by the previous theorem

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq 1.5 - 1 = 0.5.$$

So the minimum value of $P(A \cap B)$ is 0.5 ■

Example 17.4
Suppose there's 40% chance of colder weather, 10% chance of rain and colder weather, 80% chance of rain or colder weather. Find the chance of rain.

Solution.
By the addition rule we have

$$P(R) = P(R \cup C) - P(C) + P(R \cap C) = 0.8 - 0.4 + 0.1 = 0.5 \blacksquare$$

Odds
What's the difference between probabilities and odds? To answer this question, let's consider a game that involves rolling a die. If one gets the face 1 then he wins the game, otherwise he loses. The probability of winning is $\frac{1}{6}$ whereas the probability of losing is $\frac{5}{6}$. The odds of winning is 1:5(read 1 to 5). This expression means that the probability of losing is five times the probability of winning. Thus, probabilities describe the frequency of a favorable result in relation to all possible outcomes whereas the "odds in favor" compare the favorable outcomes to the unfavorable outcomes. More formally,

$$\text{odds in favor} = \frac{\text{favorable outcomes}}{\text{unfavorable outcomes}}$$

If E is the event of all favorable outcomes then its complementary, $\overline{E}$, is the event of unfavorable outcomes. Hence,

$$\text{odds in favor} = \frac{n(E)}{n(\overline{E})}$$

Also, we define the **odds against** an event as

$$\text{odds against} = \frac{\text{unfavorable outcomes}}{\text{favorable outcomes}} = \frac{n(\overline{E})}{n(E)}$$

Any probability can be converted to odds, and any odds can be converted to a probability.

Converting Odds to Probability
Suppose that the odds for an event E is a:b. Thus, $n(E) = ak$ and $n(\overline{E}) = bk$ where k is a positive integer. Since E and $\overline{E}$ are complementary, $n(S) = n(E) + n(\overline{E})$. Therefore,

$$\begin{aligned}
P(E) &= \frac{n(E)}{n(S)} \\
&= \frac{n(E)}{n(E)+n(\overline{E})} \\
&= \frac{ak}{ak+bk} = \frac{a}{a+b} \\
P(\overline{E}) &= \frac{n(\overline{E})}{n(S)} \\
&= \frac{n(\overline{E})}{n(E)+n(\overline{E})} \\
&= \frac{bk}{ak+bk} = \frac{b}{a+b}
\end{aligned}$$

Example 17.5
If the odds in favor of an event E is 5 to 4, compute $P(E)$ and $P(\overline{E})$.

Solution.
We have

$$P(E) = \frac{5}{5+4} = \frac{5}{9}$$

and

$$P(\overline{E}) = \frac{4}{5+4} = \frac{4}{9} \blacksquare$$

Converting Probability to Odds

Given $P(E)$, we want to find the odds in favor of E and the odds against E. The odds in favor of E are

$$\begin{aligned} \frac{n(E)}{n(\overline{E})} &= \frac{n(E)}{n(S)} \cdot \frac{n(S)}{n(\overline{E})} \\ &= \frac{P(E)}{P(\overline{E})} \\ &= \frac{P(E)}{1-P(E)} \end{aligned}$$

and the odds against E are

$$\frac{n(\overline{E})}{n(E)} = \frac{1-P(E)}{P(E)}$$

Example 17.6

For each of the following, find the odds in favor of the event's occurring:
(a) Rolling a number less than 5 on a die.
(b) Tossing heads on a fair coin.
(c) Drawing an ace from an ordinary 52-card deck.

Solution.

(a) The probability of rolling a number less than 5 is $\frac{4}{6}$ and that of rolling 5 or 6 is $\frac{2}{6}$. Thus, the odds in favor of rolling a number less than 5 is $\frac{4}{6} \div \frac{2}{6} = \frac{2}{1}$ or 2:1
(b) Since $P(H) = \frac{1}{2}$ and $P(T) = \frac{1}{2}$, the odds in favor of getting heads is $\left(\frac{1}{2}\right) \div \left(\frac{1}{2}\right)$ or 1:1
(c) We have P(ace) $=\frac{4}{52}$ and P(not an ace) $=\frac{48}{52}$ so that the odds in favor of drawing an ace is $\left(\frac{4}{52}\right) \div \left(\frac{48}{52}\right) = \frac{1}{12}$ or 1:12 $\blacksquare$

Remark 17.1

A probability such as $P(E) = \frac{5}{6}$ is just a ratio. The exact number of favorable outcomes and the exact total of all outcomes are not necessarily known.

Practice Problems

Problem 17.1
Which of the following are mutually exclusive? Explain your answers.
(a) A driver getting a ticket for speeding and a ticket for going through a red light.
(b) Being foreign-born and being President of the United States.

Problem 17.2
If A and B are the events that a consumer testing service will rate a given stereo system very good or good, $P(A) = 0.22$, $P(B) = 0.35$. Find
(a) $P(A^c)$;
(b) $P(A \cup B)$;
(c) $P(A \cap B)$.

Problem 17.3
If the probabilities are 0.20, 0.15, and 0.03 that a student will get a failing grade in Statistics, in English, or in both, what is the probability that the student will get a failing grade in at least one of these subjects?

Problem 17.4
If A is the event "drawing an ace" from a deck of cards and B is the event "drawing a spade". Are A and B mutually exclusive? Find $P(A \cup B)$.

Problem 17.5
A bag contains 18 coloured marbles: 4 are coloured red, 8 are coloured yellow and 6 are coloured green. A marble is selected at random. What is the probability that the ball chosen is either red or green?

Problem 17.6
Show that for any events A and B, $P(A \cap B) \geq P(A) + P(B) - 1$.

Problem 17.7
A golf bag contains 2 red tees, 4 blue tees, and 5 white tees.
(a) What is the probability of the event R that a tee drawn at random is red?
(b) What is the probability of the event "not R" that is, that a tee drawn at random is not red?
(c) What is the probability of the event that a tee drawn at random is either red or blue?

Problem 17.8
A fair pair of dice is rolled. Let E be the event of rolling a sum that is an even number and P the event of rolling a sum that is a prime number. Find the probability of rolling a sum that is even or prime?

Problem 17.9
If events A nd B are from the same sample space, and if P(A)=0.8 and P(B)=0.9, can events A and B be mutually exclusive?

Problem 17.10
If the probability of a boy's being born is $\frac{1}{2}$, and a family plans to have four children, what are the odds against having all boys?

Problem 17.11
If the odds against Deborah's winning first prize in a chess tournament are 3 to 5, what is the probability that she will win first prize?

Problem 17.12
What are the odds in favor of getting at least two heads if a fair coin is tossed three times?

Problem 17.13
If the probability of rain for the day is 60%, what are the odds against its raining?

Problem 17.14
On a tote board at a race track, the odds for Gameylegs are listed as 26:1. Tote boards list the odds that the horse will lose the race. If this is the case, what is the probability of Gameylegs's winning the race?

Problem 17.15
If a die is tossed, what are the odds in favor of the following events?
(a) Getting a 4
(b) Getting a prime
(c) Getting a number greater than 0
(d) Getting a number greater than 6.

Problem 17.16
Find the odds against E if $P(E) = \frac{3}{4}$.

Problem 17.17
Find P(E) in each case.
(a) The odds in favor of E are 3:4
(b) The odds against E are 7:3

18. Conditional Probability and Independent Events

When the sample space of an experiment is affected by additional information, the new sample space is reduced in size. For example, suppose we toss a fair coin three times and consider the following events:
A : getting a tail on the first toss
B : getting a tail on all three tosses
Since

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

we have $P(A) = \frac{4}{8} = \frac{1}{2}$ and $P(B) = \frac{1}{8}$. What if we were told that event A has occurred (that is, a tail occurred on the first toss), and we are now asked to find $P(B)$. The sample space is now reduced to $\{THH, THT, TTH, TTT\}$. The probability that all three are tails given that the first toss is a tail is $\frac{1}{4}$. The notation we use for this situation is $P(B|A)$, read "the probability of B given A," and we write $P(B|A) = \frac{1}{4}$. Notice that

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

This is true in general, and we have the following:

Given two events A and B belonging to the same sample S. The **conditional probability** $P(B|A)$ denotes the probability that event B will occur given that event A has occurred. It is given by the formula

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Example 18.1
Consider the experiment of tossing a fair die. Denote by A and B the following events:

$$\begin{array}{rcl} A & = & \{\text{Observing an even number of dots on the upper face of the die}\}, \\ B & = & \{\text{Observing a number of dots less than or equal to 3 on the upper face of the die}\}. \end{array}$$

Find the probability of the event A, given the event B.

Solution.
Since $A = \{2, 4, 6\}$ and $B = \{1, 2, 3\}$, we find $A \cap B = \{2\}$ and therefore

$P(A|B) = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$ ■

If $P(B|A) = P(B)$, i.e., the occurrence of the event A does not affect the probability of the event B, then we say that the two events A and B are **independent**. In this case the above formula gives

$$P(A \cap B) = P(A) \cdot P(B).$$

This formula is known as the "multiplication rule of probabilities". If two events are not independent, we say that they are **dependent**. In this case, $P(B|A) \neq P(B)$ or equivalently $P(A \cap B) \neq P(A) \cdot P(B)$.

Example 18.2
Consider the experiment of tossing a fair die. Denote by A and B the following events:

$$\begin{aligned} A &= \{\text{Observing an even number of dots on the upper face of the die}\}, \\ B &= \{\text{Observing a number of dots less than or equal to 4 on the upper face of the die}\}. \end{aligned}$$

Are A and B independent?

Solution.
Since $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$ we have $A \cap B = \{2, 4\}$ and therefore $P(A|B) = \frac{\frac{2}{6}}{\frac{4}{6}} = \frac{1}{2} = P(A)$. Thus, A and B are independent ■

Probability Trees
Probability trees can be used to compute the probabilities of combined outcomes in a sequence of experiments.

Example 18.3
Construct the probability tree of the experiment of flipping a fair coin twice.

Solution.
The probability tree is shown in Figure 18.1■

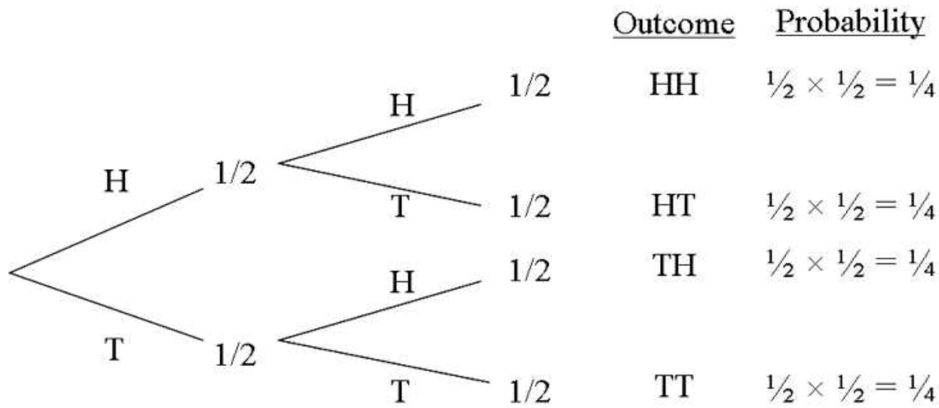


Figure 18.1

The probabilities shown in Figure 18.1 are obtained by following the paths leading to each of the four outcomes and multiplying the probabilities along the paths. This procedure is an instance of the following general property.

Multiplication Rule for Probabilities for Tree Diagrams
For all multistage experiments, the probability of the outcome along any path of a tree diagram is equal to the product of all the probabilities along the path.

Example 18.4
Suppose that out of 500 computer chips there are 9 defective. Construct the probability tree of the experiment of sampling two of them without replacement.

Solution.
The probability tree is shown in Figure 18.2 ∎

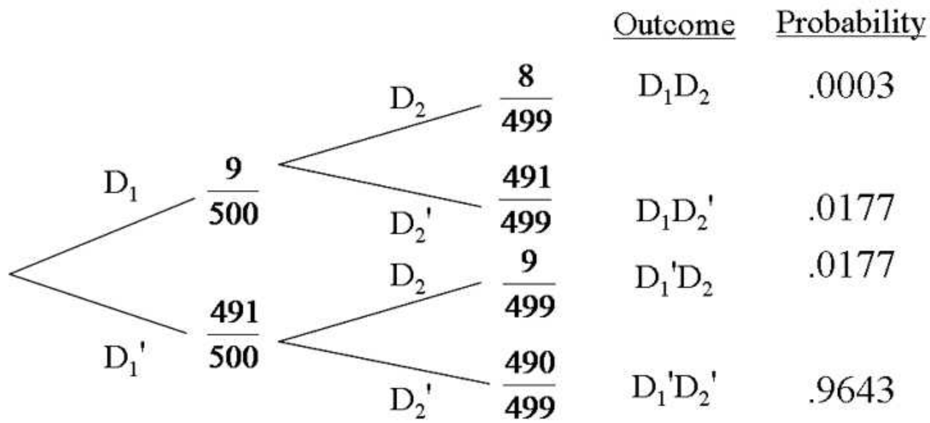

Outcome
Probability
D1
9/500
D1'
491/500
D2
8/499
D2'
491/499
D2
9/499
D2'
490/499
D1D2
.0003
D1D2'
.0177
D1'D2
.0177
D1'D2'
.9643

Figure 18.2

Practice Problems

Problem 18.1
Suppose that A is the event of rolling a sum of 7 with two fair dice. Make up an event B so that
(a) A and B are independent.
(b) A and B are dependent.

Problem 18.2
When tossing three fair coins, what is the probability of getting two tails given that the first coin came up heads?

Problem 18.3
Suppose a 20-sided die has the following numerals on its face:1, 1, 2, 2, 2, 3, 3, 4, 5, 6, 7, ,8 9, 10, 11, 12, 13, 14, 15, 16. The die is rolled once and the number on the top face is recorded. Let A be the event the number is prime, and B be the event the number is odd. Find $P(A|B)$ and $P(B|A)$.

Problem 18.4
What is the probability of rolling a 6 on a fair die if you know that the roll is an even number?

Problem 18.5
A red die and a green die are rolled. What is the probability of obtaining an even number on the red die and a multiple of 3 on the green die?

Problem 18.6
Two coins are tossed. What is the probability of obtaining a head on the first coin and a tail on the second coin?

Problem 18.7
Consider two boxes: Box 1 contains 2 white and 2 black balls, and box 2 contains 2 white balls and three black balls. What is the probability of drawing a black ball from each box?

Problem 18.8
A container holds three red balls and five blue balls. One ball is drawn and discarded. Then a second ball is drawn.
(a) What is the probability that the second ball drawn is red if you drew a

red ball the first time?
(b) What is the probability of drawing a blue ball second if the first ball was red?
(c) What is the probability of drawing a blue ball second if the first ball was blue?

Problem 18.9
Consider the following events.

A: rain tomorrow
B: You carry an umbrella
C: coin flipped tomorrow lands on heads

Which of two events are dependent and which are independent?

Problem 18.10
You roll a regular red die and a regular green die. Consider the following events.

A: a 4 on the red die
B: a 3 on the green die
C: a sum of 9 on the two dice

Tell whether each pair of events is independent or dependent.
(a) A and B (b) B and C

Problem 18.11
In a state assembly, 35% of the legislators are Democrats, and the other 65% are Republicans. 70% of the Democrats favor raising sales tax, while only 40% of the Republicans favor the increase.
If a legislator is selected at random from this group, what is the probability that he or she favors raising sales tax?

19. Conditional Probability and Bayes' Formula

It is often the case that we know the probabilities of certain events conditional on other events, but what we would like to know is the "reverse". That is, given $P(A|B)$ we would like to find $P(B|A)$.
Bayes' formula is a simple mathematical formula used for calculating $P(B|A)$ given $P(A|B)$. We derive this formula as follows. Let A and B be two events. Then

$$A = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c).$$

Since the events $A \cap B$ and $A \cap B^c$ are mutually exclusive, we can write

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A|B)P(B) + P(A|B^c)P(B^c) \end{aligned} \tag{9}$$

This result can be extended as follows: Let $U_1, U_2, \cdots, U_n$ be n pairwise mutually exclusive events in a sample space S whose union is the sample space S. For any event $E \subset S$ we have

$$P(E) = P(E \cap U_1) + P(E \cap U_2) + \cdots + P(E \cap U_n).$$

Example 19.1
The completion of a construction job may be delayed because of a strike. The probabilities are 0.60 that there will be a strike, 0.85 that the construction job will be completed on time if there is no strike, and 0.35 that the construction will be completed on time if there is a strike. What is the probability that the construction job will be completed on time?

Solution.
Let A be the event that the construction job will be completed on time and B is the event that there will be a strike. We are given $P(B) = 0.60,\ P(A|B^c) = 0.85$, and $P(A|B) = 0.35$. From Equation (9) we find

$$P(A) = P(B)P(A|B) + P(B^c)P(A|B^c) = (0.60)(0.35) + (0.4)(0.85) = 0.55 \blacksquare$$

From Equation (9) we can get Bayes' formula:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}. \tag{10}$$

In general, if $U_1, U_2, \cdots, U_n$ be n pairwise mutually exclusive events in a sample space S whose union is the sample space S then for any event $E \subset S$ such that $P(E) \neq 0$ we have

$$\begin{aligned}
P(U_i|E) =&\frac{P(E\cap U_i)}{P(E)} = \frac{P(E\cap U_i)}{P(E\cap U_1)+P(E\cap U_2)+\cdots+P(E\cap U_n)}\\
=&\frac{P(E|U_i)P(U_i)}{P(E|U_1)P(U_1)+P(E|U_2)P(U_2)+\cdots+P(E|U_n)P(U_n)}
\end{aligned}$$

Example 19.2
One urn has 3 blue and 2 white balls; a second urn has 1 blue and 3 white balls. A single fair die is rolled and if 1 or 2 comes up, a ball is drawn out of the first urn; otherwise, a ball is drawn out of the second urn. If the drawn ball is blue, what is the probability that it came out of the first urn? Out of the second urn?

Solution.
LetU_1 be the event that a fair die will be show either a 1 or a 2. Then $P(U_1) = \frac{2}{6} = \frac{1}{3}$. Let$U_2$ be the event that a fair die will show 3, 4, 5, or 6. Then $P(U_2) = \frac{4}{6} = \frac{2}{3}$. Let B denote the event that a drawn ball is blue. Using Bayes' formula, we have

$$\begin{aligned}
P(U_1|B) =&\frac{P(B|U_1)P(U_1)}{P(B|U_1)P(U_1)+P(B|U_1^c)P(U_1^c)}\\
=&\frac{\frac{3}{5}\frac{1}{3}}{\frac{3}{5}\frac{1}{3}+\frac{1}{4}\frac{2}{3}}\\
=&0.55.
\end{aligned}$$

Likewise, we find $P(U_2|B) = 0.45$ ■

Example 19.3
A new, inexpensive skin test is devised for detecting tuberculosis. To evaluate the test before it is put into use, a medical researcher randomly selects 1,000 people. Using the precise but more expensive methods already available, it is found that 8% of the 1,000 people have tested tuberculosis. Now each of the 1,000 subjects is given the new skin test and the following results are recorded: The test indicates tuberculosis in 96% of those who have it and in 2% of those who do not. Based on these results,

(a) What is the probability of a randomly chosen person having tuberculosis given that the skin test indicates the disease?
(b) What is the probability of a person not having tuberculosis given that the skin test indicates the disease?
(c) What is the probability that a person has tuberculosis given that the test indicates no tuberculosis is present?
(d) What is the probability that a person does not have tuberculosis given that the test indicates no tuberculosis is present?

Solution.
Let S denote the event that the new skin test is a success. Then $P(S) = 0.96$. Let T denote the event that a person has tuberculosis. Then $P(T) = 0.08$.
(a) We have

$$\begin{aligned}P(T|S) =& \frac{P(S|T)P(T)}{P(S|T)P(T) + P(S|T^c)P(T^c)}\\ =& \frac{(0.96)(0.08)}{0.96(0.08) + 0.02(0.092)}\\ =& 0.81.\end{aligned}$$

(b) We have $P(T^c|S) = 1 - P(T|S) = 1 - 0.81 = 0.19$.
(c) We have

$$\begin{aligned}P(T|S^c) =& \frac{P(S^c|T)P(T)}{P(S^c|T)P(T) + P(S^c|T^c)P(T^c)}\\ =& \frac{(0.04)(0.08)}{0.04(0.08) + 0.98(0.92)}\\ =& 0.0035.\end{aligned}$$

(d) $P(T^c|S^c) = 1 - P(T|S^c) = 1 - 0.0035 = 0.9965$ ■

Practice Problems

Problem 19.1
A company produces 1,000 refrigerators a week at three plants. Plant A produces 350 refrigerators a week, plant B produces 250 refrigerators a week, and plant C produces 400 refrigerators a week. Production records indicate that 5% of the refrigerators produced at plant A will be defective, 3% of the refrigerators produced at plant B will be defective, 7% of the refrigerators produced at plant C will be defective. All the refrigerators are shipped to a central warehouse. If a refrigerator at the warehouse is found to be defective,
(a) What is the probability that it was produced at plant A?
(b) What is the probability that it was produced at plant B?
(c) What is the probability that it was produced at plant C?

Problem 19.2
One urn has 4 red balls and 1 white ball; a second urn has 2 red balls and 3 white balls. A single card is randomly selected from a standard deck. If the card is less than 5 (aces count as 1), a ball is drawn out of the first urn; otherwise a ball is drawn out of the second urn. If the drawn ball is red, what is the probability that it came out of the second urn?

Problem 19.3
A small manufacturing company has rated 75% of its employees as satisfactory (S) and 25% as unsatisfactory (S^c). Personnel records show that 80% of the satisfactory workers had previous work experience (E) in the job they are now doing, while 15% of the unsatisfactory workers had no work experience (E^c) in the job they are now doing. If a person who has had previous work experience is hired, what is the approximate empirical probability that this person will be an unsatisfactory employee?

Problem 19.4
A basketball team is to play two games in a tournament. The probability of winning the first game is 0.10. If the first game is won, the probability of winning the second game is 0.15. If the first game is lost, the probability of winning the second game is 0.25. What is the probability the first game was won if the second game is lost?

Problem 19.5
To evaluate a new test for detecting Hansen's disease, a group of people

5% of which are known to have Hansen's disease are tested. The test finds Hansen's disease among 98% of those with the disease and 3% of those who don't. What is the probability that someone testing positive for Hansen's disease under this new test actually has it?

Problem 19.6
An urn contains 4 red and 5 white balls. Two balls are drawn in succession without replacement. If the second ball is white, what is the probability that the first ball was white?

Problem 19.7
An urn contains 4 red and 5 white balls. Two balls are drawn in succession without replacement. If the second ball is red, what is the probability that the first ball was red?

Problem 19.8
Urn 1 contains 7 red and 3 white balls. Urn 2 contains 4 red and 5 white balls. A ball is drawn from urn 1 and placed in urn 2. Then a ball is drawn from urn2. If the ball drawn from urn 2 is red, what is the probability that the ball drawn from urn 1 was red?

Problem 19.9
Urn 1 contains 7 red and 3 white balls. Urn 2 contains 4 red and 5 white balls. A ball is drawn from urn 1 and placed in urn 2. Then a ball is drawn from urn2. If the ball drawn from urn 2 is white, what is the probability that the ball drawn from urn 1 was white?

Problem 19.10
A company has rated 75% of its employees as satisfactory and 25% as unsatisfactory. Personnel records indicate that 80% of the satisfactory workers had previous work experience, while only 40% of the unsatisfactory workers had any previous work experience. If a person with previous work experience is hired, what is the probability that this person will be a satisfactory employee? If a person with no previous work experience is hired, what is the probability that this person will be a satisfactory employee?

Problem 19.11
A manufacturer obtains clock-radios from three different subcontractors: 20% from A, 40% from B, and 40% from C. The defective rates for these

subcontractors are 1%, 3%, and 2%, respectively. If a defective clock-radio is returned by a customer, what is the probability that it came from subcontractor A? From B? From C?

Problem 19.12
A computer store sells three types of microcomputer, brand A, brand B, brand C. Of the computers sell, 60% are brands A, 25% are brand B, 15% are brand C. They have found that 20% of the brand A computers, 15% of the brand B computers, and 5% of the brand C computers are returned for service during the warranty period. If a computer is returned for service during the warranty period, what is the probability that it is a brand A computer, A brand B computer? A brand C computer?

Problem 19.13
A new, simple test has been developed to detect a particular type of cancer. The test must be evaluated before it is put into use. A medical researcher selects a random sample of 1,000 adults and finds (by other means) that 2% have this type of cancer. Each of the 1,000 adults is given test, and it is found that the test indicates cancer in 98% of those who have it and in 1% of those who do not. Based on these results, what is the probability of a randomly chosen person having cancer given that the test indicates cancer? Of a person having cancer given that the test does not indicate cancer?

Problem 19.14
In a random sample of 200 women who suspect that they are pregnant, 100 turn out to be pregnant. A new pregnancy test given to these women indicated pregnancy in 92 of the 100 pregnant women and in 12 of the 100 non-pregnant women. If a woman suspects she is pregnant and this test indicates that she is pregnant, what is the probability that she is pregnant? If the test indicates that she is not pregnant, what is the probability that she is not pregnant?

Problem 19.15
One of two urns is chosen at random with one as likely to be chosen as the other. Then a ball is drawn from the chosen urn, Urn 1 contains 1 white and 4 red balls, and urn 2 has 3 white and 2 red balls.
(a) If a white ball is drawn, what is the probability that it came from urn 1?
(b) If a white ball is drawn, what is the probability that it came from urn 2?
(c) If a red ball is drawn, what is the probability that it came from urn 2?
d) If a red ball is drawn, what is the probability that it came from urn 1?

20. Random Variable, Probabiltiy Distribution, and Expected Value

By definition, a **random variable** X is a function with domain the sample space and range a subset of the real numbers. For example, in rolling two dice X might represent the sum of the points on the two dice. Similarly, in taking samples of college students X might represent the number of hours per week a student studies, a student's GPA, or a student's height.
The notation $X(s) = x$ means that x is the value associated with the outcome s by the random variable X.
There are three types of random variables: discrete random variables, continuous random variables, and mixed random variables.
A **discrete** random variable is one which takes a finite number of values. A **continuous** random variable is one which takes an infinite number of possible values. A **mixed** random variable is partially discrete and partially continuous.
In this chapter we will just consider discrete random variables.

Example 20.1
State whether the random variables are discrete or continuous.
(a) A coin is tossed ten times. The random variable X is the number of tails that are noted.
(b) The random variable Y that counts the life expectancy of a circuit in a computer..

Solution.
(a) X can only take the values 0, 1, ..., 10, so X is a discrete random variable.
(b) Y can take any positive real value, so Y is a continuous random variable ■

Example 20.2
Toss a coin 3 times: $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.
Let X = # of Heads in 3 tosses. Find the range of X.

Solution.
We have

$$\begin{array}{llllllll} X(HHH) = 3 & X(HHT) = 2 & X(HTH) = 2 & X(HTT) = 1 \\ X(THH) = 2 & X(THT) = 1 & X(TTH) = 1 & X(TTT) = 0 \end{array}$$

Thus, the range of X consists of $\{0, 1, 2, 3\}$ ■

We use upper-case letters X, Y, Z, etc. to represent random variables. We use small letters x, y, z, etc to represent possible values that the corresponding random variables X, Y, Z, etc. can take. The statement $X = x$ defines an event consisting of all outcomes with X-measurement equal to x which is the set $\{s \in S : X(s) = x\}$. For instance, considering the random variable of the previous example, the statement "X = 2" is the event $\{HHT, HTH, THH\}$. Because the value of a random variable is determined by the outcomes of the experiment, we may assign probabilities to the possible values of the random variable. For example, $P(X = 2) = \frac{3}{8}$.

Example 20.3
Consider the experiment consisting of 2 rolls of a fair 4-sided die. Let X be a random variable, equal to the maximum of 2 rolls. Complete the following table

x	1	2	3	4
P(X=x)				

Solution.

x	1	2	3	4
P(X=x)	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

■

For a discrete random variable X, we define the **probability distribution** or the **probability mass function** by the equation

$$p(x) = P(X = x).$$

That is, a probability mass function (pmf) gives the probability that a discrete random variable is exactly equal to some value.
The pmf can be an equation, a table, or a graph that shows how probability is assigned to possible values of the random variable.

Example 20.4
Suppose a variable X can take the values 1, 2, 3, or 4. The probabilities associated with each outcome are described by the following table:

x	1	2	3	4
$p(x)$	0.1	0.3	0.4	0.2

Draw the probability histogram.

Solution.
The probability histogram is shown in Figure 20.1 ∎

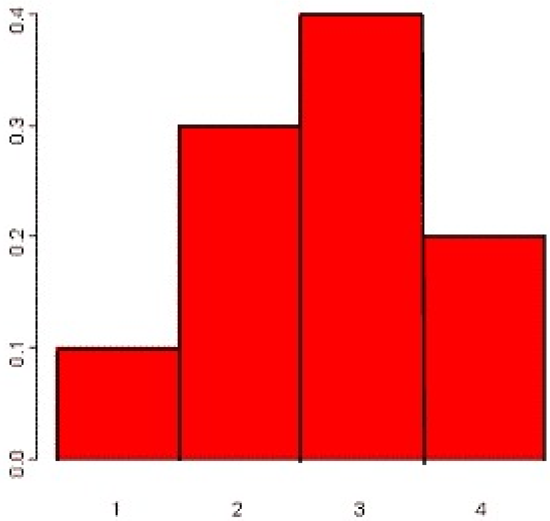


Figure 20.1

The Expected value of a discrete random variable
A cube has three red faces, two green faces, and one blue face. A game consists of rolling the cube twice. You pay \$2 to play. If both faces are the same color, you are paid \$5(that is you win \$3). If not, you lose the \$2 it costs to play. Will you win money in the long run? Let W denote the event that you win. Then $W = \{RR, GG, BB\}$ and

$$P(W) = P(RR) + P(GG) + P(BB) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{6} = \frac{7}{18} \approx 39\%.$$

Thus, $P(L) = \frac{11}{18} = 61\%$. Hence, if you play the game 18 times you expect to win 7 times and lose 11 times on average. So your winnings in dollars will be $3 \times 7 - 2 \times 11 = -1$. That is, you can expect to lose \$1 if you play the game 18 times. On the average, you will lose \$ $\frac{1}{18}$ per game (about 6 cents). This can be found also using the equation

$$3 \times \frac{7}{18} - 2 \times \frac{11}{18} = -\frac{1}{18}$$

If we let X denote the winnings/losings of this game then the range of X consists of the two numbers 3 and -2 which occur with respective probability 0.39 and 0.61. Thus, we can write

$$E(X) = 3 \times \frac{7}{18} - 2 \times \frac{11}{18} = -\frac{1}{18}.$$

We call this number the expected value of X. More formally, let the range of a discrete random variable X be a sequence of numbers $x_1, x_2, \cdots, x_k$, and let $p(x)$ be the corresponding probability mass function. Then the **expected value** of X is

$$E(X) = x_1 p(x_1) + x_2 p(x_2) + \cdots + x_k p(x_k).$$

The expected value of X is also known as the **mean** value.

Example 20.5
Suppose that an insurance company has broken down yearly automobile claims for drivers from age 16 through 21 as shown in the following table.

Amount of claim	Probability
\$ 0	0.80
\$ 2000	0.10
\$ 4000	0.05
\$ 6000	0.03
\$ 8000	0.01
\$ 10000	0.01

How much should the company charge as its average premium in order to break even on costs for claims?

Solution.
Let X be the random variable of the amount of claim. Finding the expected value of X we have

$$E(X) = 0(.80)+2000(.10)+4000(.05)+6000(.03)+8000(.01)+10000(.01) = 760$$

Since the average claim value is \$760, the average automobile insurance premium should be set at \$760 per year for the insurance company to break even ■

Example 20.6
A man determines that the probability of living 5 more years is 0.85. His insurance policy pays \$1,000 if he dies within the next 5 years. Let X be the random variable that represents the amount the insurance company pays out in the next 5 years.
(a) What is the probability distribution of X?
(b) What is the most he should be willing to pay for the policy?

Solution.
(a) $P(X = 1000) = 0.15$ and $P(X = 0) = 0.85$.
(b) $E(X) = 1000 \times 0.15 + 0 \times 0.85 = 150$. Thus, his expected payout is \$150, so he should not be willing to pay more than \$150 for the policy ■

Practice Problems

Problem 20.1
Let X be a random variable with probability distribution given below.

x	1	2	3	4	4	6
$p(x)$	0.14	0.13	0.18	0.20	0.11	0.24

Find the expected value of X.

Problem 20.2
A carton of 20 watch batteries contains 2 dead ones. A random sample of 4 is selected from 20 and tested. Let X be the random variable associated with the number of dead batteries found in a sample.
(a) Find the probability distribution of X.
(b) Find the expected number of dead batteries in a sample.

Problem 20.3
A spinner device is numbered from 0 to 5, and each of the 6 numbers is as likely to come up as any other. A player who bets \$1 on any given number wins \$4 (gets the bet back) if the pointer comes to rest on the chosen number, otherwise, the \$1 bet is lost. What is the expected value of the game?

Problem 20.4
Suppose you are interested in insuring a car stereo system for \$500 against theft. An insurance company charges a premium of \$60 for coverage 1 year, claiming an empirically determining probability of 0.1 that the stereo will be stolen some time during the year. What is your expected return from the insurance company if you take out this insurance?

Problem 20.5
Suppose a random sample of 2 light bulbs is selected from a group of 8 bulbs that contain 3 defective bulbs.
(a) Construct the probability Distribution Table.
(b) What is the expected value?

Problem 20.6
Suppose 1000 raffle tickets are sold at a price of \$10 each. Two first place tickets will be drawn, 5 second place tickets will be drawn and 10 third place tickets will be drawn. The first place prize is a \$200 VCR, the second place

prize is a $100 printer, and the third place prize is a $50 gift certificate.
(a) Construct the probability distribution table (payoff table).
(b) What is the expected return?

Problem 20.7
After paying $4 to play, a single fair die is rolled and you are paid back the number of dollars corresponding to the number of dots facing up. For example, if a 5 turns up, $5 is returned to you for a net gain, or payoff, of $1; if a 1 turns up, $1 is returned for a net gain of −$3 and so on. What is the expected value of the game? Is the game fair (i.e. expected value is zero)?

Problem 20.8
A friend offers the following game: She wins $1 from you if, on four rolls of a single die, a 6 turns up at least once; otherwise, you win $1 from her. What is the expected value of the game to you? To her?

Problem 20.9
A pair of dice is rolled once. Suppose you lose $10 if a 7 turns up and you win $11 if an 11 or 12 turns up. How much should you win or lose if any other number turns up for the game to be fair?

Problem 20.10
A game has an expected value to you of $100. It costs $100 to play, but if you win you receive $100,000 for a net gain of $99,900 (since it cost $100 to play). What is the probability of winning

Problem 20.11
5,000 tickets are sold at $1.00 each for a charity raffle. Tickets are to be drawn at random and monetary prizes awarded as follows. 1 prize of $500 for a net gain of $499, 3 prizes of $100 for a net gain of $99, 5 prizes of $20 for a net gain of $19, and 20 prizes of $5.00 for a net gain of $4.00. What is the expected value of this raffle if you buy 1 ticket?

Problem 20.12
The annual premium for a $5,000 insurance policy against the theft of a painting is $150. If the (empirical) probability that the painting will be stolen during the year is 0.01, what is your expected return from the insurance company if you take out this insurance?

Problem 20.13
In tossing two coins, what is the expected number of heads?

Problem 20.14
Two coins are flipped. You win \$2 if either 2 heads or 2 tails turn up; you lose \$3 if a head and a tail turn up. What is the expected value of the game?

Statistics

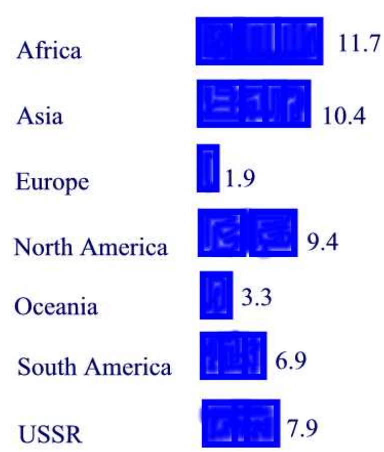
21. Graphical Representations of Data

Visualization techniques are ways of creating and manipulating graphical representations of data. We use these representations in order to gain better insight and understanding of the problem we are studying - pictures can convey an overall message much better than a list of numbers.
In this section we describe some graphical presentations of data.

Bar Graphs
Bar graphs consist of bars that are nonoverlapping rectangles of equal width and they are equally spaced. The bars can be vertical or horizontal. The length of a bar represents the quantity we wish to compare.

Example 21.1
The areas of the various continents of the world (in millions of square miles) are as follows:11.7 for Africa; 10.4 for Asia; 1.9 for Europe; 9.4 for North America; 3.3 Oceania; 6.9 South America; 7.9 Soviet Union. Draw a bar chart representing the above data and where the bars are horizontal.

Solution.



Areas (in millions of square miles) of the variuos continents of the world

Example 21.2
Create a vertical graph bar for the data given in the table.

Year	Debt (in Billions of $)
1960	284.1
1970	370.1
1980	907.7
1990	3,233.3
2000	5,674.2

U.S. Public Debt

Solution.
The bar graph is shown Figure 21.1 below ■

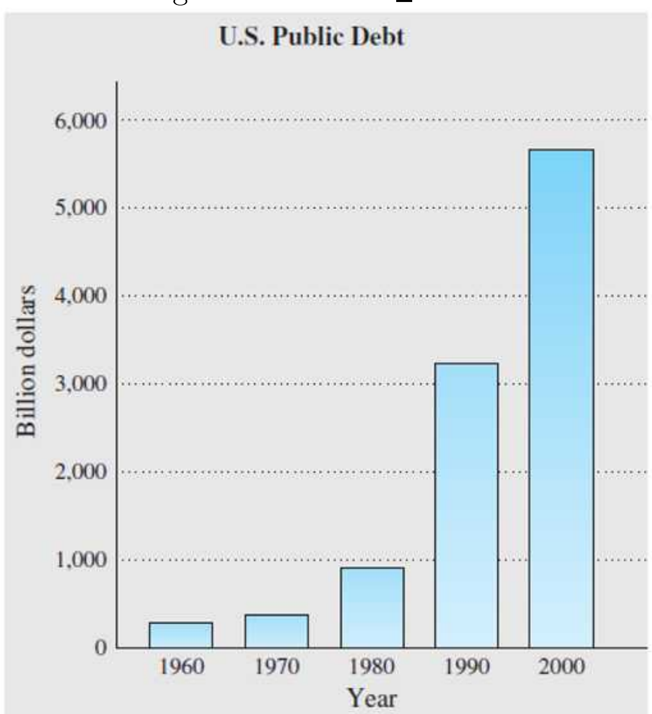


Figure 21.1

Broken-Line Graph
A **broken-line graph** can be obtained from a vertical bar graph by joining the midpoints of the tops of consecutive bars with staight lines.

Example 21.3
Create the broken-line graph of the data in the previous example.

Solution.
The broken-line graph is shown in Figure 21.2 below ■

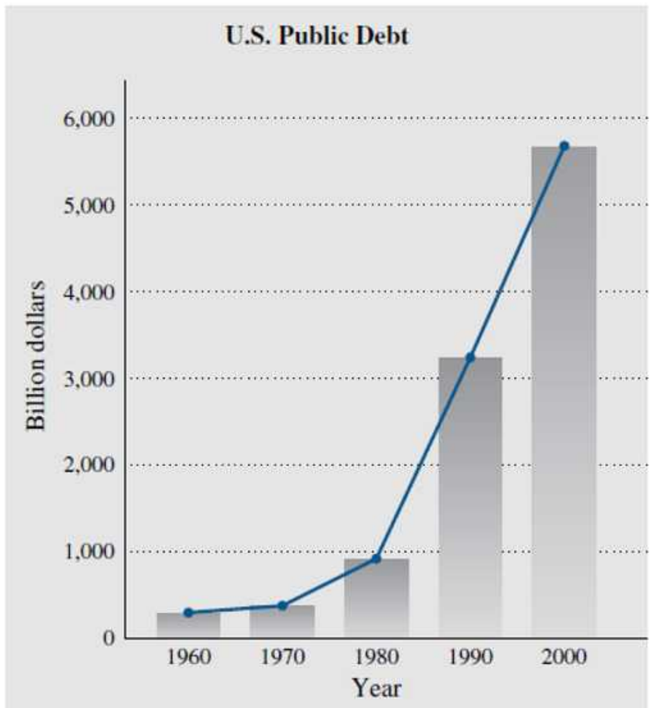


Figure 21.2

Pie Chart
Another popular pictorial representation of data is the **pie chart** or the **circle graph**. This kind of graph is needed to show percentages effectively. To construct a pie chart we first convert the distribution into a percentage distribution. Then, since a complete circle corresponds to 360 degrees, we obtain the central angles of the various sectors by multiplying the percentages by 3.6.

Example 21.4
A survey of 1000 adults uncovered some interesting housekeeping secrets.

When unexpected company comes, where do we hide the mess? The survey showed that 68% of the respondents toss their mess in the closet, 23% shove things under the bed, 6% put things in the bath tub, and 3% put the mess in the freezer. Make a circle graph to display this information.

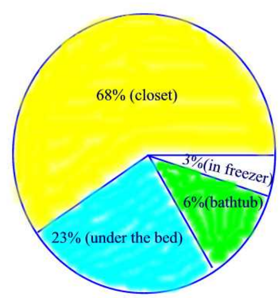
Solution.
We first find the central angle corresponding to each case:

$$\begin{array}{lrcr} \textit{in closet} & 68 \times 3.6 & = & 244.8 \\ \textit{under bed} & 23 \times 3.6 & = & 82.8 \\ \textit{in bathtub} & 6 \times 3.6 & = & 21.6 \\ \textit{in freezer} & 3 \times 3.6 & = & 10.8 \end{array}$$

Note that

$$244.8 + 82.8 + 21.6 + 10.8 = 360.$$

The pie chart is given below ■



Where to hide the mess?

Frequency Distribution
When we deal with large sets of data, a good overall picture and sufficient information can be often conveyed by distributing the data into a number of **classes** or **class intervals** and to determine the number of elements belonging to each class, called **class frequency.** For instance, the following table shows some test scores from a math class.

65	91	85	76	85	87	79	93
82	75	100	70	88	78	83	59
87	69	89	54	74	89	83	80
94	67	77	92	82	70	94	84
96	98	46	70	90	96	88	72

It's hard to get a feel for this data in this format because it is unorganized. To construct a frequency distribution,

- Compute the class width $CW = \frac{Largest\ data\ value - smallest\ data\ value}{Desirable\ number\ of\ classes}$.

- Round CW to the next highest whole number so that the classes cover the whole data. If necessary, add additional class intervals to cover all the values.

The low number in each class is called the **lower class limit**, and the high number is called the **upper class limit.**
Suppose we want to organize the above data into six class intervals. Then $CW = 8$. However, this width will not cover the whole data values. For this reason we will consider seven class intervals instead of six. With the above information we can construct the following table called **frequency distribution**.

Class	Frequency
45.5 - 55.5	2
55.5 - 65.5	2
65.5- 75.5	8
75.5 - 85.5	12
85.5 - 95.5	12
95.5 - 105.5	4

Relative Frequency Distribution
One further extension to the frequency distribution is to look at the percentage of values that show up in each class. This is called a **relative frequency distribution**. Here's how the above data would be presented in this way.

Class	Frequency	Relative Frequency	Percent
45.5 -55.5	2	2/40	5%
55.5 - 65.5	2	2/40	5%
65.5 - 75.5	8	8/40	20%
75.5 - 85.5	12	12/40	30%
85.5 - 95.5	12	12/40	30%
95.5 - 105.5	4	4/40	10%

Cumulative Frequency

The final frequency distribution that we will discuss is the **cumulative frequency distribution.** The cumulative frequency for a class is the sum of the frequencies for that class and all the previous classes. Cumulative totals can be used to determine how many data are above or below a set level. Here's a cumulative frequency distribution for the above set of data.

Class	Frequency	Cumulative
45.5 - 55.5	2	2
55.5 - 65.5	2	4
65.5 - 75.5	8	12
75.5 - 85.5	12	24
85.5 - 95.5	12	36
95.5 - 105.5	4	40.

Histograms

Once frequency distributions are constructed, it is usually advisable to present them graphically. The most common form of graphical representation is the **histogram**.

In a histogram, each of the classes in the frequency distribution is represented by a vertical bar whose height is the class frequency of the ineterval. The horizontal endpoints of each vertical bar correspond to the class endpoints. A histogram of the math scores is given in Figure 21.3

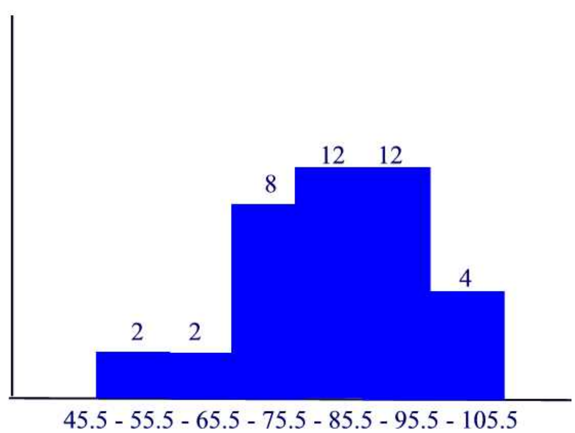


Figure 21.3

The following histogram shapes commonly occur (See Figure 21.4):

- **Symmetrical** bell-shaped distribution.
- **Uniform** or **rectangular** distribution. This happens when every class has equal frequency.
- **skewed left** (i.e. negatively skewed) or **skewed right** (i.e. positively skewed) distribution: These terms refer to a histogram in which one tail is stretched longer than the other.
- **Bimodal** distribution is the one in which two classes with largest frequencies are seperated by at least one class.

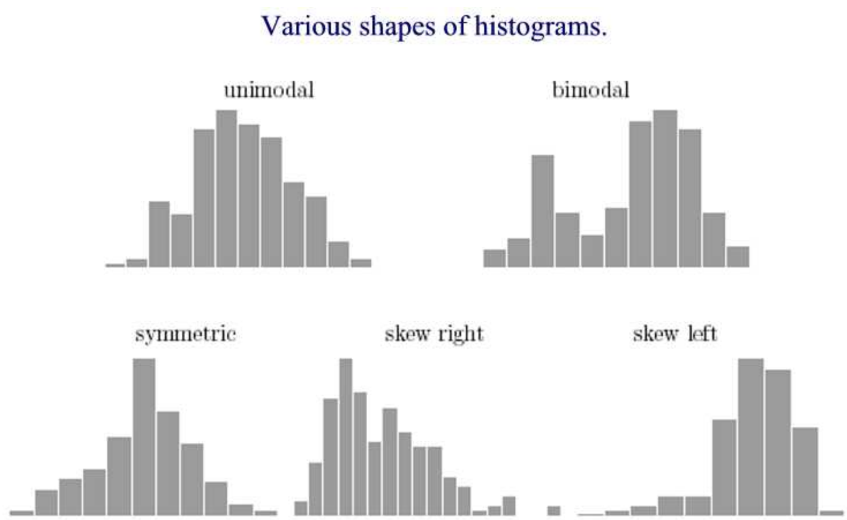


Figure 21.4

Frequency Polygon
Another, less widely used form of graphical presentation of frequency distribution is the **frequency polygon.** It is made by connecting in order the top midpoints of the bars in a histogram. These midpoints are called **class marks**. A class mark of a frequency class is obtained by adding the lower class limit and the upper class limit and divide by 2.
Figure 21.5 shows a line graph of the math scores problem.

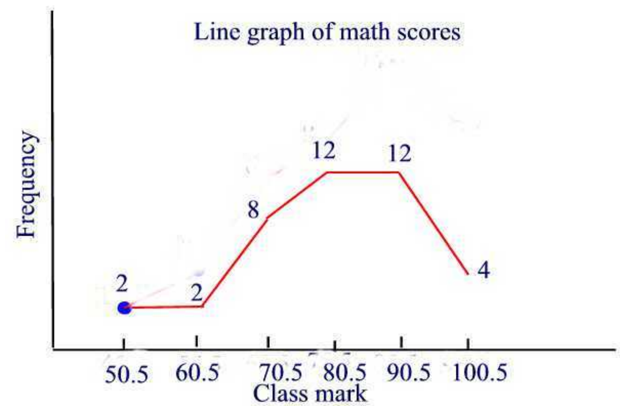


Figure 21.5

Line graphs are useful if you wish to compare two distributions. Figure 21.5 shows the line graph of the math scores problem. In Figure 21.6, the pulse rate of a person is shown to change over time.

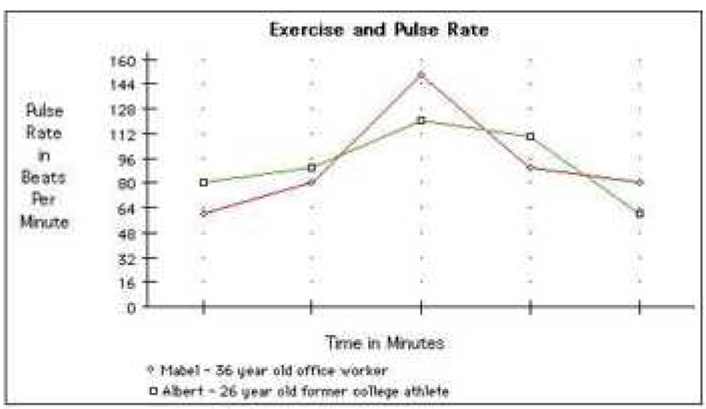


Figure 21.6

Practice Problems

Problem 21.1
The figures for total population, male and female population of the UK at decade intervals since 1959 are given below:

Year	*Total UK Resident Population*	*Male*	*Female*
1959	51,956,000	25,043,000	26,913,000
1969	55,461,000	26,908,000	28,553,000
1979	56,240,000	27,373,000	28,867,000
1989	57,365,000	27,988,000	29,377,000
1999	59,501,000	29,299,000	30,202,000

Construct a bar chart representing the data.

Problem 21.2
The following data gives the number of murder victims in the U.S in 1978 classified by the type of weapon used on them. Gun, 11,910; cutting/stabbing, 3,526; blunt object, 896; strangulation/beating, 1,422; arson, 255; all others 705. Construct a bar chart for this data. Use vertical bars.

Problem 21.3
The figures for total population at decade intervals since 1959 are given below:

Year	*Total UK Resident Population*
1959	51,956,000
1969	55,461,000
1979	56,240,000
1989	57,365,000
1999	59,501,000

Construct a bar chart for this data.

Problem 21.4
In the United States, approximately 45% of the population has blood type O; 40% type A; 11% type B; and 4% type AB. Illustrate this distribution of blood types with a pie chart.

Problem 21.5

The following table represent a survey of people's favorite ice cream flavor

Flovor	*Number of people*
Vanilla	21.0%
Chocolate	33.0%
Strawberry	12.0%
Raspberry	4.0%
Peach	7.0%
Neopolitan	17.0%
Other	6.0%

Plot a pie chart to represent this data.

Problem 21.6

A newly qualified teacher was given the following information about the ethnic origins of the pupils in a class.

Ethnic origin	*No. of pupils*
White	12
Indian	7
BlackAfrican	2
Pakistani	3
Bangladeshi	6
TOTAL	30

Plot a pie chart representing the data.

Problem 21.7

The table below shows the ingredients used to make a sausage and mushroom pizza.

Ingredient	%
Sausage	7.5
Cheese	25
Crust	50
TomatoSause	12.5
Mushroom	5

Plot a pie chart for this data.

Problem 21.8
(a) Describe the purpose of a frequency distribution.
(b) Describe the type of data that could be usefully described with a histogram.

Problem 21.9
Suppose a sample of 38 female university students was asked their weights in pounds. This was actually done, with the following results:

130	108	135	120	97	110
130	112	123	117	170	124
120	133	87	130	160	128
110	135	115	127	102	130
89	135	87	135	115	110
105	130	115	100	125	120
120	120				

(a) Suppose we want 9 class intervals. Find CW.
(b) Construct a frequency distribution.
(c) Construct the corresponding histogram.
(d) Construct the relative frequency distribution.
(e) Construct a line graph.

Problem 21.10
The table below shows the response times of calls for police service measured in minutes.

34	10	4	3	9	18	4
3	14	8	15	19	24	9
36	5	7	13	17	22	27
3	6	11	16	21	26	31
32	38	40	30	47	53	14
6	12	18	23	28	33	
3	4	62	24	35	54	
15	6	13	19	3	4	
4	20	5	4	5	5	
10	25	7	7	42	44	

(a) Construct a frequency distribution and the corresponding histogram.
(b) Construct the relative frequency distribution and the relative frequency

histogram.
(c) Construct a line graph.

Problem 21.11
A business magazine was conducting a study into the amount of travel required for mid-level managers across the US. Seventy-five managers were surveyed for the number of days they spent traveling each year.

Days Traveling	Frequency
0 - 6	15
7 - 13	21
14 - 20	27
21 - 27	9
28 - 34	2
34 and above	1

Construct a relative frequency distribution.

Problem 21.12
A nutritionist is interested in knowing the percent of calories from fat which Americans intake on a daily basis. To study this, the nutritionist randomly selects 25 Americans and evalustes the percent of calories from fat consumed in a typical day. The results of the study are as follows

34%	18%	33%	25%	30%
42%	40%	33%	39%	40%
45%	35%	45%	25%	27%
23%	32%	33%	47%	23%
27%	32%	30%	28%	36%

(a) Construct a frequency distribution for the percent of calories from fat.
(b) Construct the relative frequency distribution for the percent of calories from fat.
(c) Construct the cumulative frequency distribution for the percent of calories from fat.
(d) Construct a histogram of the relative frequency distribution.
(e) Construct a line graph.

Problem 21.13
The table below shows the exam scores of a Physics test.

95	72	88	90	76	85	60	99	72	84
66	78	90	74	85	65	89	93	91	85
82	75	88	93	53	84	87	58	72	94

(a) Construct a frequency distribution with $CW = 4$ and the corresponding histogram.
(b) Construct the relative frequency distribution and the relative frequency histogram.
(c) Construct a line graph.

Problem 21.14
33 students took a math test. The test was out of 40 points. The following table shows the frequency distribution.

Score	Frequency
24	1
25	1
28	2
29	2
30	1
31	1
32	3
33	1
34	6
35	3
36	4
37	8

(a) Construct the corresponding histogram and show that the shape is skewed to the left.

Problem 21.15
Given a frequency table of ages from a large survey.

AGE	Freq	Rel.Freq	Cum.Freq.
3	2	0.3%	0.3%
4	9	1.4%	1.7%
5	28	4.3%	6.0%
6	37	5.7%	11.6%
7	54	8.3%	19.9%
8	85	13.0%	32.9%
9	94	14.4%	47.2%
10	81	12.4%	59.6%
11	90	13.8%	73.4%
12	57	8.7%	82.1%
13	43	6.6%	88.7%
14	25	3.8%	92.5%
15	19	2.9%	95.4%
16	13	2.0%	97.4%
17	8	1.2%	98.6%
18	6	0.9%	99.5%
19	3	0.5%	100.0%
Total	654	100.0%	

Construct the corresponding histogram. Does the histogram have a bimodel shape?

Problem 21.16

An investigator interested in finger-tapping behavior conducts the following study: Students are asked to tap as fast as they can with their ring finger. The hand is cupped and all fingers except the one being tapped are placed on the surface. Either the right or the left hand is used, at the preference of the student. At the end of 15 seconds, the number of taps for each student is recorded. Example data using 18 subjects are presented below:

53	35	67	48	63	42
48	55	33	50	46	45
59	40	47	51	66	53

(a) Construct the frequency distribution with 12 classes.
(b) Construct the corresponding histogram. Show that the distribution is symmetrical bell shaped distribution.

Problem 21.17

The weekly sales from a sample of Hi-Tec electronic supply stores were organized into a frequency distribution. The mean of weekly sales was computed to be \$105900, the median \$105000, and the mode \$104500.

(a) Sketch the sales in the form of a smoothed frequency polygon. Note the location of the mean, median, and mode on the X-axis.

(b) Is the distribution symmetrical, positively skewed, or negatively skewed?

22. Measures of Central Tendency

It is often necessary to represent a set of data by means of a single number which, in its way, is descriptive of the entire set. Such a number is called an **average**. Exactly what sort of number we choose depends on the particular characteristic we want to describe. In one study we may be interested in an extreme (smallest or largest) value among the data, in another, in the center or the middle of a set of data. The following measures are also known as the **measures of central tendency**.

Mode
Several types of averages can be defined. The easiest one to compute is the **mode**. It is defined as the value which occurs with the highest frequency in the data. Thus, it is used when one is interested in the most common value in a distribution. A mode may not exist and usually this happens when no data value occurs more frequently than all the others. Also a mode may not be unique.

Example 22.1
The final grades of a class of six graduate students were A, C, B, C, A, B. What is the mode?

Solution.
Since the grades occur at the same frequency, there is no mode for this data ■

Example 22.2
A sample of the records of motor vehicle bureau shows that 18 drivers in a certain age group received $3, 2, 0, 0, 2, 3, 3, 1, 0, 1, 0, 3, 4, 0, 3, 2, 3, 0$ traffic tickets during the last three year. Find the mode?

Solution.
The mode consists of 0 and 3 since they occur six times on the list ■
The definition of the mode given above is for ungrouped data. For finding the mode of grouped data (i.e., data organized into a frequency distribution), first of all we have to determine the **modal class.** The class interval whose frequency is maximum is known by this name. the mode lies in between this class. Then the mode is calculated by the following formula:

$$\text{Mode} = L + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$$

where
L = lower limit of modal class
f_1 = frequency of modal class
f_0 = frequency of class preceding the modal class
f_2 =frequency of class following the modal class
h = width of model class.

Example 22.3
Find the mode for the following grouped data, which describe, in thousands, the number of people in each age group.

Age	Frequency
0-19	78097
20-39	78481
40-59	73071
60-79	36259
80-99	9157

Solution.
The model class is 20-39. Its mode is

$$\text{Mode} = 20 + \frac{78481 - 78097}{2(78481) - 78097 - 73071} \times 19 = 21.26 \blacksquare$$

Median
Another average is the **median**. The median usually is found when we have an ordered distribution. It is computed as follows. We arrange the numerical data from smallest to largest. If n denotes the size of the set of data then the median can be found by using the **median rank**

$$MR = \frac{n+1}{2}.$$

If MR is a whole number then the median is the value in that position. If MR ends in .5, we take the sum of the adjacent positions and divide by 2. Unlike the mode, the median always exists and is unique. But it may or may not be one of the given data values. Note that extreme values (smallest or largest) do not affect the median.

Example 22.4
Among groups of 40 students interviewed at each of 10 different colleges, $18, 13, 15, 12, 8, 3, 7, 14, 16, 3$ said that they jog regularly. Find the median.

Solution.
First, arrange the numbers from smallest to largest to obtain

$$3 \quad 3 \quad 7 \quad 8 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 18$$

Next, compute the median rank $MR = \frac{10+1}{2} = 5.5$. Hence, the median is $\frac{12+13}{2} = 12.5$ ■

Example 22.5
Nine corporations reported that in 1982 they made cash donations to $9, 16, 11, 10, 13, 12, 6, 9,$ and 12 colleges. Find the median number.

Solution.
Arranging the numbers from smallest to largest to obtain

$$6 \quad 9 \quad 9 \quad 10 \quad 11 \quad 12 \quad 12 \quad 13 \quad 16$$

The median rank is $MR = \frac{9+1}{2} = 5$. The median is 11 ■
The above definition of the median is for ungrouped data. For grouped data, we first find the **median class**. This is the class whose cumulative frequency is greater (and nearest to) $\frac{n}{2}$, where n is the total frequency. The median is found by the formula

$$\text{Median} = L + \frac{\frac{n}{2} - CF}{f} \times h$$

where
$L =$ lower limit of median class
$n =$ total frequency
$CF =$ cumulative frequency of class preceeding the median class
$f =$ frequency of median class
$h =$ width of median class.

Example 22.6
Find the median for the following grouped data, which describe, in thousands, the number of people in each age group.

Age	Frequency
0-19	78097
20-39	78481
40-59	73071
60-79	36259
80-99	9157

Solution.
The median class is 20-39. Its median is

$$\text{Mode} = 20 + \frac{137532.5 - 78097}{78481} \times 19 = 34.39 \blacksquare$$

Arithmetic Mean
Another most widely used average is the **arithmetic mean** or simply **mean**. The mean of a set of N numbers $x_1, x_2, ..., x_N$, denoted by $\overline{x}$, is defined as

$$\overline{x} = \frac{x_1 + x_2 + ... + x_N}{N}.$$

Unlike the median, the mean can be affected by extreme values since it uses the exact value of each data.

Example 22.7
If nine school juniors averaged 41 on the verbal portion of the PSAT test, at most how many of them can have scored 65 or more?

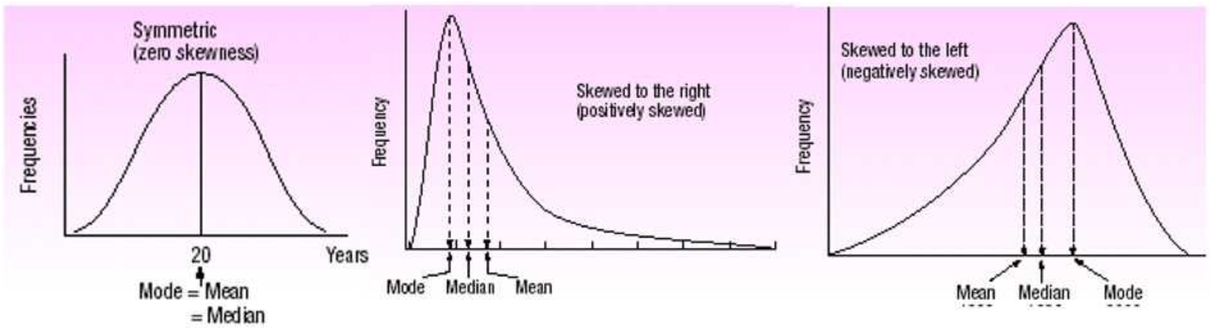
Solution.
We have that $\overline{x} = 41$ and $N = 9$ so that $x_1 + x_2 + \cdots + x_9 = 41 \times 9 = 369$. Since $6 \times 65 = 390 > 369$ and $5 \times 65 = 325$, at most 6 students can score more than 65 $\blacksquare$

Example 22.8
If the numbers $x_1, x_2, ..., x_N$ occur with frequencies $m_1, m_2, ..., m_N$ respectively then what is the mean in this case?

Solution.
The mean is given by

$$\overline{x} = \frac{m_1x_1 + m_2x_2 + ... + m_Nx_N}{m_1 + m_2 + \cdots + m_N} \blacksquare$$

The figure below give the relationships among the measures of central tendency



Example 22.9
The **geometric mean** of n positive numbers is the nth root of their product. Thus, the geometric mean of 3 and 12 is $\sqrt{3 \cdot 12} = 6$. The geometric mean is used mainly to average ratios.
Find the geometric mean of 8 and 32.

Solution.
The average mean of 8 and 32 is $\sqrt{8 \times 32} = \sqrt{256} = 16$ ■

Weighted Mean
When averaging quantities, it is often necessary to account for the fact that not all of them are equally important in the phenomenon being described. Thus, in order to give quantities being averaged their proper degree of importance, it is necessary to assign them **weights**, and then calculate the **weighted mean**. In general, if $w_1, w_2, ..., w_N$ are the weights associated to the quantities $x_1, x_2, ..., x_N$ then the weighted mean is

$$\overline{x}_w = \frac{w_1x_1 + w_2x_2 + ... + w_Nx_N}{w_1 + w_2 + ... + w_N}.$$

Example 22.10
If a final examination in a course is weighted 3 times as much as a quiz and a student has a final examination grade of 85 and quiz grades of 70 and 90 then what is the weighted mean ?

Solution.
The weighted mean is

$$\frac{70 + 90 + 3(85)}{1 + 1 + 3} = 83 \blacksquare$$

Example 22.11
Given that in 1978 commercial fishermen caught 87.7 million pounds of clams, 449.1 million pounds of crabs, 39 million pounds of lobsters, 33.3 million pounds of scallops, and 434.8 million pounds of shrimp. The prices per pound are : \$1.24 (clams), \$.21 (crabs), \$1.77 (lobsters), \$2.54 (scallops), and \$2.11 shrimp. Use the weights and prices to determine the average price per pound.

Solution.
The average price per pound is
$((87.7)(1.24) + (449.1)(0.21) + (39)(1.77) + (33.3)(2.54) + (434.8)(2.11)) \div 1043.9 =$ $\$1.22/lb$ ■

Remark 22.1
We will use $\overline{x}$ denote the sample mean whereas we use μ to denote the population mean.

Mean of grouped data
To find the sample mean of **grouped data** (i.e., data organized into a frequency distribution), we find the midpoint of each of the classes in the frequency distribution and then we weight each of these midpoints by the frequency of that class. Thus, the mean is given by the formula

$$\overline{x} = \frac{\sum_{i=1}^{k} x_i f_i}{n}$$

where k is the number of frequency classes, f_i is the i^{th} class frequency, and n is the total frequency.

Example 22.12
Find the population mean of the following frequency distribution.

Class Interval	Frequency
20-30	6
31-40	18
41-50	11
51-60	11
61-70	3
71-80	1

Solution.
The required calculations for the mean are presented in the table below.

Class Interval	Frequency (f)	Class Midpoint (x)	xf
20-30	6	25	150
31-40	18	35	630
41-50	11	45	495
51-60	11	55	605
61-70	3	65	195
71-80	1	75	75
	50		2150

The mean is

$$\mu = \frac{2150}{50} = 43 \blacksquare$$

Practice Problems

Problem 22.1
The 20 meetings of a square dance club were attended by 26, 25, 28, 23, 25, 24, 24, 23, 26, 26, 28, 26, 24, 32, 25, 27, 24, 23, 24, and 22 of its members. Find the mode.

Problem 22.2
Find the mode of the set of data 21, 22, 21, 23, 22, 23.

Problem 22.3
The following are numbers of passengers on 25 runs of a ferryboat:52, 84, 40, 57, 61, 65, 77, 64, 62, 35, 82, 58, 50, 78, 103, 71, 75, 41, 53, 66, 60, 95, 58, 49, and 89. Find the median.

Problem 22.4
Find the median of the data set

$$3, 5, 2, 8, 9, 3, 12.$$

Problem 22.5
If the mean annual salary paid to the top of three executives of a firm is $\$96,000$, can one of them receive an annual salary of $\$300,000$?

Problem 22.6
For the 12 months of 1982, a police department reported 3, 2, 4, 4, 9, 7, 8, 5, 2, 3, 7, and 6 armed robberies. Find the mean of armed robberies per month.

Problem 22.7
An instructor counts the final examination in a course four times as much as of the four one-hour examinations. What is the average grade of a student who received grades of $74, 80, 61$, and 77 in the four one- hour examinations and 83 in the final examination?

Problem 22.8
In 1980 a college paid its 52 instructors a mean salary of $\$13,200$, its 96 assistant professors a mean salary of $\$15,800$, its 67 associate professors a mean salary of $\$18,900$, and its 35 full professors a mean salary of $\$23,500$. What was the mean salary paid to all the teaching staff of this college?

Problem 22.9
In the Fall of 2000 the statistics class gathered data on the number of siblings for each member of the class. One student was an only child and had no siblings. One student had 13 brothers and sisters. The complete data set is as follows: $1, 2, 2, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 7, 8, 9, 10, 12, 12, 13$. What is the mode?

Problem 22.10
The age of students in a statistics class is given by the following list:$18, 19, 19, 20, 20$ $, 21, 21, 21, 21, 22, 22, 22, 22, 23, 23, 24, 24, 25, 25, 26$. Find the mode.

Problem 22.11
Find the median for the sibling data $1, 2, 2, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 7, 8, 9, 10, 12, 12, 13$.

Problem 22.12
(a) A company is considering a move into a regional market for specialty soft drinks. In analyzing the size containers that his competitors are currently offering, would the company be more interested in the mean, median, or mode of their containers.
(b) The creative director for an advertising agency is trying to target an ad campaign that will be shown in one city only. Would he be more interested in the mean or median family income in the city?

Problem 22.13
(a) A young economist was assigned the task of comparing the interest rates on ninety day Certificates of Deposit CD's in three major cities. Should she compare the mean, median, or modal interest for the banks in the three cities?
(b) A telephone company is interested in knowing how customers rate their service: excellent, good, average, or poor. Would the company be more interested in studying the mean, median, or mode of the customer service ratings?

Problem 22.14
The following are the daily high temperatures for a southern city in July (measured in degrees Fahrenheit).

84	85	84	88	94	100
97	102	97	89	89	90
88	95	91	95	99	93
97	99	90	94	90	88
91	88	106	99	102	85

(a) Calculate the average of the daily high temperatures.
(b) Calculate the median of the daily high temperatures.
(c) Calculate the mode of the daily high temperatures.

Problem 22.15
The weekly salaries of six employees at McDonalds are \$140, \$220, \$90, \$180, \$140, \$200. For these six salaries, find: (a) the mean (b) the median (c) the mode

Problem 22.16
Andy has grades of 84, 65, and 76 on three math tests. What grade must he obtain on the next test to have an average of exactly 80 for the four tests?

Problem 22.17
A storeowner kept a tally of the sizes of suits purchased in her store. Which of the mean, median, or mode should the storeowner use to describe the average suit sold? Justify your answer.

Problem 22.18
The mean price of 5 items is \$7.00. The prices of the first four items are \$6.50, \$8.00, \$5.50, and \$6.00. How much does the fifth item cost?

Problem 22.19
The mean of a set of 7 numbers is 13. What is the sum of the numbers?

Problem 22.20
The mean of a set of data is 174.25, and the sum of the data is 1,394. How many numbers are in the set?

Problem 22.21
The grade point averages of 10 students are listed below. Find the median grade point average.

3.15	3.62	2.54	2.81	3.97
1.85	1.93	2.63	2.50	2.80

Problem 22.22
Calculate the GPA (weighted mean) for the following data: Biology, 5 credits, A- (use 3.667); Chemistry, 4 credits, B+ (use 3.333); College Algebra, 3 credits, A (use 4.000); and Health, 2 credits, C (use 2.000); Debate, 2 credits, B (use 3.000). Express your results to three decimal places.

Problem 22.23
Find the geometric mean of 9 and 12 to the nearest hundredth.

Problem 22.24
The Carter Construction Company pays its hourly employees \$6.50, \$7.50, or \$8.50 per hour. There are 26 hourly employees, 14 are paid at the \$6.50 rate, 10 at the \$7.50 rate, and 2 at the \$8.50 rate. What is the mean hourly rate paid the 26 employees?

Problem 22.25
Springers sold 95 Antonelli mens suits for the regular price of \$400. For the spring sale the suits were reduced to \$200 and 126 were sold. At the final clearance, the price was reduced to \$100 and the remaining 79 suits were sold. What was the weighted mean price of an Antonelli suit?

Problem 22.26
In June an investor purchased 300 shares of Oracle stock at \$20 per share. In August she purchased an additional 400 shares at \$25 per share. In November she purchased an additional 400 shares, but the stock declined to \$23 per share. What is the weighted mean price per share?

Problem 22.27
All the students in advanced Computer Science 411 are considered the population. Their course grades are 92, 96, 61, 86, 79, and 84.

(a) Compute the mean course grade.
(b) Is the mean you computed in (a) a statistic or a parameter? Why?

Problem 22.28
Andrews and Associates specialize in corporate law. They charge \$100 per hour for researching a case, \$75 per hour for consultations, and \$200 per hour for writing a brief. Last week one of the associates spent 10 hours consulting with her client, 10 hours researching the case, and 20 hours writing the brief. What was the weighted mean hourly charge for her legal services?

Problem 22.29
Average weekly employment insurance benefits, by category, are: \$279, \$253, \$290, \$400, \$92, and \$351. (a) What is the median monthly benefit?
(b) How many observations are below the median? Above it?

Problem 22.30
The return on investment earned by Atkins Construction Company for four successive years was: 30 percent, 20 percent, -40 percent, and 200 percent. What is the geometric mean rate of return on investment?

Problem 22.31
Suppose your midterm test score is 94 and your final exam score is 90. Using the weights of 41% for the midterm and 59% for the final exam, compute the weighted average of your scores.

Problem 22.32
Suppose your final grade in your organic chemistry class is based on several things: a lab score, scores on two major tests, and your score on the final exam. There are 100 points available for each score. However, the lab score is worth 13% of your total grade, each major test is worth 26%, and the final exam is worth 35%.

Compute the weighted average for the following scores: 72 on the lab, 84 on the first major test, 90 on the second major test, and 80 on the final exam.

Exercise 22.1
The age distribution and frequency of people doing volunteer work is shown for a random sample of 530 volunteers.

Age	Number of Volunteers
14- 17	122
18-24	80
25-44	159
45-64	106
65-80	63

Approximate the mean and the median of the above distribution.

Exercise 22.2
The Marathon Walk for World peace has a 25-mile route. A random sample of participants walked the distances shown below

Distance (miles)	Number for Walkers
1- 5	14
6-10	9
11-15	11
16-20	10
21-25	6

Approximate the mean and the median of the above distribution.

Exercise 22.3
The manager of a supermarket wants to hire one more checkout clerk. To justify his request to the regional manager, the manager chose a random sample of 44 customers and timed how long each stood in line before a clerk could begin checking the customer out. The written request contained the table shown below. Approximate the mean waiting time to two decimal places.

Waiting Time (min)	Frequency (f)
0-2	4
3-5	5
6-8	7
9-11	16
12-14	12

23. Measures of Dispersion

While mean and median tell you about the center of your data, it says nothing about the "spread" of the data. **Variability** or **dispersion** measure the extent to which data are spread out. The measures of variability for data that we look at are: the range, the variance and the standard deviation.

The Range
To measure the variability between extreme values (i.e. smallest and largest values) one uses the **range**. The range is the difference between the largest and smallest values of a distribution.

Example 23.1
Find the range of each of the following samples:

Sample 1: 6,18,18,18,18,18,18,18,18,18.
Sample 2: 6,6,6,6,6,6,18,18,18,18,18.
Sample 3: 6,7,9,11,12,14,15,16,17,18.

Solution.
Sample 1: 18 - 6 = 12
Sample 2: 18 - 6 = 12
Sample 3: 18 - 6 = 12.■

As you can see from this example, each sample has a range $18 - 6 = 12$ but the spread out of values is quite different in each case. In Sample 1, the spread is uniform whereas it is not in Sample 3. This is a disadvantage of this kind of measure. The range tells us nothing about the dispersion of the values between the extreme (smallest and largest) values.

Variance and Standard Deviation
Unlike the range, the variance combines all the values in a data set to produce a measure of spread. The variance and the standard deviation are both measures of the spread of the distribution about the mean.
If μ is the population mean of set of data then the quantity $(x - \mu)$ is called the **deviation from the mean**. The **variance** of a data set is the arithmetic average of squared deviation from the mean. It is denoted by s^2 and

is given by the formula

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}.$$

Note that the variance is nonnegative, and it is zero only if all observations are the same.

The **standard deviation** is the square root of the variance:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}}.$$

The variance is the nicer of the two measures of spread from a mathematical point of view, but as you can see from the algebraic formula, the physical unit of the variance is the square of the physical unit of the data. For example, if our variable represents the weight of a person in pounds, the variance measures spread about the mean in squared pounds. On the other hand, standard deviation measures spread in the same physical unit as the original data, but because of the square root, is not as nice mathematically. Both measures of spread are useful.

A step by step approach to finding the standard deviation is:

1. Calculate the mean.
2. Subtract the mean from each observation.
3. Square each result.
4. Add these squares.
5. Divide this sum by the number of observations.
6. Take the positive square root.

The variance and standard deviation introduced above are for a population. We define the **sample variance** by the formula

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}$$

and the **sample standard deviation** by the formula

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}.$$

The reason that for s we use $n-1$ instead of n is because usually a sample does not contain extreme values and we want s to be an estimate of σ therefore by using $n-1$ we make s a little larger than if we divide by n.

Example 23.2
The owner of the Ches Tahoe restaurant is interested in how much people spend at the restaurant. He examines 10 randomly selected receipts for parties of four and writes down the following data.

44 50 38 96 42 47 40 39 46 50

(a) Find the arithmetic mean.
(b) Find the variance and the standard deviation.

Solution.
(a) The arithmetic mean is the sum of the above values divided by 10,i.e., $\overline{x} = 49.2$
Below is the table for getting the standard deviation:

x	$x - 49.2$	$(x - 49.2)^2$
44	-5.2	27.04
50	0.8	0.64
38	11.2	125.44
96	46.8	2190.24
42	-7.2	51.84
47	-2.2	4.84
40	-9.2	84.64
39	-10.2	104.04
46	-3.2	10.24
50	0.8	0.64
Total		2600.4

Hence the variance is $s^2 = \frac{2600.4}{9} \approx 289$ and the standard deviation is $s = \sqrt{289} = 17$. What this means is that most of the patrons probably spend between 49.20 - 17 =\$32.20 and 49.20+17=\$66.20. ∎

Example 23.3
A museum curator examined the Crown of Charlemagne and found the seven rubies to have the following weights (in carats):

$$19.8 \quad 43.8 \quad 36.1 \quad 52.4 \quad 63.1 \quad 20.7 \quad 46.3$$

Compute the population mean, population variance, and population standard deviation.

Solution.
The population mean is given by

$$\overline{x} = \frac{19.8 + 43.8 + 36.1 + 52.4 + 63.1 + 20.7 + 46.3}{7} = 40.31.$$

The population variance is found as follows

x	$x - 40.31$	$(x - 40.31)^2$
19.8	-20.51	420.66
43.8	3.49	12.18
36.1	-4.21	17.72
52.4	12.09	146.17
63.1	22.79	519.38
20.7	-19.61	384.55
46.3	5.99	35.88
Total		1536.54

Thus, $s^2 = \frac{1536.54}{7} \approx 219.51$ and the standard deviation is $\sigma = \sqrt{\frac{1536.54}{7}} \approx 14.82.$■

Remark 23.1
In the denominator of the formula for s^2 we use $n - 1$ instead n because statisticians proved that if s^2 is defined as above then s^2 is an unbiased estimate of the variance of the population from which the sample was selected (i.e. the expected value of s^2 is equal to the population variance).

Variance and Standard Deviation of Grouped Data
Recall that the sample mean for a grouped data is given by the formula

$$\overline{x} = \frac{\sum xf}{n}.$$

The sample standard deviation is given by the formula

$$
\begin{aligned}
s =& \sqrt{\frac{\sum (x-\overline{x})^2 f}{n-1}} \\
=& \frac{1}{n-1}\left(\sum x^2 f - \frac{(\sum xf)^2}{n}\right)^{\frac{1}{2}}.
\end{aligned} \tag{11}
$$

where x is the midpoint of a class, f is the frequency of that class, n is the size of the sample, and the summation $\sum$ is over all the classes in the distribution.
The population standard deviation is given by

$$
\begin{aligned}
s =& \sqrt{\frac{\sum (x-\overline{x})^2 f}{N}} \\
=& \frac{1}{N}\left(\sum x^2 f - \frac{(\sum xf)^2}{N}\right)^{\frac{1}{2}}.
\end{aligned} \tag{12}
$$

where N is the population size.

Example 23.4
The table below presents in grouped form the systolic blood pressure reading of 85 patients. Find the mean and the standard deviation.

Systolic Blood Pressure	Frequency
105.5 - 115.5	2
115.5 - 125.5	5
125.5 - 135.5	13
135.5 - 145.5	12
145.5 - 155.5	19
155.5 - 165.5	9
165.5 - 175.5	8
175.5 - 185.5	8
185.5 - 195.5	7
195.5 - 205.5	2

Solution.
Recall that the midpoint of a class is the average of the lowerclass boundary and the upperclass boundary. The required calculations for the mean and

the standard deviation are presented in the table below.

Interval	Frequency (f)	Midpoint (x)	xf	x^2	x^2f
105.5 - 115.5	2	110.5	220.5	12210.25	24420.50
115.5 - 125.5	5	120.5	602.5	14520.25	72601.25
125.5 - 135.5	13	130.5	1696.5	17030.25	221393.25
135.5 - 145.5	12	140.5	1686.0	19740.25	236883.00
145.5 - 155.5	19	150.5	2859.5	22650.25	430354.75
155.5 - 165.5	9	160.5	1444.5	25760.25	231842.25
165.5 - 175.5	8	170.5	1364.0	29070.25	232562.00
175.5 - 185.5	8	180.5	1444.0	32580.25	260642.00
185.5 - 195.5	7	190.5	1333.5	36290.25	254031.75
195.5 - 205.5	2	200.5	401.0	40200.25	80400.50
	85		13052.5	250052.5	2045131.25

The mean is

$$\mu = \frac{13052.5}{85} \approx 153.559$$

and the standard deviation is

$$s = \left(\frac{2045131.25 - \frac{13052.5^2}{85}}{85} \right)^{\frac{1}{2}} \approx 22.040 \blacksquare$$

Remark 23.2
A small value of σ tells us that the data is more concentrated around the mean. A large value tells us the data are more spread out.

Practice Problems

Problem 23.1
A population consists of the integers $1, 2, 3, 4, 5, 6$. Find the range, population mean, population variance, and population standard deviation.

Problem 23.2
The following are the wind velocities reported at 6 P.M.on six consecutive days: 13,8,15,11,3 and 10. Find the range, sample mean, sample variance, and sample standard deviation.

Problem 23.3
By sampling different landscapes in a national park over a 2-year period, the number of deer per square kilometer was determined. The results were (deer per square kilometer)

30	20	5	29	58	7
20	18	4	29	22	9

Compute the range, sample mean, sample variance, and sample standard deviation.

Problem 23.4
A researcher wants to find the number of pets per household. The researcher conducts a survey of 35 households. Find the range, the sample variance and standard deviation.

0	2	3	1	0
1	2	3	1	0
1	2	1	1	0
3	2	1	1	1
4	1	2	2	4
3	2	1	2	3
2	3	4	0	2

Problem 23.5
Let's compare some sample data from recent test scores(out of 100).
Test scores from Class #1: 65, 66, 67, 69, 70, 70, 70, 71, 71, 72, 73, 74, 76
Test scores from Class #2 50, 53, 54, 55, 55, 58, 59, 59, 75, 95, 98, 100, 100.
Which class scores are more grouped toward its mean?

Problem 23.6
Suppose two machines produce nails which are on average 10 inches long. A sample of 11 nails is selected from each machine.

Machine A: 6, 8, 8, 10, 10, 10, 10, 10, 12, 12, 14.
Machine B: 6, 6, 6, 8, 8, 10, 12, 12, 14, 14, 14.

Which machine is better than the other?

Problem 23.7
The age distribution and frequency of people doing volunteer work is shown for a random sample of 530 volunteers.

Age	Number of Volunteers
14- 17	122
18-24	80
25-44	159
45-64	106
65-80	63

Approximate the mean and the standard deviation of the above distribution.

Problem 23.8
Find the mean and sample standard deviation for the following grouped data, which describe, in thousands, the number of people in each age group.

Age	Frequency
019	78481
2039	78097
4059	73071
60-79	36259
80-99	9157

Problem 23.9
The Marathon Walk for World peace has a 25-mile route. A random sample of participants walked the distances shown below

Distance (miles)	Number for Walkers
1- 5	14
6-10	9
11-15	11
16-20	10
21-25	6

Approximate the mean and the standard deviation of the above distribution.

Problem 23.10
The manager of a supermarket wants to hire one more checkout clerk. To justify his request to the regional manager, the manager chose a random sample of 44 customers and timed how long each stood in line before a clerk could begin checking the customer out. The written request contained the table shown below. Approximate the mean waiting time to two decimal places.

Waiting Time (min)	Frequency (f)
0-2	4
3-5	5
6-8	7
9-11	16
12-14	12

Problem 23.11
What are the big corporations doing with their wealth? One way to answer this question is to examine profits as a percentage of assets. A random sample of 52 large corporations gave the following information.

Profit (% of assets)	Number of companies
8.6-12.5	14
12.6-16.5	22
16.6-20.5	5
20.6-24.5	9
24.6-28.5	2

Estimate the sample mean for profit as a percentage of assets.

24. Binomial Distributions

Binomial experiments (also known as **Bernoulli trials**) are problems that consist of a fixed number of trials n, with each trial having exactly two possible outcomes: **Success** and **failure.** The probability of a success is denoted by $p = P(S)$ and that of a failure by $q = P(F)$. Moreover, p and q are related by the formula

$$p + q = 1.$$

Also, we assume that the trials are **independent**, that is what happens to one trial does not affect the probability of a success in any other trial.
The prefix bi in binomial experiment refers to the fact that there are two possible outcomes (e.g., head or tail, true or false, working or defective) to each trial in the binomial experiment.

Example 24.1
Consider the experiment of asking people about their gender.
(a) What makes a trial?
(b) What is a success? a failure?

Solution.
(a) A trial consists of a response to the question of gender.
(b) A success could be Male in this case a failure is Female. Also, a success could be Female and a failure is a Male ■

The central question of a binomial experiment is to find the probability of r successes out of n trials. In the next paragraph we'll see how to compute such a probability. Now, anytime we make selections from a population **without replacement,** we do not have independent trials. For example, selecting a ball from a box that contain balls of two different colors. If the selection is without replacement then the trials are dependent.

Example 24.2
Privacy is a concern for many users of the Internet. One survey showed that 79% of Internet users are somewhat concerned about the confidentiality of their e-mail. Based on this information, we would like to find the probability that for a random sample of 12 Internet users, 7 are concerned about the privacy of their e-mail. What are the values of n, p, q, r?

Solutions.
This is a binomial experiment with 12 trials. If we assign success to an Internet user being concerned about the privacy of e-mail, the probability of success is 79%. We are interested in the probability of 7 successes. We have $n = 12, p = 0.79, q = 1 - 0.79 = 0.21, r = 7$ ■

Example 24.3
A biologist is studying a new hybrid tomato. It is known that the seeds of this hybrid tomato have probability 0.65 of germinating. The biologist plants 10 seeds. We wish to find the probability that at least 8 seeds will germinate. What are the values of n, p, q, r?

Solution.
This is a binomial experiment with n = 10 trials. Each seed planted represents an independent trial. We'll say germination is success, so the probability for success on each trial is p = 0.65, and $q = 1 - 0.65 = 0.35$. We are interested in the probability of 8 or more seeds germinating. Thus, $r = 8$ ■

Binomial Distribution Formula
As mentioned above, the central problem of a binomial experiment is to find the probability of r successes out of n independent trials.
Now, the probability of r successes in a sequence out of n independent trials is given by $p^r q^{n-r}$ and this is a result of the multiplication rule of counting. Since $C(n, r)$ counts all the number of outcomes that have r successes and $n - r$ failures then the probability of having r successes in any order is given by the **binomial distribution formula**

$$P(r) = C(n, r)p^r q^{n-r}.$$

Example 24.4
Suppose 40% of the student body at a large university are in favor of a ban on drinking in dormitories. Suppose 5 students are to be randomly sampled. Find the probability that
(a) 2 favor the ban.
(b) less than 4 favor the ban.
(c) at least 1 favor then ban.

Solution.
(a) $P(X = 2) = C(5, 2)(.4)^2(0.6)^3 \approx 0.3456.$

(b) $P(X < 4) = P(0)+P(1)+P(2)+P(3) = C(5,0)(0.4)^0(0.6)^5+C(5,1)(0.4)^1(0.6)^4+C(5,2)(0.4)^2(0.6)^3 + C(5,3)(0.4)^3(0.6)^2 \approx 0.913.$
(c) $P(X \geq 1) = 1 - P(X < 1) = 1 - C(5,0)(0.4)^0(0.6)^5 \approx 0.922$ ■

Example 24.5
A binomial experiment is repeated nine times. If the probability of a success is 0.6, find the probability of getting four successes.

Solution.
We have $n = 9, p = 0.6, q = 0.4$, and $r = 4$. Thus, $P(4) = C(9,4)(0.6)^4(0.4)^5 \approx 0.1672$ ■

Example 24.6
According to data from the Center for Disease Control, 40% of all individuals have Group A blood. If six individuals give blood, find the probability that none of the individuals have Group A blood.

Solution.
We have $n = 6, p = 0.4, q = 0.6, r = 0$ Thus, $P(0) = C(6,0)(0.4)^0(0.6)^6 \approx 0.0467$ ■

Binomial Distribution Histogram
Like the case of frequency distribution, one can make a histogram for the binomial probability distribution. The histogram is constructed by putting the r values on the horizontal axis and $P(r)$ values on the vertical axis. The width of the bar is 1 and its height is $P(r)$. The bars are centered at the r values.

Example 24.7
Construct the binomial distribution for the total number of heads in four flips of a balanced coin. Make a histogram.

Solution.
The binomial distribution is given by the following table

r	0	1	2	3	4
P(r)	1/16	4/16	6/16	4/16	1/16

The corresponding histogram is shown in Figure 24.1 ■

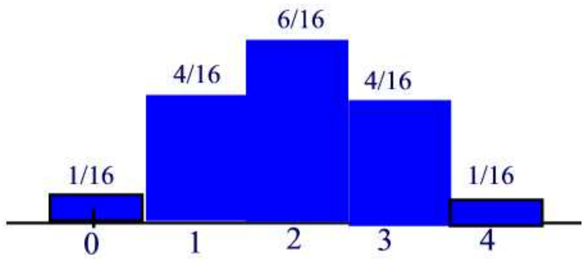


Figure 24.1

Example 24.8
Let X be the random variable that records the sum when a fair die is tossed twice. Construct the binomial distribution and then the corresponding histogram.

Solution.
The sample space of the experiment is given by

$$\begin{aligned} S = \{ \quad & (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ & (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ & (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ & (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ & (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ & (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\} \end{aligned}$$

The binomial distribution is given by the table

r	2	3	4	5	6	7	8	9	10	11	12
P(r)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Figure 24.2 shows the histogram ■

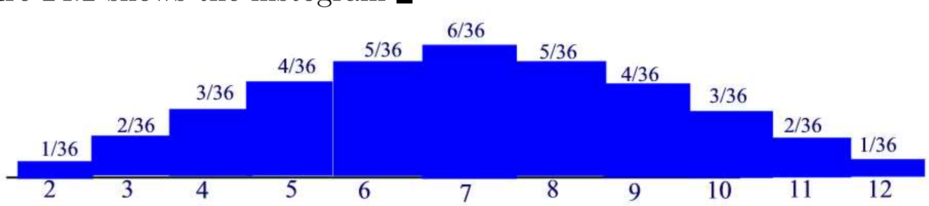


Figure 24.2

Mean, Variance, and Standard Deviation
For a single trial, the mean is $1 \cdot p + 0 \cdot (1-p) = p$ and the variance is $(0-p)^2(1-p) + (1-p)^2 p = p(1-p)$. Let X be the number of successes in a series of n independent trials, and let X_i be the number of successes on the ith of those n trials (either 0 or 1). Then

$$X = X_1 + X_2 + \cdots + X_n.$$

Therefore,

$$\mu_X = E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = p + p + \cdots + p = np.$$

Similarly, the variance is given by

$$\sigma_X^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \cdots + \sigma_{X_n}^2 = p(1-p) + p(1-p) + \cdots + p(1-p) = np(1-p).$$

The variance is given by the and

$$\sigma_X = \sqrt{npq}.$$

Example 24.9
Suppose we have 10 balls in a bowl, 3 of the balls are red and 7 of them are blue. Define success S as drawing a red ball. If we sample with replacement, p = P(S)=0.3 for every trial. Find the mean, variance, and the standard variation when $n = 20$.

Solution.
The mean is $\mu_X = np = 20(0.3) = 6$, the variance is $\sigma_X^2 = npq = 20(0.3)(0.7) = 4.2$ and the standard variation is $\sigma_X = \sqrt{4.2} \approx 2.049$ ■

Example 24.10
Among companies doing highway or bridge construction, 80% test employees for substance abuse. A study involves the random selection of 10 such companies. What is the mean and standard deviation for the number of companies in this sample that test for substance abuse?

Solution.
This is binomial, since there are 2 outcomes (test or not test), a fixed number of trials (10) and independence. So $\mu_X = 10(0.8) = 8, \sigma_X^2 = 10(0.8)(0.2) = 1.6$ and $\sigma_X = \sqrt{1.6} = 0.4$ ■

Practice Problems

Problem 24.1
The registrar of a college noted that for many years the withdrawal rate from an introductory chemistry course has been 35% each term. We wish to find the probability that 55 students out of 80 will register for the course.
(a) What makes a trial?
(b) What is a success? a failure?
(c) What are the values of n, p, q, r?

Problem 24.2
At Community Hospital, the nursing staff is large enough so that 80% of the time a nurse can respond to a room call within 3 minutes. Last night there were 73 room calls. We wish to find the probability nurses responded to 62 of them within 3 minutes.
(a) What makes a trial?
(b) What is a success? a failure?
(c) What are the values of n, p, q, r?

Problem 24.3
Harper's Index states that 10% of all adult residents in Washington D.C., are lawyers. For a random sample of 15 adult Washington, D.C., residents, we want to find the probability that 3 are lawyers.
(a) What makes a trial?
(b) What is a success? a failure?
(c) What are the values of n, p, q, r?

Problem 24.4
If the probability is 0.70 that any one registered voter (randomly selected from official rolls) will vote in a given election, what is the probability that two of five registered voters will vote in the election?

Problem 24.5
Suppose that the probability is 0.60 that a car stolen in a given city will be recovered. Find the probability that
(a) at least seven of ten cars stolen in this city will be recovered;
(b) at most three of ten cars stolen in this city will be recovered.

Problem 24.6
Find the probability that in tossing a fair coin three times there will appear (a) 3 heads, (b)2 heads and 1 tail, (c) 2 tails and 1 head, and (d) 3 tails.

Problem 24.7
Find the probability that in a family of 4 children there will be (a) at least 1 boy and (b) at least 1 boy and 1 girl. Assume that the probability of a male birth is $\frac{1}{2}$.

Problem 24.8
The probability that an entering college student will graduate is 0.4. Determine the probability that out of 5 students (a) none, (b) 1, (c) at least 1, (d) all will graduate.

Problem 24.9
Find the probability of guessing correctly at least 6 of the 10 answers on a true-false examination.

Problem 24.10
An insurance salesperson sells policies to 5 men, all of identical age and in good health. According to the actuarial tables, the probability that a man of this particular age will be alive 30 years is $\frac{2}{3}$. Find the probability that in 30 years (a) all 5 men, (b) at least 3 men, (c) only 2 men, (d) none will be alive.

Problem 24.11
Privacy is a concern for many users of the Internet. One survey showed that 82% of Internet users are somewhat concerned about the confidentiality of their e-mail. Based on this information, what is the probability that for a random sample of 10 Internet users, 6 are concerned about the privacy of their e-mail?

Problem 24.12
A biologist is studying a new hybrid tomato. It is known that the seeds of this hybrid tomato have probability 0.56 of germinating. The biologist plants 10 seeds. What is the probability that at least 8 seeds will germinate?

Problem 24.13
The percentage of American men who say they would marry the same woman

if they had it to do all over again is 67%. The percentage of American women who say they would marry the same man if they had it to do all over again is 86%.
What is the probability that in a group of 9 married men, at least 7 will claim that they would marry the same woman again?

Problem 24.14
25% of all small U.S. businesses have a Web site. If you randomly select 10 small businesses, what is the probability that
(a) three or fewer of them have a Web site?
(b) six or more of them have a Web site?
(c) between two and four inclusive of them have a Web site?
(d) exactly four of them have a Web site?

Problem 24.15
Assume that 25% of fuses are defective, and the fuses in packages of six fuses are independently selected.
(a) What is the probability that (exactly) two fuses in a package of six are defective?
(b) What is the probability that fewer than two are defective? Fewer than two means 0 or 1?

Problem 24.16
Assume that 30% of a population is Hispanic. A random sample of size 4 is chosen from this population. If X is the number of Hispanics in the sample, then what is the probability that the sample contains at most 2 hispanics?

Problem 24.17
Using the binomial model, and assuming that a success occurs with probability 1/5 in each trial, find the probability that in 6 trials there are
(a) 0 successes (b) 3 successes (c) 2 failures.

Problem 24.18
The probability that a car travelling along a certain road will have a tiree burst is 0.05. Find the probability that among 15 cars:
(a) exactly one has a burst tire,
(b) at most three have a burst tire,
(c) two or more have burst tires.

Problem 24.19
An examination consists of 10 multiple-choice questions, in each of which a candidate has to deduce which one of five suggested answers is correct. A completely unprepared student may be assumed to guess each answer completely randomly. What is the probability that this student gets 8 or more questions correct?

Problem 24.20
The probability that a student that received a C in a math class goes on to receive a C or better in their next math class is 0.1. If 5 students in a class receive C's, find the probability that 3 of them go on to receive a C or better in their next math class.

Problem 24.21
The probability that a certain medication will cause a bad reaction in young children is 0.06. Find the probability that out of 13 children that receive the medication, at most 1 will have a bad reaction.

Problem 24.22
Suppose we have 10 balls in a bowl, 3 of the balls are red and 7 of them are blue. Define success S as drawing a red ball. If we sample with replacement, P(S)=0.3 for every trial. What is the probability that we select 5 red balls out of 20 trials?

Problem 24.23
Suppose you have a bag with 3 red beads and 7 black beads in it. You are going to pull out a bead at random, examine it and then put it back in the bag. You are going to do this five times. Every time you get a red bead, that is a "success" and every time you get a black bead, that is a "failure".
(a) What is the probability of getting 5 red beads?
(b) What is the probability 4 red beads and one black?
(c) What is the probability 2 red beads and 3 black beads?

Problem 24.24
A machine produces bolts, and has a chance of 0.1 of producing a faulty bolt. Calculate the probability that in a sample of 4 bolts, there will be
(a) exactly 2 faulty bolts
(b) at least 2 faulty bolts.

Problem 24.25
Suppose that whenever the TV show ER is on, 34% of televisions are tuned to this show (this is how Neilson ratings are reported). What is the probability that in 15 randomly selected TV's at least 5 T.V.'s are tuned to ER?

Problem 24.26
Suppose that each time you take a free throw shot, you have a 25% chance of making it. If you take 15 shots, What is the probability of making exactly 5 of them?

Problem 24.27
A multiple-choice exam consists of 10 questions, and each question has 3 choices. It is assumed that for every question one, and only one of the choices is the correct answer.
(a) Find n, the number of trials, p, the probability of success, and $1-p$, the probability of failure.
(b) Find the probability of answering exactly 7 questions right.
(c) Find the probability of answering 8 or more questions right.
(d) Find the probability of answering at most one question.

Problem 24.28
A coin is loaded so that $P(H) = 3/4$ and $P(T) = 1/4$. The coin is flipped 5 times and its outcome recorded. Find the probability that heads turns up at least once.

Problem 24.29
Consider a binomial distribution with $n = 5$ trials and $p = 0.50$.
(a) Construct a binomial probability distribution.
(b) Make a histogram.
(c) Compute the mean and the standard deviation of this distribution.

Problem 24.30
June solicits contributions for charity by phone. She has a history of being successful at getting a donor on 55% of her calls. If June has a quota to get at least four donors each day, how many calls must she make to be 96.4% sure of meeting the quota?

Problem 24.31
Vince is a computer software salesman who has a history of making a successful sales call 40% of the time. If vince has a sales quota of at least five

sales this week, how many sales calls must he make to be 95% sure of meeting the quota?

Problem 24.32
30% of college students own videocassette recorders. The Telektronic Company produced a videotape and sent pilot copies to 20 college students. What is the mean and standard deviation of this sample?

Problem 24.33
Find the mean, variance, and standard deviation for the number of sixes that appear when rolling 30 dice.

Problem 24.34
A certain type of missile has failure probability 0.02. Compute the mean and standard deviation of the number of failures in 50 launches.

Problem 24.35
10% of men are bald. Compute the mean and the standard deviation of the number of bald men out of 818 men.

Problem 24.36
A notice is sent to all owners of a certain type of automobile, asking them to bring their cars to a dealer to check for the presence of a particular manufacturing defect. Suppose that only .05% of such cars have the defect. Consider a random sample of 10,000 cars. What are the expected value and standard deviation of the number of cars in the sample that have the defect?

Problem 24.37
A waiter at the Green Spot Restaurant has learned, from long experience, that the probability that a lone diner will leave a tip is only 0.50. During one lunch hour he serves 10 people who are dining by themselves. Compute the mean and standard deviation of the resulting distribution.

Problem 24.38
Old Friends Information Service is a California company that is in the business of finding addresses of long-lost friends. Old Friends claims to have a 45% success rate. Suppose that you have the names of 7 friends for whom you have no addresses and decide to use Old Friends to track them. What is the expected number of friends for whom addresses will be found? What is the standard deviation?

Problem 24.39
A recent study on prisons found that about 40% of all prison parolees become repeat offenders. Chris is a social worker whose job is to counsel people on parole. Let us say success means that a person does not become a repeat offender. Chris has been given a group of 7 parolees. What is the expected number of people who will not become repeat offenders? What is the standard deviation?

Problem 24.40
A laser production facility is known to have a 75% yield; that is, 75% of the lasers manufactured by the facility pass the quality test. Suppose that today the facility is scheduled to produce 15 scheduled to produce 15 lasers. Compute the mean, variance, and the standard deviation of this experiment.

Problem 24.41
A multiple choice test has 100 questions with 4 choices each. Find the mean and the standard deviation for the number of correct answers.

Problem 24.42
Using a certain treatment, it has been found that cancer goes into remission 30% of the time. Suppose a group of 20 cancer patients begin this treatment. Find the number of patients that are expected to go into remission and the standard deviation.

Problem 24.43
suppose that 4% of all TVs made by W&B Company in 1995 are defective. Suppose eight of these TVs are randomly selected from across the country and tested. Compute the mean, variance and the standard deviation.

25. Normal Distributions

The binomial random variable is an example of a discrete random variable. This section deals with continuous random variables and continuous probability distributions.
If a random variable can take any value in a line interval then it is called **continuous random variable.** It is the result of a measurement. For instance, the height of students in a college. The height could in theory take on any value from a low of, say 3 feet to a high of, say, 7 feet.

Example 25.1
State whether the random variables are continuous or discrete:
(a) The number of books on a library shelf.
(b) The diameter of a sphere.

Solution.
(a) Discrete randon variable since the range is a part of the set of positive integers.
(b) Continuous random variable since the range is a subset of the real numbers ∎

Normal Probability Distribution
The probability distribution of a continuous random variable is called **continuous probability distribution.** An important example of a continuous probability distribution is the **normal distribution** and its graph is called a **normal curve** or a **bell-shaped curve.** The curve has the following properties:

1. The curve is bell-shaped with the highest point over the mean μ.
2. It is symmetrical about the line through μ.
3. The curve approaches the horizontal axis but never touches or crosses it.
4. The points where the curve changes concavity occur at $\mu+\sigma$ and $\mu-\sigma$.
5. The area under the curve between any two values a and b gives the probability that a random variable having the continuous distribution will take on a value on the interval from a to b.

6. The total area under the curve is assumed to be 1.(0.5 to the left of the mean and 0.5 to the right).

A normal distribution curve is given in Figure 25.1

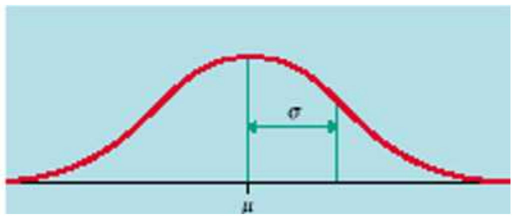


Figure 25.1

Example 25.2
Graduate Management Aptitude Test (GMAT) scores are widely used by graduate schools of business as an entrance requirement. Suppose that in one particular year, the mean score for the GMAT was 476, with a standard deviation of 107. Assuming that the GMAT scores are normally distributed, represent the following questions graphically:

(a) What is the probability that a randomly selected score from this GMAT falls between 476 and 650?
(b) What is the probability of receiving a score greater than 750?
(c) What is the probability of receiving a score of 540 or less ?
(d) What is the probability of receiving a score between 440 and 330

Solution.
The graphs are shown in Figure 25.2 ■

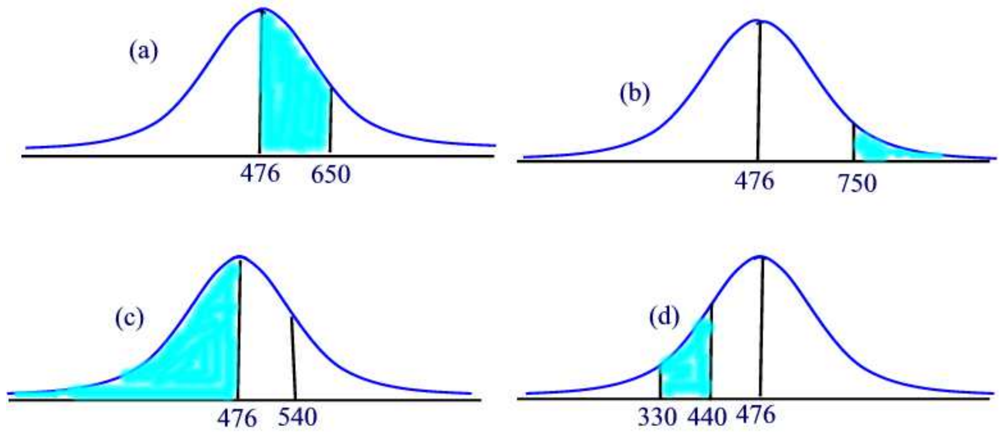


Figure 25.2

Percentage of Values in Some Commonly Used Intervals
The normal curve is more spread out when σ gets large. Now, recall that Chebyshev theorem tells us only "at least what percentage" of data that lies within k standard deviation of the mean. This result applies for any distribution. However, for normal distributions, we can get a much more precise result (See Figure 25.3)

- About 68 percent of the data values will lie within one standard deviation of the mean.
- About 95 percent of the data values will lie within two standard deviations of the mean.
- About 99.7 percent of the data values will lie within three standard deviations of the mean.

This result is sometimes referred to as the **emperical rule**, because the given percentages are observed in practice.

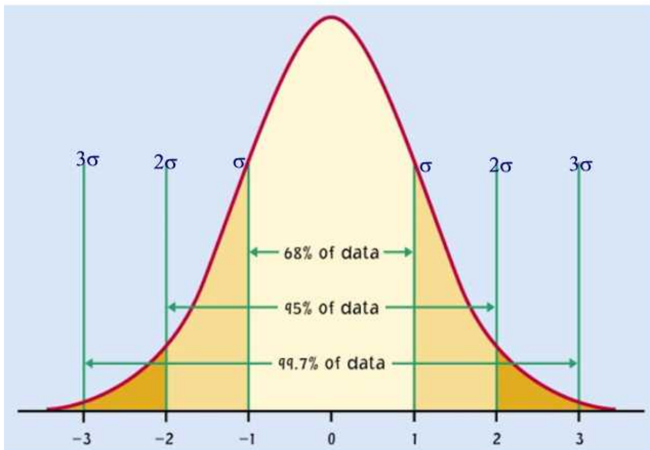


Figure 25.3

Example 25.3
The yearly wheat yield per acre on a particular farm is normally distributed with mean $\mu = 35$ bushels and standard deviation $\sigma = 8$ bushels.
(a) Shade the area under the normal curve that represents the probability that an acre will yield between 19 and 35 bushels.
(b) Is the area the same as the area between $\mu - 2\sigma$ and μ?
(c) Find the percentage of area over the interval between 19 and 35.
(d) Assume that the shaded area is 0.477, what is the probability that the yield will be between 19 and 35 bushels per acre?

Solution.
(a) The area is shown in Figure 25.4

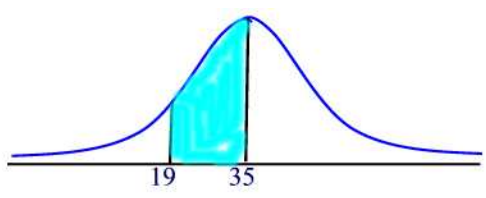


Figure 19.4

(b) We have $\mu - 2\sigma = 35 - 16 = 19$ and $\mu = 35$.
(c) By the emperical rule, the percentage of area is 0.475 which is half of 95%.
(d) The probability that the yield will be between 19 and 35 is also 0.475 ■

Areas Under the Standard Normal Curve
As we have seen, the area under the graph of a normal distribution is of great importance. In practice, we find areas under the graphs of normal curves using special tables. But it is almost impossible to construct seperate tables of normal-curve areas for all conceivable pairs of values of μ and σ. Instead, we tabulate the areas only for the normal distribution with $\mu = 0$ and $\sigma = 1$, called the **standard normal distribution** or **Z-distribution**. Table Z at the end of this section provides the area under the standard normal curve from 0 to the specified value of z.

Example 25.4
Find the area between z = 0.20 and z = 3.65.

Solution.
According to Table Z, the area between $z = 0.20$ and $z = 3.65$ is $0.4999 - 0.0793 = 0.4206$ ■

Example 25.5
(a) What is the probability that z takes values between 0 and 1.9?
(b) What is the probability that z takes values between 0 and 1.96?

Solution.
(a) $P(0 < z < 1.9) = 0.4713$.
(b) $P(0 < z < 1.96) = 0.4750$ ■

Example 25.6
Find the following probabilities and shade the corresponding areas under the standard normal curve.

(a) $P(0 < z < 1.5)$ (b) $P(z > 1.8)$
(c) $P(1.5 < z < 1.8)$ (d) $P(z < 1.8)$
(e) $P(-1.5 < z < 0)$ (f) $P(z < -1.5)$
(g) $P(-1.5 < z < 1.8)$.

Solution.
(a) $P(0 < z < 1.5) = 0.4332$.
(b) $P(z > 1.8) = 0.5 - P(0 < z \leq 1.8) = 0.5 - 0.4641 = .0359$.
(c) $P(1.5 < z < 1.8) = P(0 < z < 1.8) - P(0 < z < 1.5) = 0.4641 - 0.4332 = 0.0309$.
(d) $P(z < 1.8) = 0.5 + P(0 < z < 1.8) = 0.5 + 0.4641 = 0.9641$.
(e) $P(-1.5 < z < 0) = P(0 < z < 1.5) = 0.4332$.
(f) $P(z < -1.5) = 0.5 - P(0 < z < 1.5) = 0.5 - 0.4332 = 0.0668$.
(g) $P(-1.5 < z < 1.8) = P(0 < z < 1.5) + P(0 < z < 1.8) = 0.4332 + 0.4641 = 0.8973$ ■

Areas Under Any Normal Curve
When the normal curve is not the standard normal curve, table for the areas under the curve of the standard normal distribution can be used to find areas under the original normal curve. This can be done by performing the change of scale which converts the original measurements x, a, and b to **standard units** or **z-scores** by means of the formula

$$z = \frac{x - \mu}{\sigma}$$

where μ is the mean of the distribution and σ is the standard deviation. Thus,

$$P(a \leq x \leq b) = P(\frac{a - \mu}{\sigma} \leq z \leq \frac{b - \mu}{\sigma}).$$

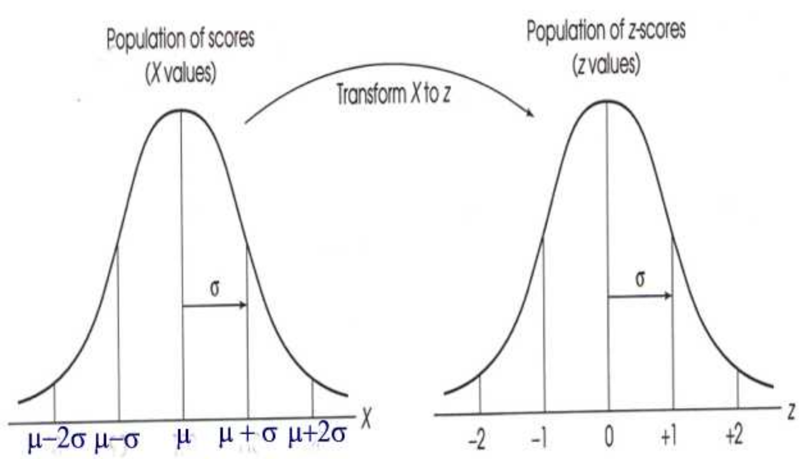


Figure 25.5

Now, if the z-score is given then the raw data can be found by solving the equation $z = \frac{x-\mu}{\sigma}$ for x to obtain

$$x = \mu + z\sigma.$$

Note that from this formula we see that the value of z tells us how many standard deviation the corresponding value of x lies above (if $z > 0$) or below (if $z < 0$) the mean of its distribution.

Example 25.7
A study was done to determine the stress levels that students have while taking exams. The stress level was found to be normally distributed with a mean stress level of 8.2 and a standard deviation of 1.34. What is the probability that at your next exam, you will have a stress level between 9 and 10?

Solution.
We want to find $P(9 < x < 10)$. Using z-scores we find

$$\begin{aligned} P(9 < x < 10) &= P\left(\frac{9-8.2}{1.34} < z < \frac{10-8.2}{1.34}\right) \\ &= P(0.60 < z < 1.34) \\ &= P(0 < z < 1.34) - P(0 < z < 0.60) \\ &= 0.9099 - 0.7257 = 0.1842. \end{aligned}$$

We conclude that there is about an 18 percent chance that the stress level will be between nine and ten ■

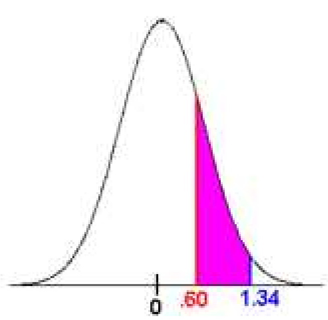


Figure 25.6

Example 25.8
Find the probability that a normal random variable with a mean of 10 and standard deviation of 20 will lie between 10 and 40.

Solution.
Applying the z-scores formula yields

$$P(10 \leq x \leq 40) = P(\frac{10-10}{20} \leq z \leq \frac{40-10}{20}) = P(0 \leq z \leq 1.5).$$

Now, using Table Z we find $P(0 \leq z \leq 1.5) = 0.4332$ ■

Example 25.9
Suppose that a national testing service gives a test in which the results are normally distributed with a mean of 400 and a standard deviation of 100. If you score a 644 on the test, what fraction of the students taking the test exceeded your score?

Solution.
Let X = a student's score on the test. Applying the z-scores formula yields

$$\begin{aligned} P(X > 644) &= P(z > \tfrac{644-400}{100}) \\ &= P(z > 2.44) \\ &= 0.5 - P(0 < z < 2.44) \\ &= 0.5 - 0.4927 = 0.0073. \end{aligned}$$

Thus only 0.73% of the students scored higher than your score of 644 ■

Normal Approximation to the Binomial Distribution
Consider the following binomial distribution problem: What is the probability that at least 70 of 100 mosquitos will be killed by a new insect spray, if the probability is 0.75 that any one of them will be killed by the spray?
The probability is given by the sum

$$P(70 \leq r \leq 100) = P(70) + P(71) + P(72) + \cdots + P(100)$$

where

$$P(r) = C_{100,r}(0.75)^r(.25)^{100-r}, 70 \leq r \leq 100.$$

Now, computing the above sum requires a tremendous amount of work. Instead, one approximates the binomial distribution by a normal distribution as follows:

(a) The mean of the normal distribution is $\mu = np = 100(0.75) = 75$.
(b) The standard deviation $\sigma = \sqrt{np(1-p)} = 4.33$.

(c) $P(70 \leq r \leq 100)$ is the area under the normal curve between 69.5. and 100.5
(d) Converting to z-scores to obtain

$$z_{69.5} = \frac{x-\mu}{\sigma} = \frac{69.5-75}{4.33} = -1.27$$

and

$$z_{100.5} = \frac{x-\mu}{\sigma} = \frac{100.5-75}{4.33} = 5.89.$$

(e) Using Table Z we find that $P(70 \leq r \leq 100) = P(-1.27 \leq z \leq 5.89) = .3980 + .4999 = 0.8979$.

Because the normal approximation is not accurate for small values of n, a good rule of thumb is to use the normal approximation only if $np > 5$ and $nq > 5$.
Remember that when using the normal distribution to approximate the binomial, we're computing the areas under bars. The bar over r goes from $r - 0.5$ to $r + 0.5$. If r is a left (resp. right) endpoint of an interval, we subtract (resp. add) 0.5 to get the corresponding normal variable. For example, $P(6 \leq r \leq 10)$ where r is a binomial variable is approximated by $P(5.5 \leq x \leq 10.5)$ where x is the corresponding normal variable.

Example 25.10
Consider a population of voters in a given state. The true proportion of voters who favor candidate A is equal to 0.40. Given a sample of 200 voters, what is the probability that more than half of the voters support candidate A?

Solution.
The mean of the distribution is equal to $200(0.4) = 80$, and the standard deviation is $\sqrt{200(0.4)(0.6)} = 6.93$. The probability that more than half of the voters in the sample support candidate A is equal to the probability that

x is greater than 100, which is equal to $1 - P(x < 100)$. Thus,

$$\begin{aligned}
P(x \geq 100) =& 1 - P(x < 100) \\
=& 1 - P(x \leq 100.5) \\
=& 1 - P(z \leq \frac{100.5 - 80}{6.93}) \\
=& 1 - P(z \leq 2.96) \\
=& 1 - (0.5 + P(0 < z < 2.96) \\
=& 1 - (0.5 + 0.4985) = 0.0015 \blacksquare
\end{aligned}$$

Example 25.11
If 10% of men are bald, what is the probability that fewer than 100 in a random sample of 818 men are bald?

Solution.
Finding the mean and standard deviation we find $\mu = np = 0.1(818) = 81.8$ and $\sigma = \sqrt{81.8(0.9)} = 8.5802$. Now, using z-score to obtain $z = \frac{100.5-81.8}{8.5802} = 2.18$. Thus, $P(x < 100) = P(x < 100.5) = P(z < 2.18) = 0.5 + P(0 < z < 2.18) = 0.5 + 0.4854 = 0.9854$ $\blacksquare$

Example 25.12
The number of heads in 100 tosses of a coin, is binomially distributed with n = 100 and p = 0.5. What is the probability that the number of heads is between 48 and 58?

Solution.
Since $np = n(1 - p) = (100)(0.5) = 50 > 5$, the normal approximation to the binomial distribution should provide a good estimate. Since $\mu = 50$ and $\sigma = \sqrt{50(0.5)} = 5$ then the probability that X will have a value between 48 and 58 is calculated as follows:

$$\begin{aligned}
P(48 < x < 58) =& P(47.5 < x < 58.5) \\
=& P(\frac{47.5 - 50}{5} < z < \frac{58.5 - 50}{5}) \\
=& P(-0.5 < z < 1.7) \\
=& P(0 < z < 0.5) + P(0 < z < 1.7) \\
=& 0.1915 + 0.4554 = 0.6469 \blacksquare
\end{aligned}$$

Practice Problems

Problem 25.1
State whether the random variables are continuous or discrete:
(a) The height of a student in your college.
(b) The number of left-handed students in each class at ATU.

Problem 25.2
Assuming that the heights of college women are normally distributed with means 65 in. and standard deviation 2.5 in. Answer the following questions:

(a) What percentage of women are taller than 65 in.?
(b) What percentage of women are shorter than 65 in.?
(c) What percentage of women are between 62.5 and 67.5 in.?
(d) What percentage of women are between 60 in and 70 in.?

Problem 25.3
What percent of the area under a normal curve lies
(a) To the right of μ?
(b) Between $\mu - 2\sigma$ and $\mu + 2\sigma$?
(c) To the right of $\mu + 3\sigma$?

Problem 25.4
Assuming that the heights of college women are normally distributed, with a mean of 64.9 inches and standard deviation of 4.3 in, what is the percentage of women that are between 64.9 in. and 69.2 in.?

Problem 25.5
Archaeological studies have used the method of tree-ring dating in an effort to determine when prehistoric people lived in a pueblo. Wood from several excavations gave a mean of (year) 1238 with a standard deviation of 23 years. The distribution of dates was more or less mound shaped and symmetrical about the mean. Use empirical rule to estimate the range of years centered about the mean in which about 95% of the data will be found.

Problem 25.6
You are the manager at a new toy store and want to determine how many Monopoly games to stock in you store. The mean number of Monopoly games that sell per month is 22 with a standard deviation of 6. Assume that this

distribution is Normal. What is the probability that next month you will sell between 10 and 34 games? If you stock 45 games, should you feel secure about not running out?

Problem 25.7
The distribution of heights of women aged 20 to 29 is approximately Normal with mean 64 inches and standard deviation 2.7 inches. Draw a Normal curve on which this mean and standard deviation are correctly located.

Problem 25.8
The distribution of heights of women aged 20 to 29 is approximately Normal with mean 64 inches and standard deviation 2.7 inches. Use the emperical rule to answer the following questions.
(a) Between what heights do the middle 95% of young women fall?
(b) What percent of young women are taller than 61.3 inches?

Problem 25.9
The length of human pregnancies from conception to birth varies according to a distribution that is approximately Normal with mean 266 days and standard deviation 16 days. Use the emperical rule to answer the following questions.
(a) Between what values do the lengths of the middle 95% of all pregnancies fall?
(b) How short are the shortest 2.5% of all pregnancies?

Problem 25.10
For the population of Canadian high school students, suppose that the number of hours of TV watched per week is normally distributed with a mean of 20 hours and a standard deviation of 4 hours. Approximately, what percentage of high school students watch
(a) between 16 and 24 hours per week?
(b) between 12 and 28 hours per week?
(c) between 8 and 32 hours per week?

Problem 25.11
Sketch the areas under the standard normal curve over the indicated intervals, and find the specified area.
(a) Between $z = 0$ and $z = 3.18$.
(b) Between $z = 0$ and $z = -2.01$.

(c) Between $z = -2.18$ and 1.34.
(d) Between $z = 0.32$ and $z = 1.92$.
(e) To the right of $z = 0$.
(f) To the right of $z = 1.52$.
(g) To the left of $z = -1.22$.

Problem 25.12
Let z be a random variable with a standard normal distribution. Find the indicated probability, and shade the corresponding area under the standard normal curve.
(a) $P(0 \leq z \leq 1.62)$.
(b) $P(-0.82 \leq z \leq 0)$.
(c) $P(-0.45 \leq z \leq 2.73)$.

Problem 25.13
Sketch the areas under the standard normal curve over the indicated intervals, and find the specified area.
(a) Between $z = 0$ and $z = 1.62$.
(b) Between $z = 0$ and $z = -1.93$
(c) Between $z = -1.20$ and 2.64.
(d) Between $z = 1.73$ and $z = 3.12$.
(e) To the right of $z = 0$.
(f) To the right of $z = 0.15$.
(g) To the left of $z = -1.32$.

Problem 25.14
Let z be a random variable with a standard normal distribution. Find the indicated probability, and shade the corresponding area under the standard normal curve.

(a) $P(0 \leq z \leq 0.54)$	(d) $P(1.73 \leq z \leq 3.12)$.
(b) $P(-2.37 \leq z \leq 0)$	(e) $P(z \geq 2.17)$.
(c) $P(-1.20 \leq z \leq 2.64)$.	(f) $P(z \leq -2.15)$.

Problem 25.15
What percent of the area under the standard normal curve falls between the following z-values?
(a) 0 and 0.67
(b) 0 and 1.64
(c) 0 and 1.960
(d) 0 and 2.57

Problem 25.16
What percent of the area under the standard normal curve falls between the following z-values?
(a) -0.97 and 0.97
(b) -0.54 and 1.82
(c) -1.95 and 2.28
(d) -2.89 and 1.59

Problem 25.17
Determine the probability for each of the following events. Sketch the associated areas.
(a) $z \leq 0$ (b) $z \leq 1$ (c) $z \geq 0$ (d) $z \geq -1$ (e) $z \leq -1$ (f) $z \geq 1$.

Problem 25.18
Determine the probability for each of the following events. Sketch the associated areas.
(a) $0 \leq z \leq 0.79$ (b) $-1.57 \leq z \leq 2.33$ (c) $z \geq 1.89$ (d) $z \geq 1.96$ (e) $z \leq -2.77$ (f) $z \geq 1$.

Problem 25.19
Determine the probability for each of the following events. Sketch the associated areas.
(a) $z \leq 1.28$ (b) $z \leq -1.96$ (c) $z \geq 1.28$ (d) $z \geq 1.96$ (e) $-1.28 \leq z \leq 1.28$ (f) $-1.96 \leq z \geq 1.96$.

Problem 25.20
Find the value of z such that 0.10 of the area under the standard normal curve lies to the right of z.

Problem 25.21
Let x have a normal distribution with $\mu = 10$ and $\sigma = 2$. Find the probability that an x value selected at random from this distribution is between 11 and 14.

Problem 25.22
Find the area under the curve between 12 and 15 for the normal distribution with $\mu = 10$ and $\sigma = 5$.

Problem 25.23
On a final examination in Statistics, the mean was 72 and the standard deviation was 15. Assuming normal distribution, determine the z-score of students receiving the grades (a) 60, (b) 93, and (c) 72.

Problem 25.24
Referring to the previous problem, find the grades corresponding to the z-score $z = 1.6$.

Problem 25.25
If $z_1 = 0.8$, $z_2 = -0.4$ and the corresponding x-values are $x_1 = 88$ and $x_2 = 64$ then find the mean and the standard deviation, assuming we have a normal distribution.

Problem 25.26
Let x have a normal distribution with $\mu = 3$ and $\sigma = 0.5$.

(a) Find the probability that an x value selected at random from this distribution is between 2.1 and 3.7 inclusive.
(b) Find the area under the curve between 2.1 and 3.7 inclusive.

Problem 25.27
A student has computed that it takes an average (mean) of 17 minutes with a standard deviation of 3 minutes to drive from home, park the car, and walk to an early morning class. Assuming normal distribution,

(a) One day it took the student 21 minutes to get to class. How many standard deviations from the average is that?
(b) Another day it took only 12 minutes for the student to get to class. What is this measurement in standard units?
(c) Another day it took him 17 minutes to be in class. What is the z-score?

Problem 25.28
Mr. Eyha's z-score on a college exam is 1.3. If the x-scores have a mean of 480 and a standard deviation of 70 points, what is his x-score?

Problem 25.29
Suppose that the volume of beer in a bottle of a certain brand is normally distributed with mean 0.5 liter and standard deviation 0.01 liter. Find the probability that a bottle will contain at least 0.48 liter.

Problem 25.30
It has been reported that the average hotel check-in time, from curbside to delivery of bags into the room, is 12.1 minutes. Mary has just left the cab that brought her to her hotel. Assuming a normal distribution with a standard deviation of 2.0 minutes, what is the probability that the time required for Mary and her bags to get to the room will be:
(a) greater than 14.1 minutes?
(b) less than 8.1 minutes?
(c) between 10.1 and 14.1 minutes?
(d) between 10.1 and 16.1 minutes?

Problem 25.31
An experiment has a binomial distribution with n=20 and p=.35. Is it appropriate to approximate the binomial distribution with the normal curve?

Problem 25.32
Find the probability of getting 6 heads and 10 tails in 16 flips of a balanced coin, using
(a) the formula for the binomial distribution;
(b) the normal approximation to the binomial distribution with $n = 16$ and $p = 0.5$.

Problem 25.33
For a binomial distribution with $n = 20$ trials and probability of success $p = 0.5$, we let $r =$ number of successes out of 20 trials.
(a) Explain why we can use a normal approximation to this binomial distribution.
(b) Use the normal approximation to estimate $P(6 \leq r \leq 12)$. How does this value compare with the corresponding probability obtained using the binomial table?

Problem 25.34
If 20 percent of the loan applications received by a bank are refused, what is the probability that among 225 loan applications at most 40 will be refused?

Problem 25.35
A decade ago high levels of lead in the blood put 84% of children at risk. Today only 6% of children in the United States are at risk of high blood-lead levels.

In a random sample of 207 children taken now, what is the probability that 19 or more have high blood-lead levels?

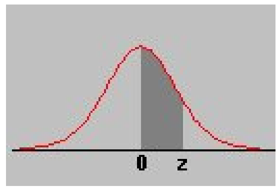
Problem 25.36

In Tampa, a local newspaper ran a full page of nearly 50 mug shots of people the police wanted to arrest for serious crimes. Within one week, the police received enough information to locate and arrest about 10% of these "wanted" people. If next month the newspaper runs a full page of 195 mug shots of fugitives, what is the probability that the police will receive enough information to locate and arrest (within one week) 26 or more fugitives?

z	0.0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4990	0.4991	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992
3.2	0.4993	0.4993	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995
3.3	0.4995	0.4995	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996
3.4	0.4996	0.4996	0.4996	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997
3.5	0.4997	0.4997	0.4997	0.4997	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.7	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.9	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
4.0	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999

Table Z

For values of z greater than or equal to 4.0, use 0.4999 to approximate the shaded area under the standard normal curve.



Answer Key

Section 1

1.1
\$112.50.

1.2
\$3,678.40.

1.3
16.2%.

1.4
$t = \frac{2}{3} = 8$ months.

1.5
\$3,000.

1.6
\$150.

1.7
18%

1.8
\$9.41.

1.9
2.149%.

1.10
\$996.16.

1.11
6.697%

1.12
\$27,027.03.

1.13
\$1,300.

1.14
6.986%.

1.15
\$1,500.

1.16
20 years.

1.17
\$3,000 should be invested at 5% and \$5,000 should be invested at 6%.

1.18
\$6,000 should be invested at 7% and \$18,000 should be invested at 11%.

1.19
\$9,000 was invested at 7% and \$6,000 at 6.5%.

1.20
69.33%

Section 2

2.1
5.1%.

2.2
5.127%.

2.3
(i) \$1,050 (ii) \$ 1,050.95 (iii) \$1,051.16 (iv) \$1,051.25 (v) \$1,051.27.

2.4
(a) 5% (b) 5.095% (c) 5.116% (d) 5.125% (e) 5.127% (f) 5.127%.

2.5
4.51 years.

2.6
19.97 years.

2.7
$103.80.

2.8
$11,240.13.

2.9
7%.

2.10
$2,283.41.

2.11
15.75 years.

2.12
Bank B.

2.13
Bank B.

2.14
13.3 years.

2.15
APY(Flagstar Bank) = 4.452%, APY(Principal Bank) = 4.459%. Principal Bank offers the greater APY.

2.16
1.95 million.

2.17
\$7,522.50.

2.18
(a) \$16,659.95 (b) \$21,472.67.

2.19
2.53%

2.20
5.7%

Section 3

3.1
$r = 0.02$ and $n = 80$.

3.2
\$33,584.46.

3.3
\$457.59.

3.4
\$6,902.95.

3.5
\$3,476.55.

3.6
\$12,931.39.

3.7
(a) \$504,504.87 (b) \$135,00 (c) \$369.504.87.

3.8
\$1,286,191.89.

3.9
\$15,100.50.

3.10
\$641.79.

3.11
\$2,078.69.

3.12
(a) \$95,094.67 per quarter (b) \$248,628.88.

3.13
(a) \$106,752.47 (b) 8.79%.

3.14
(a) \$1,200.00 (b) \$3,511.58.

3.15
\$84,895.10 from which \$24,895.40 is interest.

3.16
(a) 4.75% (b) \$189.58 per month.

3.17
\$323,967.96.

3.18
(a) \$39,664.40 (b) \$17,664.40.

3.19
3.4%

3.20
(a) \$27,437.89 (b) \$143,785.10.

Section 4

4.1
\$3,458.51.

4.2
\$586.01.

4.3
29.

4.4
2.9%

4.5
\$13,577.71.

4.6
The monthly payment is \$466.05.

Period	R	Interest portion of R	Principal portion of R	Unpaid balance
0				\$19,500
1	466.05	112.13	353.93	19,146.08
2	466.05	110.09	355.96	18,790.12
3	466.05	108.04	358.01	18,432.11

4.7
\$582.46.

4.8
(a) \$394.37 (b) 47 months.

4.9
\$35,693.18.

4.10

Rebate option is better than the 0% financing.

4.11

Period	R	Interest portion of R	Principal portion of R	Unpaid balance
0				\$5,000
1	706.29	140.00	566.29	4,433.71
2	706.29	124.14	582.15	3,851.56
3	706.29	107.84	598.45	3,253.11
4	706.29	91.09	615.20	2,637.91
5	706.29	73.86	632.43	2,005.48
6	706.29	56.15	650.14	1,355.34
7	706.29	37.95	668.34	687.00
8	706.24	19.24	687.00	0.00

4.12

Period	R	Interest portion of R	Principal portion of R	Unpaid balance
0				\$10,000
1	1,821.58	260.00	1,561.58	8,438.42
2	1,821.58	219.40	1,602.18	6,836.24
3	1,821.58	177.74	1,643.84	5,192.40
4	1,821.58	135.00	1,686.58	3,505.82
5	1,821.58	91.15	1,730.43	1,775.39
6	1,821.55	46.16	1,775.39	0.00

4.13
\$310.19.

4.14
\$82,779.47.

4.15
\$69,790.39.

4.16
$60,251.45.

4.17
$97,122.49.

4.18
(a) $R = \$414.18$ and the total interest paid $14,701.60 (b) $R = \$280.46$ and the total interest paid $32,353.60.

4.19
$98,551.

4.20
221 months.

Section 5

5.1
Inconsistent.
5.2
$x = 1, y = 2.$

5.3
Dependent system: Solution set $= \{(2t - 5, t) : t \in \mathbb{R}\}$.

5.4
$x = 4, y = 5.$

5.5
Inconsistent.

5.6
Dependent system: Solution set $= \{(\frac{5}{3}t + 5, t) : t \in \mathbb{R}\}$.

5.7
27 ounces of 25% solution and 9 ounces of 45% solution.

5.8
\$17,500 for the bonds with 4% yield and \$20,500 for the bond with yield 6%.

5.9

$$\left[\begin{array}{cc|c} -1 & 2 & -3 \\ 2 & -1 & 4 \end{array}\right]$$

5.10

$$\begin{cases} x + y = & 1 \\ 6x - 3y = & 12. \end{cases}$$

5.11
$x = 3, y = 1.$

5.12
Inconsistent.

5.13
Infinitely many solutions: Solution set $= \{(2t - 3, t) : t \in \mathbb{R}\}$.

5.14
The flight eastward is 7 hours. The difference in time zones is 6 hours.

5.15
$x = 2, y = -1.$

5.16
Infinitely many solutions: Solution set $= \{(2t - 2, t) : t \in \mathbb{R}\}$.

5.17
Inconsistent.

5.18
\$23.

5.19
She invested \$8,000 in the 7.5% account and \$4,000 in the 6% account.

5.20
15 dimes.

Section 6

6.1
(a) No, because the matrix fails condition 1 of the definition. Rows of zeros must be at the bottom of the matrix.
(b) No, because the matrix fails condition 2 of the definition. Leading entry in row 2 must be 1 and not 2.
(c) Yes. The given matrix satisfies conditions 1 - 4.

6.2

$$\left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{array}\right]$$

6.3

$$\left[\begin{array}{ccccc} 1 & 4 & -22 & 15 & 8 \\ 0 & 1 & -6 & 4 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

6.4

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \blacksquare$$

6.5

$$\left[\begin{array}{ccccc} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \blacksquare$$

6.6
$x_1 = 11t + 4, x_2 = 3t + 1, x_3 = 1 - t.$

6.7
$x_1 = -1 - t, x_2 = 1 + 3t, x_4 = -4 - 5t.$

6.8
$h \neq -12.$

6.9
$x_1 = 3, x_2 = 1, x_3 = 2.$

6.10
$x_1 = 8 + 7s, x_2 = 2 - 3s, x_3 = -5 - s.$

6.11
$x_1 = 4 - 3t, x_2 = 5 + 2t, x_3 = t, x_4 = -2.$

6.12
$x_1 = \frac{1}{9}, x_2 = \frac{10}{9}, x_3 = -\frac{7}{3}.$

6.13
$x_1 = \frac{3}{2}, x_2 = 1, x_3 = -\frac{5}{2}.$

6.14
$x_1 = 1,\ x_2 = -2,\ x_3 = 1,\ x_4 = 3.$

6.15
$x_1 = 2,\ x_2 = 1,\ x_3 = -1.$

6.16
$x_1 = 2 - 2t - 3s,\ x_2 = t,\ x_3 = 2 + s,\ x_4 = s,\ x_5 = -2.$

6.17
(knives,forks,spoons) $= (2t - 9, 49 - 3t, t)$ where $t = 5, 6, 7, \cdots$.

6.18
\$2,000 invested in savings bonds
\$4,000 invested in mutual bonds
\$4,000 invested in money market.

6.19
All possibilities:

$x_1(10ft)$	$x_2(14ft)$	$x_3(24ft)$
0	10	15
2	7	16
4	4	17
6	1	18

6.20
(a) All possibilities:

$10-\text{passenger}$	$15-\text{passenge}$	$20-\text{passenger}$
0	4	8
1	2	9
2	0	10

(b) The corporation should lease ten 20-passenger planes, no 15-passenger planes, and two 10-passenger planes.

Section 7

7.1
$a = 5, b = -3, c = 4, d = 1.$

7.2
$a = -2, b = -2, c = 0, d = 1.$

7.3

$$\begin{bmatrix} 4 & -1 \\ -1 & -6 \end{bmatrix}$$

7.4
$w = -1, x = -3, y = 0, z = 5.$

7.5
$s = 0$ and $t = 3.$

7.6
We have

$$a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} =$$
$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} =$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A.$$

7.7

$$\begin{bmatrix} 9 & 5 & 1 \\ -4 & 7 & 6 \end{bmatrix}$$

7.8
(a)

$$A(BC) = (AB)C = \begin{bmatrix} 70 & 14 \\ 235 & 56 \end{bmatrix}$$

(b)

$$A(B+C) = AB + AC = \begin{bmatrix} 16 & 7 \\ 59 & 33 \end{bmatrix}$$

7.9

$$\begin{cases} 2x_1 & - & x_2 & & & = & -1 \\ -3x_1 & + & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 3. \end{cases}$$

7.10
(a) If A is the coefficient matrix and B is the augmented matrix then

$$A = \begin{bmatrix} 2 & 3 & -4 & 1 \\ -2 & 0 & 1 & 0 \\ 3 & 2 & 0 & -4 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & -4 & 1 & 5 \\ -2 & 0 & 1 & 0 & 7 \\ 3 & 2 & 0 & -4 & 3 \end{bmatrix}$$

(b) The given system can be written in matrix form as follows

$$\begin{bmatrix} 2 & 3 & -4 & 1 \\ -2 & 0 & 1 & 0 \\ 3 & 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}.$$

7.11
$k = 1.$

7.12

$$\left\{\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & 2x_3 & = & 2 \\ 4x_1 & + & 3x_2 & + & 7x_3 & = & -1 \\ -2x_1 & + & x_2 & + & 5x_3 & = & 4. \end{array}\right.$$

7.13

$$\left[\begin{array}{cc} 9 & -4 \\ -9 & 7 \\ 5 & -2 \end{array}\right]$$

7.14
$a = 20, b = 5, c = 0, d = 4, e = 1.$

7.15

$$\left[\begin{array}{cc} 32 & -4 \\ -18 & 12 \end{array}\right]$$

7.16
(a)

$$P = \left[\begin{array}{ccc} 80 & 40 & 120 \\ 60 & 30 & 150 \end{array}\right]$$

(b)

$$F = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{3}{5} \end{array}\right]$$

(c)

$$PF = \left[\begin{array}{cc} 80 & 96 \\ 75 & 108 \end{array}\right]$$

7.17
(a) The total cost of materials is \$72,900 for model A, \$54,700 for model B, \$60,800 for model C.
(b) 3800 yd^3 of concrete, 130,000 board feet of lumber, 1,400,000 bricks, and

20,000 ft^2 of shingles are needed.
(c) The total cost for exterior materials is \$188,400.

Section 8

8.1
The given matrix is row equivalent to a matrix with a row of zeros. By Theorem 8.1, this matrix is singular.

8.2

$$\begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}$$

8.3

$$D^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

8.4
$x_1 = 5, \; x_2 = 1.$

8.5

$$A^{-1} = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

8.6

$$A^{-1} = \begin{bmatrix} -\frac{3}{5} & 2 & -\frac{11}{5} \\ \frac{3}{5} & -1 & \frac{6}{5} \\ -\frac{2}{5} & 0 & \frac{1}{5} \end{bmatrix}$$

8.7

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

8.8
$x_1 = -2, x_2 = -5, x_3 = 4.$

8.9

$$A^{-1} = \begin{bmatrix} -\frac{13}{10} & \frac{7}{5} & \frac{1}{2} \\ \frac{2}{5} & -\frac{1}{5} & 0 \\ -\frac{7}{10} & \frac{3}{5} & \frac{1}{2} \end{bmatrix}$$

8.10
The matrix does not have an inverse.

8.11
(a) 514 compact size and 286 full size (b) 714 compact size and 286 full size.

8.12
$x_1 = -3, x_2 = \frac{14}{5}, x_3 = \frac{13}{5}.$

8.13
$x_1 = 0,\ x_2 = 2, x_3 = -\frac{1}{2}.$

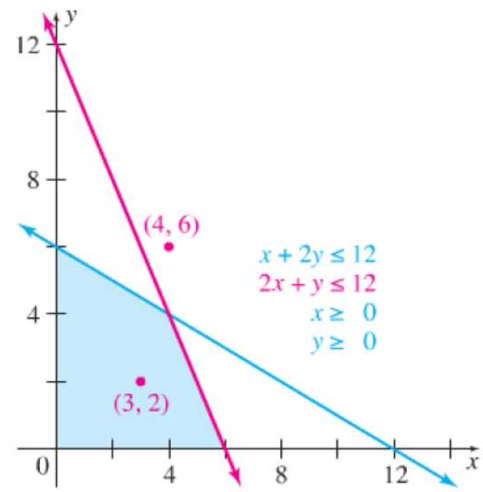
8.14
System is inconsistent.

8.15
Produce 60 model A guitars and 0 model B guitars.

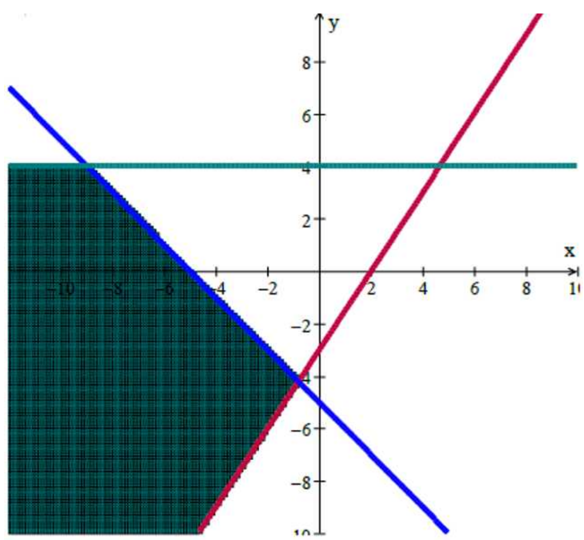
Section 9

9.1
(a) No (b) No (c) Yes.

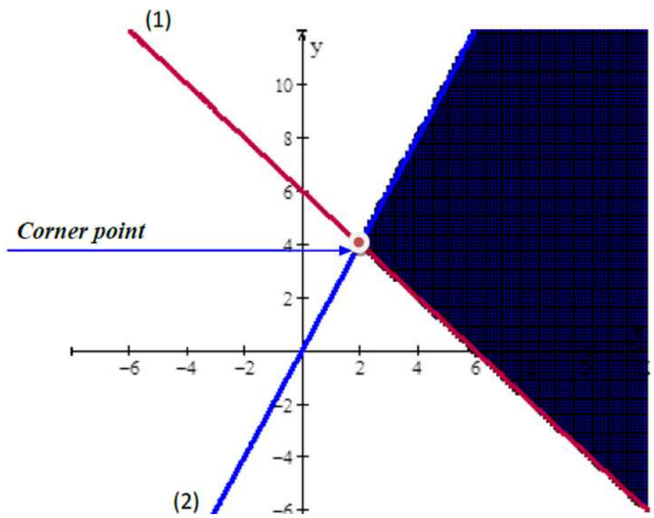
9.2


y
12
8
4
0
4
8
12
x
(4, 6)
(3, 2)
$x + 2y \le 12$
$2x + y \le 12$
$x \ge 0$
$y \ge 0$

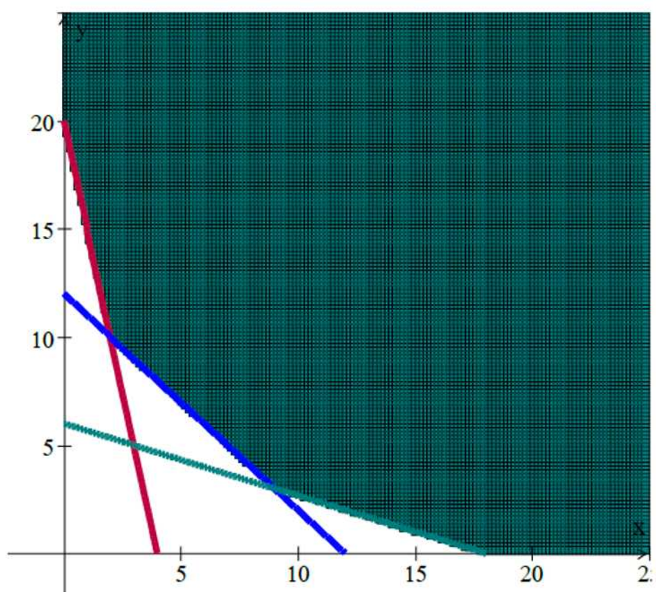
9.3


y
8
6
4
2
x
−10
−8
−6
−4
−2
2
4
6
8
−2
−4
−6
−8

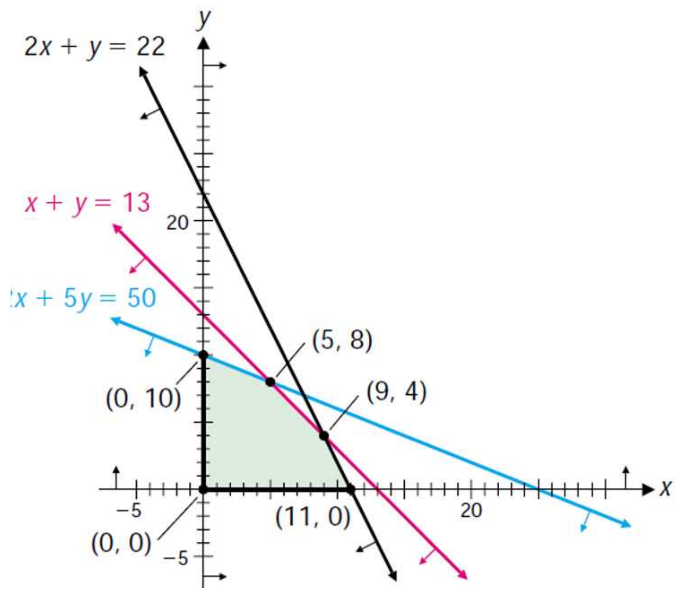
9.4



9.5

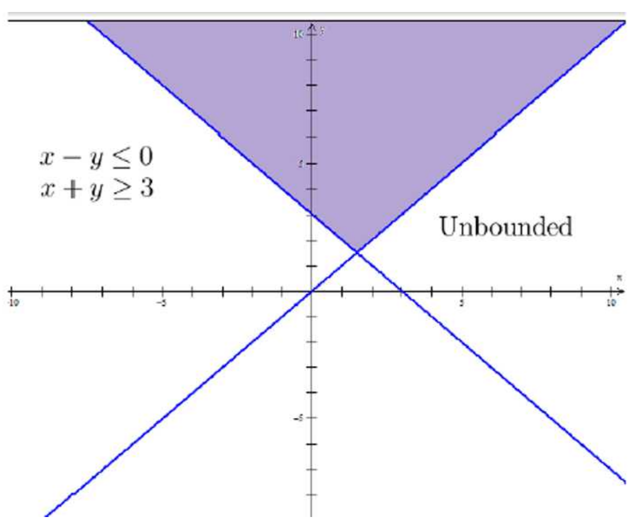


9.6



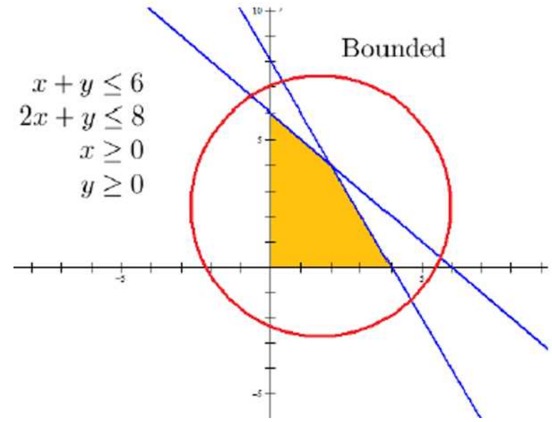
y
2x + y = 22
x + y = 13
20
2x + 5y = 50
(5, 8)
(0, 10)
(9, 4)
x
−5
20
(11, 0)
(0, 0)
−5

9.7



$x - y \leq 0$
$x + y \geq 3$
Unbounded

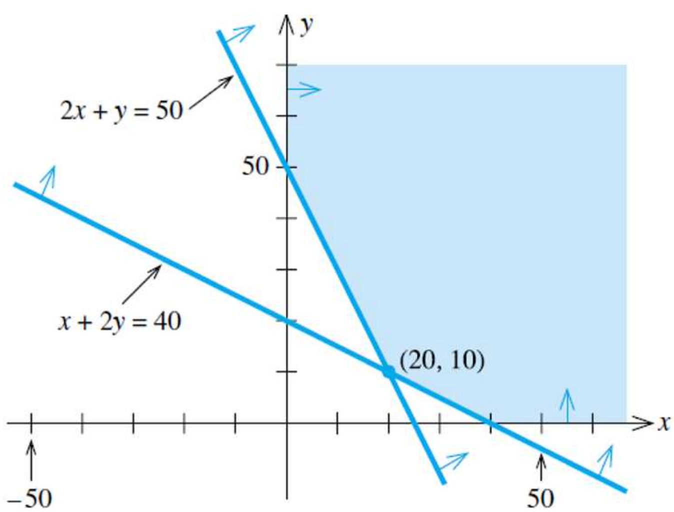
9.8



9.9
No solutions.

9.10
No solutions.

9.11



9.12
(a) - IV; (b) III; (c) I; (d) II.

9.13

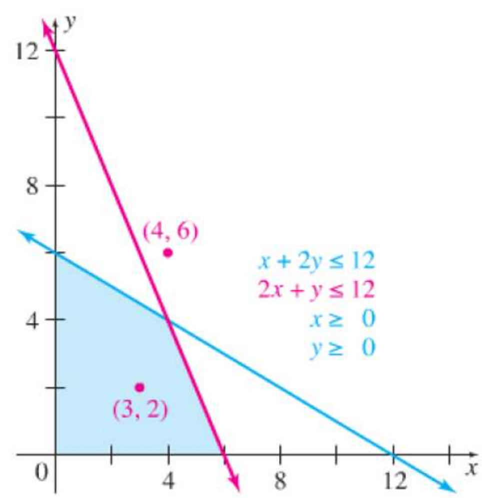
$(0, 4); (1, 4); (0.2); (2, 1).$

9.14

$$\begin{cases} 20x + 1y & \geq 460 \\ 30x + 30y & \geq 960 \\ 5x + 10y & \geq 220 \\ x, y & \geq 0. \end{cases}$$

9.15
(a) The system is

$$\begin{cases} 2x + y & \leq 12 \\ x + 2y & \leq 12 \\ x, y & \geq 0. \end{cases}$$



(b) The feasible region is

(c) 3 batches of cake cones and 2 batches of sugar cones correspond to the point $(3, 2)$. It is possible to manufacture these since the point (3, 2) is in the feasible region.
(d) 4 batches of cake cones and 6 batches of sugar cones correspond to the point $(4, 6)$. It is not possible to manufacture these since the point $(4, 6)$ is not in the feasible region.

Section 10

10.1
Maximum of 34 at $(3, 6)$ and minimum of 40 at $(2, 16)$ and $(8, 4)$.

10.2
The number of trick skis 8.75, and slalom is 11.25.

10.3
The maximum total area is $181.82 + 509.09 = 690.91$.

10.4
The minimum is $z = 30(0) + 15(18) = 270$.

10.5
The dietician should use 4 oz. of fruit and 2 oz. of nuts for a minimum of $z = 20(4) + 30(2) = 140$ calories.

10.6
Species I:$\frac{8}{7}$ units. Species II: $\frac{10}{7}$ units.

10.7
The Max. \$2 million can be achieved by investing \$5 million in Treasury bonds and 22.5 million in mutual funds Or \$10 million in Treasury bonds and 20 million in mutual funds.

10.8
A maximum profit of \$192,500,000 is obtained by growing 250,000 hectares of crop, 200,000 hectares of coffee, and 270,000 hectares of coffee.

10.9
Produce 6.4 million gal and 3.05 gal of fuel oil for a maximum revenue of \$11.5 million.

10.10
The maximum profit is \$255,000 when 300 Flexscan sets and 300 Panoramic.

10.11
The minimum value is \$2,630, 45 engines to plan I, and 32 engines to plant II.

10.12
Minimize: C = 0.2x + 0.4y
Subsject to

$$\begin{cases} 3,000x + 4,000y & \geq 36,000 \\ 1,000x + 4,000y & \geq 40,000x, y \quad 0. \end{cases}$$

8 lb of mix A, 3 lb of mix B; min $C = 0.2(8) + 0.4(3) = \$2.80$ per day.

10.13
(a)

$$x = \text{number of two-person boats produced each month}$$
$$y = \text{number of four- person boats produced each month.}$$

(b) $P = 25x + 40y$.
(c)

$$\begin{cases} 0.9x + 1.8y & \leq 864 \\ 0.8x + 1.2y & \leq 672 \\ x, y & \geq 0. \end{cases}$$

(d) 480 two-person boats and 240 four-person boats with maximum profit of \$21,600 per month.

10.14
3 units of food A and 4 units of food B will produce the minimum cost of \$1.02 per serving.

10.15
12 pigs and no goats will produce a maximum profit of \$480.

Section 11

11.1
$x_1 = 20, x_2 = 60, z = 1480$.

11.2
No optimal solution.

11.3

$x_1 = 1, x_2 = 4, z = 6.$

11.4
$x_1 = 5, x_2 = 4, x_3 = 0, z = 13.$

11.5
$x_1 = 48, x_2 = 84, x_3 = 0, z = 708.$

11.6
The maximum amount of money raised is \$1,000 per month with 6 church groups and 2 labor unions.

11.7
The maximum profit is \$104,000 and it is obtained when 1000 Royal Flush poker sets, 3000 Deluxe Diamond poker sets, and no Full House poker are assembled.

11.8
(a) The optimal solution is \$255,000 when 300 Flexscan and 300 Panoramic I sets are produced (b) $s_3 = 200$ leftover hours in the testing and packing department.

11.9
(a) When 50 loaves of raisin bread and 20 raisin cakes are baked.
(b) The maximum gross income is \$167.50.
(c) The total amount for each ingredient: Flour: 150; Sugar: 90; Raisins: 120.

11.10
40 acres of crop A, 60 acres of crop B, no crop C with maximum profit of \$17,600.

11.11
200 of Assortment I, 100 of Assortment II, 350 of Assortment III. Maximum profir: \$2,850.

11.12
The maximum profit is \$4,400 when 200 A components, zero B components and 300 C components are manufactured.

11.13
The maximum return is \$11,500 when \$50,000 is invested in government bonds, \$0 is invested in mutual bonds, \$50,000 is invested in money market funds.

11.14
maximum number of potential customers is 260,000 when 10 day-time ads, 5 prime-time-ads, and 0 late-night ads.

11.15
maximum number of interviews is 520 when 0 undergraduates, 16 graduate students, and 4 faculty members.

Section 12

12.1
$x_1 = x_2 = 2, z = 10.$

12.2
$x_1 = 1, x_2 = 0, x_3 = 4, z = 36.$

12.3
$x_1 = 25, x_2 = 50, z = 1,750,000.$

12.4
$x_1 = 6, x_2 = 10, z = 92.$

12.5
$x_1 = \frac{6}{7}, x_2 = \frac{24}{7}, z = 66.$

12.6
$x_1 = 0, x_2 = 600, x_3 = 500, x_4 = 400, z = 6,200.$

12.7
$x_1 = 1,000, x_2 = 0, x_3 = 200, x_4 = 1,600, z = 820,000.$

12.8

$x_1 = 16, x_2 = 2, x_3 = 7, z = 825.$

12.9
$x_1 = 10, x_2 = 8, x_3 = 2, z = 428.$

12.10
Plant A: Outlet I = 0, Outlet II = 600
Plant B: Outlet I = 500, Outlet II = 400.
Minimum cost is $6,200.

12.11
The minimal purchase cost is $820,000 for 1000 single-sided and 0 double-sided - Associated Electronics, 200 single-sided and 1600 double-sided - Digital Drives.

12.12
For the farmer, to meet the minimum monthly requirements at a minimal cost, should blend 16 yd^3 of A, 2 yd^3 of B, and 7 yd^3 of C; and the minimum cost is $825.00.

12.13
10 oz L, 8 oz M, 2 oz N; min cholesterol intak is 428 units.

Section 13

13.1
$\{a, b, c, d\} = \{d, b, a, c\}$ and $\{d, e, a, c\} = \{a, a, d, e, c, e\}$.

13.2
(a) B is not a subset of A since $j \in B$ but $j \notin A$.
(b) $C \subseteq A$.
(c) $C \subseteq C$.
(d) C is a proper subset of A since $C \subseteq A$ and $c \in A$ but $c \notin C$.

13.3
(a) $3 \in \{1, 2, 3\}$.
(b) $1 \not\subseteq \{1\}$.
(c) $\{2\} \subseteq \{1, 2\}$

(d) $\{3\} \in \{1, \{2\}, \{3\}\}$
(e) $1 \in \{1\}$
(f) $\{2\} \in \{1, \{2\}, \{3\}\}$
(g) $\{1\} \subseteq \{1, 2\}$
(h) $1 \notin \{\{1\}, 2\}$
(i) $\{1\} \subseteq \{1, \{2\}\}$
(j) $\{1\} \subseteq \{1\}$.

13.4
(a) $A \cup B = \{a, b, c, d, f, g\}$.
(b) $A \cap B = \{b, c\}$.
(c) $A - B = \{d, f, g\}$.
(d) $B - A = \{a\}$.

13.5
(a) True (b) False (c) False (d) False (e) True (f) False (g) True (h) True.

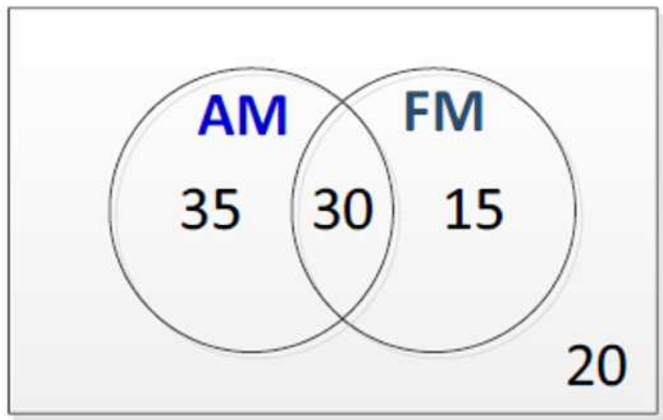
13.6
Let $A = \{1\}, B = \{2\}$, and $C = \{3\}$. Then $A \cap C = B \cap C = \emptyset$ and $A \neq B$.

13.7
(a) $A \cap B = \{2, 4, 6\}$ (b) $A \cup B = \{2, 3, 4, 5, 6, 8\}$ (c) $B^c = \{1, 7, 8, 9\}$.

13.8
(a) $\{1, 3, 9\}$ (b) $\{2.3.4.6\}$ (c) $\{1, 3, 7, 9\}$ (d) $\{1, 8\}$.

13.9
(a) $\{1, 2, 3, 4, 5, 7\}$ (b) $\{1, 3\}$ (c) $\emptyset$ (d) $\{2, 4, 6, 8, 9\}$.

13.10



13.11
(a) This set includes all applicants who are both male and employed; that is, employed male applicants.
(b) This set includes all applicants who are female (not male) or married, all married female applicants are in this set.
(c) This set includes all applicants who are female and married, this is the set of all married female applicants.

13.12
(a) 44 (b) 35 (c) 9.

13.13
(a) 12 (b) 35 (c) 5.

13.14
(a) 450 (b) 50 (c) 150 (d) 200 (e) 200.

13.15
(a) 12 (b) 13 (c) 32 (d) 3 (e) 15.

Section 14

14.1
17,576,000.

14.2
4536.

14.3
(a) 17,576 different possibilities (b) 15,600 possibilties.

14.4
3125

14.5
14 words.

14.6
120 possibilities.

14.7
6 possibilities.

14.8
(a) 9 (b) 18.

14.9
12 outcomes.

14.10
13 choices.

14.11
(a) 8 (b) 28 (c) 28 (d) 39.

14.12
35,052,000.

14.13
(a) 15 (b) 30 (c) 16 (d) 4 (e) 9.

14.14

65.

14.15
14.

Section 15

15.1
(a) 288 (b) 479001600 (c) 24 (d) 18 (e) 336 (f) 40320.

15.2
(a) 42 (b) 40320 (c) 600.

15.3
(a) 456976 (b) 358800.

15.4
(a) 15600000 (b) 11232000.

15.5
(a) 64000 (b) 59280.

15.6
(a) 479001600 (b) 604800.

15.7
(a) 5 (b) 20 (c) 60 (d) 120

15.8
20 different ways.

15.9
(a) 362880 (b) 15600.

15.10
(a) 21 (b) 1 (c) 300.

15.11

11480.

15.12
300 different handshakes.

15.13
10.

15.14
28 different ways.

15.15
(a) 720 different ways (b) 30 ways (c) 20 different ways.

15.16
4060 different ways

15.17
125970 different ways.

Section 16

16.1
$S = \{HH, HT, TH, TT\}$.

16.2
(a) $\{TTH, HTT, THT\}$
(b) $\{TTT, TTH, HTT, THT\}$
(c) $\{TTH, HTT, THT, THH, HTH, HHT, HHH\}$.

16.3
(a) $A = \{3, 4, 5, 6\}$. $P(A) = \frac{4}{6} = \frac{2}{3}$
(b) $B = \{TTH, HTT, THT\}$. $P(B) = \frac{3}{8}$
(c) $C = \emptyset$. $P(C) = 0$.

16.4
(a) $S = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$

(b) $E^c = \{(1,3),(1,4),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),$ $(4,2),(4,3),(4,4)\}$
(c) $P(\{(2,4),(3,3),(3,4),(4,2),(4,3),(4,4)\}) = \frac{6}{16} = \frac{3}{8}$. The probability that the total score is less than 6 is $1 - \frac{3}{8} = \frac{5}{8}$
(d) $\frac{12}{16} = \frac{3}{4}$
(e) $P(\{(1,2),(1,3),(1,4),(2,1),(2,3),(3,1),(3,2),(4,1)\}) = \frac{8}{16} = \frac{1}{2}$.

16.5
(a) 40 (b) 0.8 (c) 0.6

16.6
(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$.

16.7
(a) $\frac{1}{8}$ (b) $\frac{3}{8}$ (c) $\frac{4}{8} = \frac{1}{2}$ (d) $\frac{7}{8}$

16.8
(a) The sample space consists of the 12 months of the year.
(b) $A = \{January, June, July\}$
(c) $B = \{June, July\}$
(d) $C = \{March, May, November\}$.

16.9
(a) $A = \{2,4,6,8,10,12,14,16,18,20,22,24\}$ so that $P(A) = \frac{12}{25}$
(b) $B = \emptyset$ so that $P(B) = 0$
(c) $C = S$ so that $P(C) = 1$
(d) $D = \{2,3,5,7,11,13,17,19,23\}$ so that $P(D) = \frac{9}{25}$
(e) $E = \{2\}$ so that $P(E) = \frac{1}{25}$.

16.10
(a) Sincce $P(\{face\ card\}) = \frac{12}{52} = \frac{3}{13}$ and $P(\{not\ a\ face\ card\}) = \frac{40}{52} = \frac{10}{13}$, the outcomes are not equally likely.
(b) Since $P(club) = P(diamond) = P(heart) = P(spade) = \frac{1}{4}$, the outcomes are equally likely.
(c) Since $P(black) = P(red) = \frac{1}{2}$, the outcomes are equally likely.
(d) Since $P(King) = \frac{4}{52} = \frac{1}{13}$ and $P(even\ card) = \frac{20}{52} = \frac{5}{13}$, the outcomes are not equally likely.

16.11
(a) $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
(b) $A = \{0, 1, 2, 3, 4\}$
(c) $B = \{1, 3, 5, 7, 9\}$
(d) $C = \{0, 1, 3, 4, 5, 6, 7, 8, 9\}$
(e) $P(A) = \frac{5}{10} = \frac{1}{2}, P(B) = \frac{1}{2}, P(C) = \frac{9}{10}$.

16.12
(a) Since there are five vowels then the probability is $\frac{5}{26}$ (b) $\frac{21}{26}$.

16.13
(a) The factors of 35 in the figure are 1, 5, and 7 so that the probability is $\frac{3}{8}$
(b) The multiples of 3 in the figure are 3 and 6 so that the probability is $\frac{2}{8} = \frac{1}{4}$
(c) The outcomes of this event are 2,4,6,8 so that the probability is $\frac{4}{8} = \frac{1}{2}$
(d) $P(11) = 0$
(e) The composite numbers, i.e, not prime numbers, are 4,6,8 so that the probability is $\frac{3}{8}$
(f) The only number that is neither prime nor composite is 1 so that the probability is $\frac{1}{8}$.

16.14
(a) $S = \{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT\}$
(b) $E = \{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT\}$
(c) $F = \{HHHH, HHHT, HHTH, HTHH, THHH\}$.

16.15
(a) Possible (b) impossible (c) impossible.

16.16
(a) $E = \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)\}$ so $P(E) = \frac{18}{36} = \frac{1}{2}$
(b) If F is event that the sum is 10 then $F = \{(4, 6), (6, 4), (5, 5)\}$ so that the probability that the sum is not 10 is $1 - \frac{3}{36} = 1 - \frac{1}{12} = \frac{11}{12}$
(c) $G = \{(1, 1), (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4), (4, 1), (4, 3), (5, 2), (5, 6), (6, 1), (6, 5)\}$ so that $P(G) = \frac{15}{36} = \frac{5}{12}$
(d) $H = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1),$

$(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(5,1),(5,2),(5,3),(6,1),(6,2)\}$
so that $P(H) = \frac{26}{36} = \frac{13}{18}$
(e) $1 - \frac{13}{18} = \frac{5}{18}$.

16.17
(a) $\frac{3}{8}$ (b) $\frac{3}{6} = \frac{1}{2}$.

16.18
85.

16.19
30%

16.20
11.1

16.21
0.0069.

16.22
(a) 0.014 (b) 0.333.

16.23
(a) 0.012 (b) $\frac{C(17,4)}{C(27,4)}$.

16.24
(a) 1680 (b) $\frac{3}{56}$.

16.25
(a) $\frac{6000}{17576000}$ (b) $\frac{1000}{17576000}$.

Section 17

17.1
(a) Not mutually exclusive since a driver can get a ticket for speeding and going through a red light.
(b) Mutually exclusive since the President of the Unites States has to be born in the US.

17.2
(a) $P(A^c) = 1 - 0.22 = 0.78$
(b) $P(A \cup B) = P(A) + P(B) = 0.22 + 0.35 = 0.57$
(c) $P(A \cap B) = 0.$

17.3
0.32.

17.4
No since there is a card that is spade and ace. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$.

17.5
$\frac{5}{9}$.

17.6
Since $P(A \cap B) \leq 1$, we have $-P(A \cap B) \geq -1$. Add $P(A) + P(B)$ to both sides to obtain $P(A) + P(B) - P(A \cap B) \geq P(A) + P(B) - 1$. But the left hand side is just $P(A \cup B)$.

17.7
(a) Because the bag contains a total of 11 tees and 2 tees are red then $P(R) = \frac{2}{11}$
(b) $P(not\ red) = \frac{9}{11}$
(c) Since the two events are mutually exclusive, $P(R \cup B) = P(R) + P(B) = \frac{2}{11} + \frac{4}{11} = \frac{6}{11}$.

17.8
Since $E = \{(1,1),(1,3),(1,5),(2,2),(2,4),(2,6),(3,1),(3,3),(3,5),(4,2),(4,4),(4,6),(5,1),(5,3),(5,5),(6,2),(6,4),(6,6)\}$ we have $P(E) = \frac{18}{36} = \frac{1}{2}$. Since $P = \{(1,1),(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),$
$(3,4),(4,1),(4,3),(5,2),(5,6),(6,1),(6,5)\}$ we have $P(P) = \frac{15}{36} = \frac{5}{12}$. Also, $P(E \cap P) = \frac{1}{36}$ so that

$$P(E \cup P) = P(E) + P(P) - P(E \cap P) = \frac{18}{36} + \frac{15}{36} - \frac{1}{36} = \frac{32}{36} = \frac{8}{9} \blacksquare$$

17.9
If A and B are mutually exclusive then $P(A \cap B) = 0$ so that $P(A \cup B) = P(A) + P(B) = 0.8 + 0.9 = 1.7$. But $P(A \cup B)$ is a probability and so it can not exceed 1. Hence, A and B can not be mutually exclusive.

17.10
15 : 1.

17.11
$\frac{5}{8}$.

17.12
1:1.

17.13
2:3.

17.14
$\frac{1}{27}$.

17.15
(a) 5:1 (b) 1:1 (c) 1 : 0 (d) 0 : 1.

17.16
1:3.

17.17
(a) $\frac{3}{7}$ (b) $\frac{3}{10}$.

Section 18

18.1
(a) Independent (b) Dependent.

18.2
0.25.

18.3

$P(A|B) = P(B|A) = \frac{2}{3}$.

18.4
$\frac{1}{3}$.

18.5
$\frac{1}{6}$.

18.6
0.25.

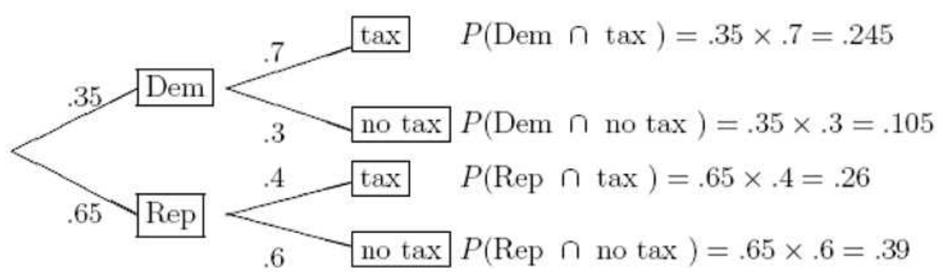
18.7
0.3.

18.8
(a) $\frac{2}{7}$ (b) $\frac{5}{7}$ (c) $\frac{4}{7}$.

18.9
A and B are depedent. A and C, B and C are independent.

18.10
(a) Independent (b) Dependent.

18.11
Figure below shows a tree diagram for this problem.



The first and third branches correspond to favoring the tax. We add their probabilities :$P(\text{tax}) = 0.245 + 0.26 = 0.505$.

Section 19

19.1
(a) 0.33 (b) 0.14 (c) 0.53.

19.2
0.53.

19.3
0.26.

19.4
0.112.

19.5
0.632.

19.6
0.5.

19.7
0.375.

19.8
0.745.

19.9
0.34.

19.10
0.86 and 0.50.

19.11
0.91; 0.545; 0.364.

19.12
0.73; 0.23; 0.05.

19.13

0.667; 0.000412.

19.14
0.88; 0.92.

19.15
(a) 0.25 (b) 0.75 (c) 0.33 (d) 0.67.

Section 20

20.1
3.72.

20.2
(a)

x	0	1	2
$p(x)$ 0.632	0.337	0.032	

(b) 0.4.

20.3
The player will lose an average of about 17¢ per game.

20.4
−\$10.

20.5

x	0	1	2
$p(x)$ 0.3571	0.5357	0.1071	

(b) 0.75.

20.6

x	−10	40	90	190
$p(x)$ 0.983	0.01	0.005	0.002	

(b) $-\$8.60$.

20.7
-50¢The game is not fair.

20.8
$-\$0.0036$; $\$0.0036$.

20.9
Win \$1.

20.10
0.002.

20.11
$E(X) = -80$¢.

20.12
$-\$100$.

20.13
1.

20.14
-50¢.

Section 21

21.1

Index