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74 Functions of Two Variables

In many real world problems one encounters a quantity that depends on more than one input. For example, if an amount of money $P$ is invested at a simple annual interest rate $r$ for a period of $t$ years then the balance at the end of $t$ years is given by the formula $B = A(1 + r)^t$. Thus, $B$ can be regarded as a function of three variables $A, r,$ and $t$. In function notation we will write

$$B = f(A, r, t).$$

Multivariable calculus is the study of functions of more than one variable. In this course we will mainly focus on functions having two or three variables. However, functions of four, five, or more variables do occur in models of the physical world and the results presented in the course also apply to such functions.

The purpose of this section is to make you familiar with functions in two variables. You will learn how to (1) represent a function of two variables in terms a formula, (2) represent a function in two variables by a table, and (3) represent a function in two variables graphically.

First, we introduce the definition of a function of two variables: A scalar-valued function of two real variables $x$ and $y$ is a rule, $f$, that associates with each choice of $x$ and $y$ a single real number $f(x, y)$ called the value of $f$ at $(x, y)$. The set $\text{Dom}(f) = \{(x, y) : f(x, y) \text{ exists}\}$ is called the domain of $f$. The range of $f$ is the set of all possible values of $f(x, y)$ for each $(x, y)$ in the domain of $f$.

**Example 74.1**

Consider the function $f(x, y) = \sqrt{1 - x^2 - y^2}$.

(a) Find $f(\frac{1}{2}, \frac{1}{2})$ and $f(2, 1)$.

(b) Find the domain and the range of the function $f$.

**Solution.**

(a) We have $f(\frac{1}{2}, \frac{1}{2}) = \sqrt{1 - \frac{1}{4} - \frac{1}{4}} = \sqrt{\frac{1}{2}}$. But $f(2, 1) = \sqrt{1 - 4 - 1} = \sqrt{-4}$ which is an imaginary number. This means that $(2, 1)$ is not in the domain of $f$.

(b) The domain consists of all points $(x, y)$ that satisfy the inequality $x^2 + y^2 \leq 1$. That is, the domain is the disk centered at the origin and with radius 1. The range is the closed interval $0 \leq z \leq 1$. □
Example 74.2
Find the domain and range of the function \( f(x, y) = \frac{1}{\sqrt{x-y}} \).

Solution.
The domain is \( \{(x, y) \mid y < x\} \) because of the square root in the denominator and the range is \( \{z \mid z > 0\} \).

Example 74.3
Find the domain and range of the function \( f(x, y) = \sqrt{\frac{x}{y}} \).

Solution.
The domain is \( \{(x, y) \mid xy \geq 0 \text{ and } y \neq 0\} \) and the range is \( \{z \mid z \geq 0\} \).

Representing a function by a formula
A car rental company charges $50 a day and 15 cents a mile for its cars. If \( C \) is the total cost of renting a car then \( C \) is a function of two variables, namely, the number of days, \( d \), and the number of miles driven, \( m \). As in the case of a function of one variable, we will represent the cost function in the form \( C(d, m) \) instead of just writing \( C \). This way of representing \( C \) helps in interpreting notations such as \( C(5, 300) \). The cost for renting a car for 5 days and driving it for 300 miles is \( C(5, 300) \).

According to the verbal description of the cost function, a formula of \( C \) would be

\[
C(d, m) = 40d + 0.15m.
\]

Thus, the cost of renting a car for three days and driving it for 104 miles is given by

\[
C(3, 104) = 40(3) + (0.12)(104) = 132.48.
\]

In the function notation \( C(d, m) \), \( d \) and \( m \) are called the independent variables and \( C \) is called the dependent variable since it depends on both \( d \) and \( m \).

Numerical Representation
The temperature adjusted for wind-chill is a temperature which tells you how cold it feels, as a result of the combination of wind and temperature. (See table below) Thus, the temperature adjusted for wind-chill is a function of wind speed and temperature.
Thus, if the temperature is $0^\circ F$ and the wind speed is 15 mph, then the wind-chill is $-31^\circ F$.

**Example 74.4**

Using the table above answer the following questions.

(a) If the temperature is $35^\circ F$, what wind speed makes it feel like $22^\circ F$?
(b) If the wind is blowing at 15 mph, what temperature feels like $0^\circ F$?

**Solution.**

(a) According to the table, a wind of 10 mph.
(b) At 15 mph, when the temperature drops from $25^\circ F$ to $20^\circ F$ the wind-chill drops from $2^\circ$ to $-5^\circ$. That is, for a decrease of one degree in temperature there is a decrease of $\left(\frac{2}{5}\right)^\circ F = 1.4^\circ F$ in the wind-chill. Thus, approximately $23.5^\circ F$ feels like $0^\circ F$.

**Graphical Representation**

Graphs of functions in two variables will be discussed in details in the next section. For the time being, we introduce the three dimensional space and learn how to locate points in this system as well as finding the distance between any two points.

**The Three Dimensional Coordinate System**

A three dimensional coordinate system consists of three rectangular axes that intersect at one point called the **origin**. The location of a point $P$ in three dimensional space may be specified by an ordered set of numbers $(x, y, z)$. The coordinate system is illustrated in Figure 74.1.
Remark 74.1
A useful way to visualize a three dimensional system is in terms of a room. The origin is a corner at floor level where two walls meet the floor. The $z$–axis is the vertical intersection of the two walls; the $x$– and $y$–axes are the intersections of each wall with the floor. Points with negative coordinates lie behind the wall in the next room or below the floor. The walls are known as the coordinate planes. Thus, we have three coordinate planes, namely, the $xy$–, $xz$–, and $yz$– coordinate planes. See Figure 74.2. The coordinate planes divide the three dimensional space into eight regions called octants. Thus, Figure 74.1 shows the first octant.

Example 74.5
Plot the points $P(0, 3, 5), Q(5, -5, 7)$, and $R(0, 10, 0)$.

Solution.
The three points are shown in Figure 74.3.
Note that if a point has one of its coordinates equal to 0, it lies in one of the coordinate planes. Thus, a point in the $xy$–plane has coordinates $(x, y, 0)$. If a point has two of its coordinates equal to 0, it lies on one of the coordinate axes. For example, the point $(0, y, 0)$ is a point on the $y$-axis.

**Example 74.6**

(a) You are two units below the $xy$–plane and in the $yz$–plane. What are your coordinates?

(b) You are standing at the point $(4, 5, 2)$, looking at the point $(0, 5, 3)$. Are you looking up or down?

**Solution.**

(a) You are located at the point $(0, y, -2)$ where $y$ is an arbitrary number.

(b) Plotting the two points you will find out that you are looking up.

**Example 74.7**

What does the graph of the equation $z = 4$ look like?

**Solution.**

All points with $z$-coordinate equals to 4 lie on a plane above and parallel to
the $xy$–plane, as shown in Figure 74.4.

**Distance Between Two points**

Consider the two points $P(a, b, c)$ and $G(x, y, z)$. We want to find the distance between these two points. See Figure 74.5. Applying the Pythagorean formula twice we find

$$|PG|^2 = |PF|^2 + |FG|^2 = |PE|^2 + |EF|^2 + |FG|^2$$

$$= (x - a)^2 + (y - b)^2 + (z - c)^2$$

Thus,

$$d(P, G) = \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2}. $$
Example 74.8
Find the equation of the surface of a sphere centered at $(4, -3, 2)$ and with radius 5.

Solution.
The distance between any point $(x, y, z)$ on the sphere and the center is given by

$$(x - 4)^2 + (y + 3)^2 + (z - 2)^2 = 25.$$ 

It is important to understand that this equation represents only the surface of the sphere. The solid ball enclosed by the sphere is given by

$$(x - 4)^2 + (y + 3)^2 + (z - 2)^2 \leq 25.$$
75 Graphs of Functions of Two Variables

If you recall that the graph of a function \( f \) of one variable \( x \) is the set of all points \((x, y)\) in the two dimensional plane such that \( y = f(x) \) and \( x \) in the domain of \( f \). That is, the graph is a curve in the 2-D system. In a similar way, the graph of a function \( f \) of two variables \( x \) and \( y \) is the set of all ordered points \((x, y, z)\) such that \( z = f(x, y) \) and \((x, y)\) is in the domain of \( f \). The graph is a surface in the 3-D space.

Example 75.1
Sketch the graph of \( z = f(x, y) = \sqrt{1 - x^2 - y^2} \).

Solution.
Using a graphic tool such as a computer we find the surface shown in Figure 75.1. This surface is the upper half of the sphere centered at the origin and with radius 1.

![Figure 75.1](image)

As you can see, graphing functions in space manually is quite difficult and is not an easy matter. You will not need to do this. We usually graph function of two variables using a graphical device such as a computer or a graphing calculator.

Example 75.2
Sketch the graph of \( z = f(x, y) = x^2 + y^2 \).

Solution.
The graph is the surface shown in Figure 75.2. This surface is called a
paraboloid and is a bowl-shaped surface

Example 75.3
Sketch the graph of \( z = f(x, y) = 6 - x^2 - y^2 \).

Solution.
First note that \( z = 6 - (x^2 + y^2) \). The graph of this function is a reflection of the previous paraboloid about the \( xy \)-plane followed by a vertical translation 6 units up along the \( z \)-axis as shown in Figure 75.3.
Example 75.4
Sketch the graph of $z = f(x, y) = 12 - 3x - 4y$.

Solution.
The graph is a plane shown in Figure 75.4. A function of the form $z = f(x, y) = c + ax + by$ is called a linear function. The graph is a plane in the 3-D space. To graph the plane one usually finds the intersection points with the three axes and then graphs the triangle that connects those three points. This triangle will be a portion of the plane and it will give us a fairly decent idea on what the plane itself should look like. A more discussion of linear functions of two variables will be covered in Section 77.

Cross-Sections
A very useful way to describe a function of two variables is to generate cross sections of the function by fixing one of the variables and varying the other. By holding one of the variables fixed, we effectively reduce the function from two to a single variable. The resulting graph is a curve that represents a ”slice” through the graph of the function at the location of the fixed variable.

Example 75.5
Describe the cross-sections of $z = f(x, y) = x^2 + y^2$ with $y$ fixed and then with $x$ fixed.
Solution.
The cross-sections with $y$ fixed at $y = b$ are given by

$$z = f(x, b) = x^2 + b^2.$$  

These are parabolas opening upward, located in the plane parallel to the $xz$-plane at $y = b$, and with a minimum at $z = b^2$. Similar argument for the cross-sections at $x = a$. See Figure 75.5.

![Figure 75.5](image)

**Functions with Missing Variables: Cylinders**

A cylinder is a surface traced out by translation of a plane curve along a straight line in space. For example, the right circular cylinder $x^2 + y^2 = 1$ shown in Figure 75.6(a) is the translation of the circle centered at the origin and with radius 1 in the $xy$-plane along the $z$-axis. A similar argument for the cylinder $y^2 + z^2 = 1$ shown in Figure 75.6(b).
Notice that the variable \( z \) is missing. In fact, when a variable is missing, the cylinder is obtained by moving a plane curve along a line parallel to the axis of the missing variable.

**Example 75.6**
Use a computer or a graphing calculator to graph the cylinder \( z = f(x, y) = x^2 \).

**Solution.**
The graph is shown in Figure 75.7.
Graphs provide one way of visualizing functions of two variables. Another important way of visualizing such functions is by drawing their contour diagrams.

Given a function of two variables \( z = f(x, y) \). The cross-section between the surface and a horizontal plane is called a **level curve** or a **contour curve**. Thus, level curves have algebraic equations of the form \( f(x, y) = k \) for all possible values of \( k \). A **contour diagram** or **contour map** of a function \( f(x, y) \) is a 2-dimensional graph showing several level curves in the \( xy \)-plane corresponding to several values of \( k \).

**Example 76.1**

Draw a contour diagram of \( z = f(x, y) = \sqrt{x^2 + y^2} \) showing several level curves.

**Solution.**

The surface representing the given function is a cone centered at the origin as shown in Figure 76.1(a). Horizontal planes crossing this surface trace circles in these planes. Thus, the level curves are circles centered at the origin in the \( xy \)-plane. Figure 76.1(b) shows level curves for \( k = 0, 1, 2, 3, 4, 5 \).

**Remark 76.1**

One can create a contour diagram from a surface and vice versa. If the surface is given, then we create contour diagrams by joining all the points at
the same height on the surface and dropping the curve into the $xy-$ plane. On the other hand, if the contour diagram is given with a constant value assigned to each contour curve, we obtain the surface by lifting each contour curve to a height equal to its assigned value.

**Example 76.2**
Find equations for the level curves of $f(x, y) = x^2 - y^2$ and draw a contour diagram for $f$.

**Solution.**
The level curves are the curves of the form $x^2 - y^2 = k$, for all possible values of $k$. For $k = 0$ we find the two perpendicular lines $y = -x$ and $y = x$. For $k \neq 0$ the graphs of $x^2 - y^2 = k$ are hyperbolas with asymptotes consisting of the lines $y = \pm x$. Figure 76.2 shows the contour diagram.

![Contour Diagram](image)

**Figure 76.2**

**Example 76.3**
Draw a contour diagram of the function $f(x, y) = 2x + 3y + 6$ showing several level curves.

**Solution.**
The contour diagram is a set of parallel lines of common slope $-\frac{2}{3}$.
Example 76.4
Find the geometrical shape of the level curves of a function $z = f(x, y)$ defined by the table of data

<table>
<thead>
<tr>
<th>y \ x</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Solution.
We notice from the table that $z = x^2 + y^2$. Thus, the contour diagram consists of circles centered at the origin. See Figure 76.1(b) ■

Cobb-Douglas Production Model
The Cobb-Douglas model was developed in 1928 to model the production of the US economy. Using government estimates of $P$ (yearly production), $K$ (capital investment), and $L$ (the total labor force), they found the following relationship between these quantities:

$$P = 1.01L^{0.75}K^{0.25}$$

A contour plot of this function is shown in Figure 76.4 below.

The above contour diagram shows that reducing labor increases capital investment in order to maintain the same level of production i.e. supposedly more investment is need in machinery to make up for the lack of available
labor. Conversely, increasing labor reduces capital investment since supposedly the work is being done by the labor force.

**Applications: Topographic Maps**

A topographic map, simply put, is a two-dimensional representation of a portion of the three-dimensional surface of the earth. Such maps consist of contour lines that indicate elevations. That is, contour lines are lines drawn on a map connecting points of equal elevation. If you walk along a contour line you neither gain or lose elevation.

**Example 76.5**

Using the topographic map below estimate the elevation of each of the 5 points $A - E$. (Assume elevations are given in feet)

![Figure 76.5](image)

**Solution.**

Point $A$ sits right on the 0 ft contour line. So the elevation of $A$ is 0 ft. That of $B$ is 10 ft.

Point $C$ does not sit directly on a contour line so we can not determine the elevation precisely. We do know that point $C$ is between the 10ft and 20 ft contour lines so its elevation must be greater than 10 ft and less than 20 ft. Because point $C$ is midway between these contour lines we can estimate the elevation is about 15 feet (Note this assumes that the slope is constant between the two contour lines, this may not be the case).

We are even less sure of the elevation of point $D$ than point $C$. Point $D$ is inside the 20 ft. contour line indicating its elevation is above 20 ft. Its elevation has to be less than 30 ft. because there is no 30 ft. contour line shown. But how much less? There is no way to tell. The elevation could be 21 ft, or it could be 29 ft. There is now way to tell from the map. We will
pick 25 ft for $D$.

Just as with point $C$ above, we need to estimate the elevation of point $E$ somewhere between the 0 ft and 10 ft contour lines it lies in between. Because this point is closer to the 10 ft line than the 0 ft line we estimate an elevation closer to 10. In this case 8 ft. seems reasonable. Again this estimation makes the assumption of a constant slope between these two contour lines.

Other practical uses of contours are on weather maps where lines of constant temperature (call isotherms, iso=equal and therm=heat) or lines of constant pressure (called isobars) are often drawn.
77 Linear Functions of Two Variables

Linear functions are central to single variable calculus; they are equally important in multivariable calculus.

We have already encountered linear functions in Example 75.3. A linear function in two variables is any function of the form \( f(x, y) = ax + by + c \), where \( a, b, c \) are constants. The graph of a linear function is a plane.

For a linear function in one variable, the graph is a line with a constant slope. For a linear function in two variables, the situation is different. To elaborate, if a plane is tilted then the slope depends on the direction in which we walk. For example, Figure 77.1 shows two directions one can take. In each direction the slope is the same but the two slopes are different.

![Figure 77.1](image)

Thus, if we walk parallel to the \( x \)-axis, we always find ourselves walking up or down with the same slope (see Figure 77.2); the same is true if we walk parallel to the \( y \)-axis.
From this we conclude that the slope in the $x$–direction is $\frac{\Delta z}{\Delta y} = a$ and the slope in the $y$–direction is $\frac{\Delta z}{\Delta z} = b$.

**Example 77.1**

A plane cuts the $z$–axis at $z = 5$, has slope 2 in the $x$ direction and slope $-1$ in the $y$ direction. What is the equation of the plane?

**Solution.**

We have $c = 5, a = 2$, and $b = -1$. Thus, the equation of the plane is $z = 2x - y + 5$ ■

**Recognizing a Linear Function Defined by a Table**

From the discussion above, for a fixed $x$ the variable $z$ is a linear function of $y$ and for a fixed $y$ it is a linear function of $x$. Thus, a linear function can be recognized from its table by the following features:

- For a fixed $x$, equal increments in $y$ leads to equal increments in $z$. That is, each row is linear. Similarly, each column is linear.
- All the rows have the same slope $b$.
- All the columns have the same slope $a$.

**Example 77.2**

The table below contains values of a linear function. Fill in the blank and give a formula for the function.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>1.5</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.5</td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$w$</td>
<td>-0.5</td>
<td></td>
</tr>
</tbody>
</table>

**Solution.**

All rows must have the same slope. It follows that

$$\frac{w + 0.5}{2.0 - 1.5} = \frac{1.5 - 0.5}{2.0 - 1.5}.$$  

Solving for $w$ we find $w = 0.5$. Thus, $a = \frac{0.5 - 1.5}{3 - 2} = -1$ and $b = \frac{1.5 - 0.5}{2.0 - 1.5} = 2$. Thus, $z = f(x, y) = -x + 2y + c$. Since $f(2, 1.5) = 0.5$ then $0.5 = -2 + 2(1.5) + c$ and this gives $c = -0.5$ so the equation of the function is $z = f(x, y) = -x + 2y - 0.5$ ■
Contour Diagram of a Plane
If a tilted plane is crossed by another plane parallel to the \( xy \)-plane then their intersection will be a line. Thus the contour lines of a plane are just parallel lines. We can also see this algebraically by setting \( f(x, y) = k \) where \( k \) is a constant. Solving for \( y \) we find

\[
y = \frac{-a}{b}x + \frac{k - c}{b}, \quad b \neq 0
\]

which is the equation of a line in the \( xy \)-plane. It follows that the contour diagram of a linear function consists of parallel lines in the \( xy \)-plane.

Example 77.3
Plot the contour diagram of \( z = 2x + y + 4 \) showing few contour lines.

Solution.
We plot the contour lines with the values \( c = -4, 0, 4, 8, 16 \) obtaining

\[
\begin{align*}
2x + y + 4 &= -4 \quad & y &= -2x - 8 \\
2x + y + 4 &= 0 \quad & y &= -2x - 4 \\
2x + y + 4 &= 4 \quad & y &= -2x \\
2x + y + 4 &= 8 \quad & y &= -2x + 4 \\
2x + y + 4 &= 18 \quad & y &= -2x + 8
\end{align*}
\]

A contour diagram is given in Figure 77.3
Notice that a contour diagram is linear if the contour lines are parallel lines, equally spaced for equally spaced values of $z$.

**Finding the Formula for a Linear Function from its Contour Diagram**

Suppose we are given the contour diagram of a linear function $z = f(x, y)$ as shown in Figure 77.4. We would like to find the formula for $f(x, y)$.

![Figure 77.4](image)

Suppose we start at the origin on the $z = 0$ contour. Moving 1.5 units in the $x$–direction takes us to the $z = 1.5$ contour. Thus, the slope in the $x$–direction is $a = \frac{\Delta z}{\Delta x} = \frac{1.5}{1.5} = 1$. Similarly, moving 1.5 units in the $y$–direction takes us to $z = -1.5$ contour. Hence, $b = \frac{\Delta z}{\Delta x} = \frac{-1.5}{1.5} = -1$. Since $f(0, 0) = 0$ then $c = 0$. Thus, the equation of the plane is $z = x - y$.
Functions of Three Variables

Functions of three variables appear in many applications. For instance, the temperature $T$ at a point on the surface of the earth depends on the longitude $x$ and the latitude $y$ of the point and on the time $t$, so we could write $T = f(x, y, t)$ so that $T$ is a function of three variables.

Functions of three variables are defined in the same way as functions of two variables. We say that $f$ is a function of the variables $x, y, z$ if $f$ is a rule that assigns to every ordered triples $(x, y, z)$ a unique number $w = f(x, y, z)$. The domain of $f$ is the set of all triples $(x, y, z)$ such that $f(x, y, z)$ exists. Thus, the domain is a subset of 3-D. The range of $f$ is the collection of all numbers $f(x, y, z)$ where $(x, y, z)$ is in the domain of $f$.

Example 78.1

Find the domain and range of the function $w = f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$.

Solution.

The domain consists of all triples $(x, y, z)$ satisfying the inequality $x^2 + y^2 + z^2 \leq 1$. Thus, the domain is the unit ball centered at the origin and with radius 1. The range is the interval $0 \leq w \leq 1$.

The graph of $f$ is the set of the 4-D space consisting of all points of the form $(x, y, z, f(x, y, z))$.

We have seen that a function of a single variable has a graph that represents a curve in the xy-plane i.e. a two dimensional space or 2-D space. Likewise the graph of a function of two variables is a surface in 3-D space i.e. a Cartesian coordinate system having three axes. Going to a function of three variables gives us a surface in 4-D space which can’t be drawn. This makes visualizing functions with three or more variables much more difficult. For functions of three variables however, their contours are surfaces in 3-D space, so a least we can visualize them by graphing these surfaces.

By a level surface of a function $w = f(x, y, z)$ we mean a 3-D surface of the form $f(x, y, z) = c$, where $c$ is a constant. The function $f$ can be represented by the family of level surfaces obtained by allowing $c$ to vary. Keep in mind that the value of the function is constant on each level surface.

Example 78.2

What do the level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$ look like?
Solution.
The level surfaces are given by the equations $x^2 + y^2 + z^2 = c$ where $c \geq 0$. The set of level surfaces for this function are just a concentric set of spheres of different radii. Figure 78.1 shows the inside of the lower half spheres.

![Figure 78.1](image)

Example 78.3
What do the level surfaces of $f(x, y, z) = x^2 + y^2$ look like?

Solution.
Level surfaces have equations of the form

$$x^2 + y^2 = c$$

where $c \geq 0$. Each surface is a circular cylinder of radius $\sqrt{c}$ around the $z$–axis. The level surfaces are concentric cylinders as shown in Figure 78.2. Note that $f$ has smaller values on the narrow cylinders near the $z$–axis and larger values on the wider ones.
Example 78.4
What do the level surfaces of \( f(x, y, z) = x^2 + y^2 - z^2 \) look like?

Solution.
Level surfaces are given by the equation \( x^2 + y^2 - z^2 = c \). For \( c < 0 \) the level surface is a surface known as a hyperboloid with two sheets. If \( c = 0 \) we obtain a cone. If \( c > 0 \) the surface is called a hyperboloid with one sheet. See Figure 78.3.
Catalog of Surfaces

Hyperboloid of two sheets
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \]

Parabolic cylinder
\[ y = ax^2 \]

Plane
\[ ax + by + cz = d \]

Hyperbolic Paraboloid
\[ z = -\frac{x^2}{a^2} + \frac{y^2}{b^2} \]

Hyperboloid of one sheet
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \]

Elliptic Paraboloid
\[ z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

Cone
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \]

Ellipsoid
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

Elliptic Cylinder
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]
Finally we note that the graph of a two-variable function $z = f(x, y)$ can be thought of as one member in a family of level surfaces representing the three-variable function:

$$g(x, y, z) = f(x, y) - z$$

Indeed, the graph of $z = f(x, y)$ is just the level surface $g(x, y, z) = 0$. Conversely, a single level surface $g(x, y, z) = c$ can be regarded as the graph of a function $f(x, y)$ if it is possible to solve for $z$ in terms of $x$ and $y$. For example, the level surface $x^2 + y^2 - z + 3 = 0$ is just the graph of the function $z = f(x, y) = x^2 + y^2 + 3$. 
6 Limits and Continuity of Functions of Two Variables

Let $h$ denote the elevation of the terrain above sea level. Then $h$ can be considered as a function of longitude and altitude. If we pick a point on a flat terrain we notice that the values of nearby points are close to the value of the point. We say that $h$ is continuous at that point. On the other hand, there are places on the earth’s surface where the elevation changes abruptly. We give a special name to such places: cliffs. Near either the base or the top of a cliff the terrain may be fairly level and smooth. However as we approach the cliff, there is a large and abrupt change in elevation. Within a few feet the terrain elevation may change by hundreds of feet (either upwards or downwards). This ‘break’ or sudden change in the ground elevation can be considered a discontinuity.

The above example illustrates the ideas of continuity and discontinuity. Roughly speaking, a function is said to be continuous at a point if its values at places near the point are close to the value at the point. If this is not the case then we say that the function is discontinuous.

In this section, we present a formal discussion of the concept of continuity of functions of two variables. Our discussion is not limited to functions of two variables, that is, our results extend to functions of three or more variables. The definition of continuity requires a discussion of the concept of limits:

We say that $L$ is the limit of a function $f$ at the point $(a, b)$, written

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if $f(x, y)$ is as close to $L$ as we please whenever the distance from the point $(x, y)$ to the point $(a, b)$ is sufficiently small, but not zero.

Using $\epsilon \delta$ definition we say that $L$ is the limit of $f(x, y)$ as $(x, y)$ approaches $(a, b)$ if and only if for every given $\epsilon > 0$ we can find a $\delta > 0$ such that for any point $(x, y)$ where $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$.

What does this mean in words? To say that $L$ is the limit of $f(x, y)$ as $(x, y) \to (a, b)$ means that for any given $\epsilon > 0$, we can find an open punctured disk (i.e. without the center and the boundary) centered at $(a, b)$ such that for any point $(x, y)$ inside the disk the difference $f(x, y) - L$ is within $\epsilon$, i.e., $L - \epsilon < f(x, y) < L + \epsilon$. Figure 79.1 illustrates this.
Example 6.1
Let \( f(x, y) = x^2 + y^2 \). Is \( \lim_{(x,y) \to (2,1)} f(x, y) = 4? \)

Solution.
Let \( \epsilon = 0.1 \). Is there a \( \delta > 0 \) such that all the points \((x, y)\) inside the open disk with radius \( \delta \) and centered at \((2, 1)\) satisfy \( 3.9 < f(x, y) < 4.1? \) Clearly, any such open disk will share points with the open disk centered at \((2, 1)\) and with radius 0.2. But any point \((x, y)\) in this latter disk satisfies \((x - 2)^2 + (y - 1)^2 < 0.04\) or \(x^2 + y^2 - 2(x + y) + 5 < 0.04\). Since \(0.8 < x < 1.2\) and \(1.8 < y < 2.2\) then \(5.2 < 2(x + y) < 6.8\). This implies that \(f(x, y) = x^2 + y^2 < 2(x + y) - 4.96 < 6.8 - 4.96 = 1.84\). Hence the point does not satisfy the double inequality \(3.9 < f(x, y) < 4.1\) and as a conclusion \( \lim_{(x,y) \to (2,1)} f(x, y) \neq 4 \). 

As in the case of functions of one variable, limits of functions of two variables possess the following properties:
- The limit, if it exists, is unique.
- The limit of a sum, difference, product, is the sum, difference, product of limits.
- The limit of a quotient is the quotient of limits provided that the limit in the denominator is not zero.

We can now define what we mean by continuity in terms of limit. Intuitively, we expect our definition to support the idea that there are no ‘breaks’ or gaps in the function if it is continuous. The continuity of functions of two
variables is defined in the same way as for functions of one variable:
A function \( f(x, y) \) is **continuous** at the point \((a, b)\) if the following two conditions are satisfied:
(a) \( f(a, b) \) exists;
(b) \( \lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) \).

A function is **continuous on a region** \( R \) in the \( xy \)-plane if it is continuous at each point in \( R \). A function that is not continuous at \((a, b)\) is said to be **discontinuous** at \((a, b)\).

Since the condition \( \lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) \) means that \( f(x, y) \) is close to \( f(a, b) \) when \((x, y)\) is close to \((a, b)\) we see that our definition does indeed correspond to our intuitive notion that the graph of \( f(x, y) \) has no gaps or breaks. Thus a graph of \( f(x, y) \) is unbroken everywhere.

**Remark 6.1**
In the case of functions of one variable, if a function \( f(x) \) is continuous at \( x = a \) then the limit of \( f(x) \) as \( x \) approaches \( a \) from either the left or right must be \( f(a) \). Similar situation occurs for a function \( f(x, y) \) of two variables with the difference that the point \((x, y)\) can approach \((a, b)\) in infinite directions. Hence, if you can find two directions toward \((a, b)\) with two different limits then the function is discontinuous at \((a, b)\).

Like the case of functions of one variable, it can be shown that sums, products, quotients (where denominator function is not zero), and compositions of continuous functions are also continuous.

**Example 6.2**
Show that
\[
f(x, y) = \begin{cases} 
\frac{x^2 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\
0 & (x, y) = (0, 0)
\end{cases}
\]
is continuous at \((0, 0)\).

**Solution.**
This function is clearly continuous everywhere except at (possibly) \((0, 0)\). Let \( \epsilon > 0 \) be given. Can we find a \( \delta > 0 \) such that if \( 0 < \sqrt{x^2 + y^2} < \delta \) then \( |f(x, y) - 0| < \epsilon \)? Let \( \delta = \frac{\epsilon}{2} \) and suppose that
0 < \sqrt{x^2 + y^2} < \delta. Then, using the fact that \( x^2 \leq x^2 + y^2 \), i.e. \( \left| \frac{x^2}{x^2 + y^2} \right| \leq 1 \) we have

\[
|f(x, y) - 0| = \left| \frac{x^2 y}{x^2 + y^2} \right| = \left| \frac{x^2}{x^2 + y^2} \right| |y| \leq |y| \leq \sqrt{x^2 + y^2} < \delta = \varepsilon < \varepsilon.
\]

Hence, \( f(x, y) \) is continuous at \((0, 0)\)

**Example 6.3**

Show that

\[
f(x, y) = \begin{cases} 
\frac{x^2 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\
0 & (x, y) = (0, 0)
\end{cases}
\]

is discontinuous at \((0, 0)\).

**Solution.**

Again this function is clearly continuous everywhere except (possibly) at \((0, 0)\). Now let’s look at the limit as \((x, y)\) approaches \((0, 0)\) along two different paths. First, let’s approach \((0, 0)\) along the \(x\)-axis, i.e. \( y = 0 \).

\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{(x,0) \to (0,0)} \frac{x^2}{x^2 + 0} = 1.
\]

Now, let’s approach \((0, 0)\) along the \(y\)-direction, i.e. \( x = 0 \).

\[
\lim_{(x,y) \to (0,0)} f(x, y) = \lim_{(0,y) \to (0,0)} \frac{0}{0 + y^2} = 0.
\]

Since the limit is not the same along the two different directions we conclude that \( f(x, y) \) is discontinuous at \((0, 0)\).
80 Introduction to Vectors

In the previous calculus classes we have seen that the study of motion involved the introduction of a variety of quantities which are used to describe the physical world. Examples of these quantities include distance, displacement, speed, velocity, acceleration, force and mass. Some of these are characterized by a single number and others require a number and a direction. In general, objects in the physical world are divided into two categories: scalars and vectors.

**Scalars** are objects that can be modeled or characterized using a single number. Examples of scalars in the study of motion are distance, speed, and mass.

**Vectors** are physical objects or quantities that require both a ”distance” and a ”direction” for their specification. In the study of motion, the quantities displacement, velocity, acceleration, and force are examples of vectors. The purpose of this section is to understand some fundamentals about vectors.

**Notation and Terminology**

By a **displacement vector** from a point $A$ to a point $B$ we mean an arrow with its tail at $A$ and its tip at $B$ as shown in Figure 80.1.

![Figure 80.1](image)

The direction of the vector is the direction of the arrow. The distance between $A$ and $B$ is known as the **magnitude** or **length** of the vector. We will represent this vector by $\vec{AB}$ and its magnitude by $||\vec{AB}||$. In many instances, the endpoints of a vector are ignored and the vector is just denoted by $\vec{v}$. In this case, the magnitude of the vector will be denoted by $||\vec{v}||$.

We define the **zero vector** to be a displacement vector with zero length. We will denote it by $\vec{0}$. The zero vector has no direction.

A vector of length 1 is called a **unit vector**.

**Algebra of Vectors: Addition and Difference of Two Vectors**
Starting from the bookstore, suppose that along a certain direction one can reach a classroom after traveling a distance of 500 ft and then 300 ft in another direction to reach the pool. By drawing the two displacement vectors one can draw a third vector representing the walk from the bookstore straight to the pool as shown in Figure 80.2.

![Figure 80.2](image1)

If we let \( \vec{v} \) and \( \vec{w} \) represent the vectors from the bookstore to the classroom and the classroom to the pool, respectively, then the vector from the bookstore to the pool is just the sum \( \vec{v} + \vec{w} \) as shown in Figure 80.3. This clearly makes sense, because a person can get to the pool from the bookstore by walking there directly or by first walking to the classroom and then to the pool. Either way he or she eventually reaches the pool.

![Figure 80.3](image2)

Now, suppose the displacement vectors \( \vec{u} \) and \( \vec{v} \) from the bookstore to both the classroom and the pool are known. What is the displacement vector \( \vec{x} \) from the classroom to the pool? See Figure 80.4. Since \( \vec{u} + \vec{x} = \vec{v} \) then we define \( \vec{x} \) to be the difference \( \vec{x} = \vec{u} - \vec{v} \).

![Figure 80.4](image3)
Multiplication of a Vector by a Scalar
If a vector \( \vec{u} \) and a scalar \( \alpha \) are given we would like to know what does \( \alpha \vec{u} \) stand for? The result is a vector of magnitude \( |\alpha| \) times the magnitude of \( \vec{u} \) and in the same direction of \( \vec{u} \) if \( \alpha > 0 \) and opposite direction if \( \alpha < 0 \). Figure 80.5 shows three vectors, \( \vec{u}, \frac{1}{2} \vec{u}, \) and \( -2\vec{u} \).

For example, the difference \( \vec{u} - \vec{v} \) is just the sum \( \vec{u} + (-1)\vec{v} \) as shown in Figure 80.6.

Parallel Vectors
The operation of multiplying a vector by a scalar leads to the following definition: We say that two vectors \( \vec{u} \) and \( \vec{v} \) are parallel if and only if \( \vec{u} = \alpha \vec{v} \) for some scalar \( \alpha \).

The Components of Vectors
Consider the three dimensional Cartesian coordinate system. We associate a direction with each of the three axes of this system. In particular we define three unit vectors \( \vec{i}, \vec{j}, \) and \( \vec{k} \) to point along the direction of the positive
numbers along each of our three axes as shown in Figure 80.7.

Now, let $\vec{v}$ be a vector in the three dimensional system with tail $A = (a, b, c)$ and tip $B = (a', b', c')$.

According to Figure 80.8 we can write

$$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}.$$ 

But $\overrightarrow{AC} = (a' - a)\vec{i} + (b' - b)\vec{j}$ and $\overrightarrow{CB} = (c' - c)\vec{k}$. Hence

$$\overrightarrow{AB} = (a' - a)\vec{i} + (b' - b)\vec{j} + (c' - c)\vec{k}.$$ 

Letting $\alpha = a' - a$, $\beta = b' - b$, and $\gamma = c' - c$ we can write

$$\vec{v} = \alpha\vec{i} + \beta\vec{j} + \gamma\vec{k}.$$
We call $\alpha \vec{i}, \beta \vec{j},$ and $\gamma \vec{k}$ the components of $\vec{v}$. Expressing a vector $\vec{v}$ in terms of its components is referred to as **resolving $\vec{v}$ into components**.

Now, if the tail of a vector is the origin of the coordinate system then we call the vector a **position vector**. In this case, if $A = (a, b, c)$ are the coordinates of the tip point of the vector then we can write

$$\vec{v} = \overrightarrow{OA} = a\vec{i} + b\vec{j} + c\vec{k}.$$ 

Resolving vectors allows us to manipulate them algebraically. In particular we can define the operations of addition, subtraction, and scalar multiplication in terms of the vector components. If $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}, \vec{v} = a'\vec{i} + b'\vec{j} + c'\vec{k}$ and $\alpha$ is a scalar then

$$\vec{u} + \vec{v} = (a + a')\vec{i} + (b + b')\vec{j} + (c + c')\vec{k},$$

$$\vec{u} - \vec{v} = (a - a')\vec{i} + (b - b')\vec{j} + (c - c')\vec{k},$$

$$\alpha \vec{u} = \alpha a\vec{i} + \alpha b\vec{j} + \alpha c\vec{k}.$$ 

Also, the magnitude of a vector can be expressed in terms of its components. Consider a vector $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$. Let $\overrightarrow{OP}$ be a position vector with the same direction as $\vec{v}$ and the same magnitude. This implies that $P = (a, b, c)$. But then the magnitude of $\vec{v}$ is just the distance between the origin and the point $P$. That is,

$$||\vec{v}|| = \sqrt{a^2 + b^2 + c^2}.$$ 

**Remark 80.1**

All the above apply as well for the 2-D case by just letting $z = 0$.

**Example 80.1**

Perform the operation $(4\vec{i} + 2\vec{j}) - (3\vec{i} - \vec{j})$.

**Solution.**

We have

$$(4\vec{i} + 2\vec{j}) - (3\vec{i} - \vec{j}) = (4 - 3)\vec{i} + (2 - (-1))\vec{j} = \vec{i} + 3\vec{j} \blacksquare$$

**Example 80.2**

Find the length of the vector $\vec{v} = \vec{i} - \vec{j} + 2\vec{k}$.

**Solution.**

The length is $||\vec{v}|| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6} \blacksquare$
Example 80.3
Show that the vectors \( \vec{u} = \vec{i} - \vec{j} + 3\vec{k} \) and \( \vec{v} = 4\vec{i} - 4\vec{j} + 12\vec{k} \) are parallel.

Solution.
Since \( \vec{v} = 4\vec{u} \) then the two vectors are parallel \( \blacksquare \)

Remark 80.2
An alternative representation of a vector \( \vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k} \) is \( \vec{v} = (v_1, v_2, v_3) \).

Resolving a Two-Dimensional Vector into Components
Suppose we are given the direction and the length of a vector \( \vec{u} \) and let \( \theta \) be the angle (in radians) the vector makes with the positive \( x \)-axis, measured counterclockwise as in Figure 80.9. If we write \( \vec{u} = a\vec{i} + b\vec{j} \) and \( u = ||\vec{u}|| \) then using trigonometric functions we can write

\[
\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{a}{u} \quad \text{and} \quad \sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{b}{u}
\]

Hence,

\[
a = u \cos \theta, \quad b = u \sin \theta
\]

\[
\vec{u} = u \cos \theta \vec{i} + u \sin \theta \vec{j}
\]

![Figure 80.9](image)

Remark 80.3
If the components of a vector are given, say \( \vec{v} = v_1\vec{i} + v_2\vec{j} \) then the angle \( \theta \) can be determined by using the formula

\[
\theta = \tan^{-1}\left( \frac{v_2}{v_1} \right)
\]
Example 80.4
Resolve the vector \( \vec{u} \) into components if \( u = ||\vec{u}|| = 8 \) and \( \theta = 40^\circ \).

Solution.
Substituting into the above formula we find
\[
\vec{u} = u \cos \theta \hat{i} + u \sin \theta \hat{j} = 8 \cos \left( \frac{2\pi}{9} \right) \hat{i} + 8 \sin \left( \frac{2\pi}{9} \right) \hat{j}
\]

Unit Vectors
By a unit vector we mean any vector of magnitude equals to 1. From any nonzero vector \( \vec{v} \), we can obtain a unit vector by setting
\[
\vec{u} = \frac{\vec{v}}{||\vec{v}||}.
\]
To see this, write \( \vec{v} = a\hat{i} + b\hat{j} + c\hat{k} \). Then \( ||\vec{v}|| = \sqrt{a^2 + b^2 + c^2} \). Moreover
\[
\vec{u} = \frac{a}{\sqrt{a^2 + b^2 + c^2}} \hat{i} + \frac{b}{\sqrt{a^2 + b^2 + c^2}} \hat{j} + \frac{c}{\sqrt{a^2 + b^2 + c^2}} \hat{k}
\]
Therefore,
\[
||\vec{u}|| = \sqrt{\frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2}} = \frac{\sqrt{a^2 + b^2 + c^2}}{\sqrt{a^2 + b^2 + c^2}} = 1.
\]

Example 80.5
Find a unit vector from the point \( P(1,2) \) and toward the point \( Q(4,6) \).

Solution.
Let \( \vec{v} = \overrightarrow{PQ} = (4 - 1)\hat{i} + (6 - 2)\hat{j} = 3\hat{i} + 4\hat{j} \). Then
\[
||\vec{v}|| = \sqrt{3^2 + 4^2} = 5.
\]
Hence, a unit vector is
\[
\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{3}{5} \hat{i} + \frac{4}{5} \hat{j}
\]

Example 80.6
Show that for any numbers \( a \) and \( b \) we have \( a\hat{i} + b\hat{j} = b\hat{j} + a\hat{i} \).
Solution.
This is clear from Figure 80.10 where, for the sake of argument, we chose \(a > 0\) and \(b > 0\) ■

Figure 80.10
81 Properties of Vector Arithmetic

As noted in the previous section vectors are used extensively in modeling the physical world. For example, physics uses vectors to represent displacement, velocity, acceleration, and force. As we have seen in Section 80, displacement (i.e. location) is a vector quantity since it requires both length and direction to be accurately defined. Similarly, velocity, acceleration, and force are vectors.

**Velocity and Speed**

Just as distance (always a nonnegative scalar) and displacement have distinctly different meanings, so do speed and velocity. The notion of velocity contains the concepts of how fast an object is going and the direction in which that object is moving or traveling.

**Speed** is a scalar quantity which refers to “how fast an object is moving.” A fast-moving object has a high speed while a slow-moving object has a low speed. An object with no movement at all has a zero speed.

**Velocity** is a vector quantity. Its length represents the speed of the object and its sign indicates the direction in which the object is traveling.

**Example 81.1**

Which is traveling faster, a car whose velocity is $21\vec{i} + 35\vec{j}$, or a car whose velocity vector is $40\vec{i}$, assuming that the units are the same for both directions.

**Solution.**

We need to calculate the speed of each car, i.e. the length of each vector.

$$||21\vec{i} + 35\vec{j}|| = \sqrt{21^2 + 35^2} = \sqrt{1666} \approx 40.8$$

$$||40\vec{i}|| = \sqrt{40^2} = 40$$

so the first car is faster.

**Relative Velocity and Riverboat Problems**

On occasion objects move within a medium which is moving with respect to an observer. For example, an airplane usually encounters a wind - air which is moving with respect to an observer on the ground below. As another example, a motor boat in a river is moving amidst a river current - water which
is moving with respect to an observer on dry land. In such instances as this, the magnitude of the velocity of the moving object (whether it be a plane or a motor boat) with respect to the observer on land will not be the same as the speedometer reading of the vehicle. That is to say, the speedometer on the motor boat might read 20 mi/hr; yet the motor boat might be moving relative to the observer on shore at a speed of 25 mi/hr. Motion is relative to the observer. Thus, the velocity vector of a plane relative to the ground is the sum of the plane’s velocity vector (relative to air) and the wind velocity vector. Similar conclusions for riverboat problems.

**Example 81.2**
A boat captain wants to travel due south at 40 knots (speed relative to ground). If the current is moving northwest at 16 knots, in what direction and magnitude should he work the engine (relative to water)?

**Solution.**
We choose a coordinate system with the $y-$axis pointing South-North and the $x-$axis pointing West-East. The boat is moving due south with velocity (relative to ground) $\vec{u} = -40\hat{j}$. In addition, the current is moving due northwest with a velocity $\vec{w}$. See Figure 81.1. Writing $\vec{w}$ in components we find

$$w_1 = ||\vec{w}|| \cos \frac{3\pi}{4} = -8\sqrt{2} \quad \text{and} \quad w_2 = ||\vec{w}|| \sin \frac{3\pi}{4} = 8\sqrt{2}$$

$$\vec{w} = -8\sqrt{2}\hat{i} + 8\sqrt{2}\hat{j}.$$ 

If we let $\vec{v}$ be the engine’s vector (relative to water) then we have

$$\vec{u} = \vec{v} + \vec{w}$$

or

$$\vec{v} = \vec{u} - \vec{w} = 8\sqrt{2}\hat{i} - (40 + 8\sqrt{2})\hat{j}.$$ 

The magnitude of $\vec{v}$ is

$$||\vec{v}|| = \sqrt{(8\sqrt{2})^2 + (40 + 8\sqrt{2})^2} \approx 52.5.$$ 

The direction of the engine is

$$\theta = \arctan \left( -\frac{40 + 8\sqrt{2}}{8\sqrt{2}} \right) \approx -1.35 \text{ rad} \quad \blacksquare$$
Acceleration
Acceleration is the rate at which an object’s velocity is changing with time. An object is accelerating if its velocity is changing with time. Like velocity it is a vector. It has both magnitude and direction. For acceleration, its magnitude represents how fast an object is speeding up (or slowing down). For example, if $a(t)$ denotes acceleration then $a(3) = 5 \text{m/sec}^2$ tells us that between the third and fourth seconds the object’s velocity is going to increase by about 5 m/s. Thus, during that time interval is object is speeding up.
An important example of acceleration that one encounters is the acceleration due to gravity. This is a vector pointing downward with magnitude 9.81 m/sec$^2$.

Force
Force is yet another example of a vector quantity. In order to move an object, something must push on it. The strength the "push" is the magnitude of the force applied. However the direction of the push is important. If you wish to move a chair in a room, it's much more effective to push on it sideways then to push down on it! In both cases, the same magnitude of force may be applied to the chair, but the results will be substantially different. Hence force has direction as well as magnitude and is therefore a vector.

Example 81.3
The motion of a pendulum with a mass attached to one end involves three forces: gravity, the tension of the rod, and air resistance. Sketch all the three forces acting on the mass.
Solution.
There are three forces acting on the mass: gravity $\vec{g}$, which is pulling the mass straight down, tension $\vec{T}$ in the rod, which is pulling the mass in the direction of the rod, and air resistance $\vec{r}$, which is pulling the mass in a direction opposite to its direction of motion. All three forces are acting on the mass as shown in Figure 81.2.

![Figure 81.2](image)

We next discuss some basic properties of vector arithmetic. These properties are valid for any number of components.

**Theorem 81.1**
If $\vec{u}, \vec{v}$, and $\vec{w}$ are three vectors (with the same number of components) and $a$ and $b$ are two scalars then we have the following properties:

\[
\begin{align*}
\vec{u} + \vec{v} &= \vec{v} + \vec{u} \\
\vec{u} + (\vec{v} + \vec{w}) &= (\vec{u} + \vec{v}) + \vec{w} \\
\vec{u} + \vec{0} &= \vec{u} \\
1\vec{u} &= \vec{u} \\
(a + b)\vec{u} &= a\vec{u} + b\vec{v} \\
a(\vec{u} + \vec{v}) &= a\vec{u} + a\vec{v}
\end{align*}
\]

**Proof.**
These properties can be easily proved using components. We will prove the first one leaving the rest for the reader. Write $\vec{u} = u_1\vec{i} + u_2\vec{j}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j}$. 

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Then
\[
\vec{u} + \vec{v} = u_1\vec{i} + u_2\vec{j} + v_1\vec{i} + v_2\vec{j} \\
= u_1\vec{i} + v_1\vec{j} + u_2\vec{i} + v_2\vec{j} \\
= v_1\vec{i} + u_1\vec{j} + v_2\vec{i} + u_2\vec{j} \\
= v_1\vec{i} + v_2\vec{j} + u_1\vec{i} + u_2\vec{j} \\
= \vec{v} + \vec{u}
\]
where in the process we used the result from Example 80.6

**Vectors in n Dimensions**

So far we have confined ourselves to vectors with two or three components. Vectors having four and more components do occur both mathematically and in practical (real world) applications. Vectors in \(n\) components are used to list various quantities in an organized way. For example, the vector
\[
\vec{v} = (v_1, v_2, v_3, v_4)
\]
might represent the prices of four different ingredients required to make a particular product.

For the most part, the course will concentrate on vectors having either two or three components, but in general the mathematics will apply to vectors of any number of components.

**Example 81.4**

There are five students in a class. Their scores on the midterm (out of 100) are given by the vector \(\vec{v} = (73, 80, 91, 65, 84)\). Their scores on the final (out of 100) are given by \(\vec{w} = (82, 79, 88, 70, 92)\). If the final exam counts twice as much as the midterm, find a vector giving the final scores of the students in this class.

**Solution.**
The final scores of the students in this class are given by the vector
\[
\frac{1}{3}(\vec{v} + 2\vec{w}) = \frac{1}{3}(237, 238, 267, 205, 268) \approx (79, 79.33, 89, 68.33, 89.33)
\]
82 Multiplication of Vectors: The Scalar or Dot Product

Up to this point we have defined what vectors are and discussed basic notation and properties. We have also defined basic operations on vectors such as addition, subtraction, and scalar multiplication.

Now, is there such thing as multiplying a vector by another vector? The answer is yes. As a matter of fact there are two types of vector multiplication. The first one is known as scalar or dot product and produces a scalar; the second is known as the vector or cross product and produces a vector. In this section we will discuss the former one leaving the latter one for the next section.

One of the motivation for using the dot product is the physical situation to which it applies, namely that of computing the work done on an object by a given force over a given distance, as shown in Figure 82.1

![Figure 82.1](image)

Indeed, the work $W$ is given by the expression

$$W = ||\vec{F}|| \cdot ||\vec{PQ}|| \cos \theta$$

where $||\vec{F}|| \cos \theta$ is the component of $\vec{F}$ in the direction of $\vec{PQ}$.

Thus, we define the dot product of two vectors $\vec{u}$ and $\vec{v}$ to be the number

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \cdot ||\vec{v}|| \cos \theta, \quad 0 \leq \theta \leq \pi$$

where $\theta$ is the angle between the two vectors as shown in Figure 82.2

![Figure 82.2](image)
The above definition is the geometric definition of the dot product. We next provide an algebraic way for computing the dot product. Indeed, let \( \vec{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k} \) and \( \vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \). Then \( \vec{v} - \vec{u} = (v_1 - u_1) \hat{i} + (v_2 - u_2) \hat{j} + (v_3 - u_3) \hat{k} \). Moreover, we have

\[
||\vec{u}||^2 = u_1^2 + u_2^2 + u_3^2 \\
||\vec{v}||^2 = v_1^2 + v_2^2 + v_3^2 \\
||\vec{v} - \vec{u}||^2 = (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2 \\
= v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 + v_3^2 - 2v_3u_3 + u_3^2
\]

Now, applying the Law of Cosines to Figure 82.3 we can write

\[
||\vec{v} - \vec{u}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}|| ||\vec{v}|| \cos \theta.
\]

Thus, by substitution we obtain

\[
v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 + v_3^2 - 2v_3u_3 + u_3^2 = u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - 2||\vec{u}|| ||\vec{v}|| \cos \theta
\]

or

\[
||\vec{u}|| ||\vec{v}|| \cos \theta = u_1v_1 + u_2v_2 + u_3v_3
\]

so that we can define the dot product algebraically by

\[
\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.
\]

### Example 82.1

Compute the dot product of \( \vec{u} = \frac{1}{\sqrt{2}} \hat{i} + \frac{1}{\sqrt{2}} \hat{j} + \frac{1}{\sqrt{2}} \hat{k} \) and \( \vec{v} = \frac{1}{2} \hat{i} + \frac{1}{2} \hat{j} + \hat{k} \) and the angle between these vectors.
Solution.
We have
\[ \vec{u} \cdot \vec{v} = \frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} + \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{2\sqrt{2}} + \frac{1}{2 \sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}. \]

We also have
\[ ||\vec{u}||^2 = \left( \frac{1}{\sqrt{2}} \right)^2 + \left( \frac{1}{\sqrt{2}} \right)^2 + \left( \frac{1}{\sqrt{2}} \right)^2 = \frac{3}{2} \]
\[ ||\vec{v}||^2 = \left( \frac{1}{2} \right)^2 + \left( \frac{1}{2} \right)^2 + 1 = \frac{3}{2} \]

Thus,
\[ \cos \theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \ ||\vec{v}||} = \frac{2\sqrt{2}}{3}. \]

Hence,
\[ \theta = \cos^{-1} \left( \frac{2\sqrt{2}}{3} \right) \approx 0.34 rad \approx 19.5^\circ. \]

Remark 82.1
The algebraic definition of the dot product extends to vectors with any number of components.

Next, we discuss few properties of the dot product.

Theorem 82.1
For any vectors \( \vec{u}, \vec{v}, \) and \( \vec{w} \) and any scalar \( \lambda \) we have
(i) Commutative law: \( \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \)
(ii) Distributive law: \( (\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \)
(iii) \( \vec{u} \cdot (\lambda \vec{v}) = (\lambda \vec{u}) \cdot \vec{v} = \lambda (\vec{u} \cdot \vec{v}) \)
(iv) Magnitude: \( ||\vec{u}||^2 = \vec{u} \cdot \vec{u} \)
(v) Two nonzero vectors \( \vec{u} \) and \( \vec{v} \) are orthogonal or perpendicular if and only if \( \vec{u} \cdot \vec{v} = 0 \)
(vi) Two nonzero vectors \( \vec{u} \) and \( \vec{v} \) are parallel if and only if \( \vec{u} \cdot \vec{v} = \pm ||\vec{u}|| \ ||\vec{v}||. \)

Proof.
Write \( \vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}, \) \( \vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}, \) and \( \vec{w} = w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k}. \) Then
(i) \( \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_2v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \vec{v} \cdot \vec{u} \).

(ii) \((\vec{u} + \vec{v}) \cdot \vec{w} = ((u_1 + v_1)\vec{i} + (u_2 + v_2)\vec{j} + (u_3 + v_3)\vec{k}) \cdot (w_1\vec{i} + w_2\vec{j} + w_3\vec{k}) = (u_1 + v_1)w_1 + (u_2 + v_2)w_2 + (u_3 + v_3)w_3 = u_1w_1 + u_2w_2 + u_3w_3 + v_1w_1 + v_2w_2 + v_3w_3 = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \).

(iii) \( \vec{u} \cdot (\lambda \vec{v}) = (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \cdot (\lambda v_1\vec{i} + \lambda v_2\vec{j} + \lambda v_3\vec{k}) = \lambda (u_1v_1 + u_2v_2 + u_3v_3) = \lambda(\vec{u} \cdot \vec{v}) \).

(iv) \( ||\vec{u}||^2 = \vec{u} \cdot \vec{u} \cos 0 = \vec{u} \cdot \vec{u} \).

(v) If \( \vec{u} \) and \( \vec{v} \) are perpendicular then the cosine of their angle is zero and so the dot product is zero. Conversely, if the dot product of the two vectors is zero then the cosine of their angle is zero and this happens only when the two vectors are perpendicular.

(vi) If \( \vec{u} \) and \( \vec{v} \) are parallel then the cosine of their angle is either 1 or -1. That is, \( \vec{u} \cdot \vec{v} = \pm ||\vec{u}|| ||\vec{v}|| \). Now go backward for the converse \( \blacksquare \)

Remark 82.2
Note that the unit vectors \( \vec{i}, \vec{j}, \vec{k} \) associated with the coordinate axes satisfy the equalities
\[ \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 1 \quad \text{and} \quad \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0 \]

Finding equations of planes using normal vectors
Recall that the equation of a plane is given by the linear expression \( z = ax + by + c \). Knowing the \( x \)–slope, \( y \)–slope, and the \( z \)–intercept we were able to find the equation of a plane. Now, we will try to find the equation of a plane given a point in the plane and a vector normal to the plane. But first we need to define by what we mean by a normal vector.

A normal vector to a plane is a vector that is orthogonal to every vector of the plane. If \( \vec{n} = a\vec{i} + b\vec{j} + c\vec{k} \) is normal to a given plane and \( P_0 = (x_0, y_0, z_0) \) is a fixed point in the plane then for any point \( P = (x, y, z) \) in the plane the vectors \( \vec{n} \) and \( \overrightarrow{P_0P} \) are orthogonal. See Figure 82.4. In terms of dot product this means that
\[ \overrightarrow{P_0P} \cdot \vec{n} = 0 \]
Resolving \( \overrightarrow{P_0P} \) in terms of components we find
\[ \overrightarrow{P_0P} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} \]
In this case, the dot product \( \overrightarrow{P_0P} \cdot \vec{n} = 0 \) leads to the equation
\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \]

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which is the equation of our required plane.

\[ 2(x - 1) + 2(y - 0) + (-1)(z + 1) = 0 \]

or

\[ z = 2x + 2y - 3 \]

Example 82.3
Find a normal vector to the plane \(-x + 3y + 2z = 7\).

Solution.
Since the coefficients of \(\vec{i}, \vec{j}, \) and \(\vec{k}\) in a normal vector are the coefficients of \(x, y,\) and \(z\) in the equation of the plane, a normal vector is \(\vec{n} = -\vec{i} + 3\vec{j} + 2\vec{k}\).

Projection of a vector onto a line
The orthogonal projection of a vector along a line is obtained by taking a vector with same length and direction as the given vector but with its tail on the line and then dropping a perpendicular onto the line from the tip of the vector. The resulting segment on the line is the vector’s orthogonal projection.
projection or simply its projection. See Figure 82.5.

![Figure 82.5](image)

Figure 82.5

Now, if $\vec{u}$ is a unit vector along the line of projection and if $\vec{v}_{\text{parallel}}$ is the vector projection of $\vec{v}$ onto $\vec{u}$ then

$$\vec{v}_{\text{parallel}} = ||\vec{v}|| \cos \theta \vec{u} = (\vec{v} \cdot \vec{u}) \vec{u}.$$

See Figure 82.6. Also, the component perpendicular to $\vec{u}$ is given by

$$\vec{v}_{\text{perpendicular}} = \vec{v} - \vec{v}_{\text{parallel}}.$$

![Figure 82.6](image)

Figure 82.6

It follows that the vector $\vec{v}$ can be resolved in terms of $\vec{v}_{\text{parallel}}$ and $\vec{v}_{\text{perpendicular}}$

$$\vec{v} = \vec{v}_{\text{parallel}} + \vec{v}_{\text{perpendicular}}.$$

**Example 82.4**

Write the vector $\vec{v} = 3\vec{i} + 2\vec{j} - 6\vec{k}$ as the sum of two vectors, one parallel, and one perpendicular to $\vec{w} = 2\vec{i} - 4\vec{j} + \vec{k}$.  

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Solution.
Let \( \vec{u} = \frac{\vec{w}}{||\vec{w}||} = \frac{2}{\sqrt{21}} \vec{i} - \frac{4}{\sqrt{21}} \vec{j} + \frac{1}{\sqrt{21}} \vec{k} \). Then,

\[
\vec{v}_{\text{parallel}} = (\vec{v} \cdot \vec{u})\vec{u} = \left( \frac{6}{\sqrt{21}} - \frac{8}{\sqrt{21}} - \frac{6}{\sqrt{21}} \right) \vec{u} = -\frac{16}{21} \vec{i} + \frac{32}{21} \vec{j} - \frac{8}{21} \vec{k}.
\]

Also,

\[
\vec{v}_{\text{perpendicular}} = \vec{v} - \vec{v}_{\text{parallel}} = \left( 3 + \frac{16}{21} \right) \vec{i} + \left( 2 - \frac{32}{21} \right) \vec{j} + \left( -6 + \frac{8}{21} \right) \vec{k}
\]

\[
= \frac{79}{21} \vec{i} + \frac{10}{21} \vec{j} - \frac{118}{21} \vec{k}
\]

Hence,

\[
\vec{v} = -\frac{8}{21} \vec{w} + \left( \frac{79}{21} \vec{i} + \frac{10}{21} \vec{j} - \frac{118}{21} \vec{k} \right) \boxed{.}
\]
83 Multiplying Vectors: The Cross Product

In this section we discuss another way of multiplying two vectors to obtain a vector, the cross product. We should note that the cross product requires both of the vectors to be three dimensional vectors.

Now, any two three dimensional vectors determine a parallelogram. We will see below that the definition of cross product involves this parallelogram.

Consider first the parallelogram obtained by two vectors $\vec{u}$ and $\vec{v}$ and let $\theta$ be the angle between these two vectors as shown in Figure 83.1.

The area of this parallelogram is given by

$$\text{Area of parallelogram} = \text{Base} \times \text{Height} = ||\vec{v}|| \cdot ||\vec{u}|| \sin \theta.$$ 

We define the cross product of two vectors $\vec{u}$ and $\vec{v}$ to be the vector which is perpendicular to both $\vec{u}$ and $\vec{v}$ with a magnitude equal to the area of the parallelogram they span. The corresponding formula is

$$\vec{u} \times \vec{v} = (||\vec{v}|| \cdot ||\vec{u}|| \sin \theta) \vec{n}$$

where $0 \leq \theta \leq \pi$ is the angle between the two vectors and $\vec{n}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$.

The first problem that we encounter here is the normal vector $\vec{n}$. For any two vectors $\vec{u}$ and $\vec{v}$, if $\vec{n}$ is a normal vector so is $-\vec{n}$. So which vector is the correct one? This is determined by the following rule, known as the right-hand rule:

The orientation of $\vec{n}$ is determined by placing $\vec{u}$ and $\vec{v}$ tail-to-tail, flattening the right hand, extending it in the direction of $\vec{u}$, and then curling the fingers in the direction leading to $\vec{v}$. The thumb then points in the direction of $\vec{n}$.
It follows from the definition, that if $\vec{u}$ and $\vec{v}$ are parallel, then $\sin \theta = 0$ and therefore $\vec{u} \times \vec{v} = \vec{0}$. In particular, $\vec{u} \times \vec{u} = \vec{0}$.

**Example 83.1**
Find all possible cross products of the unit vectors $\vec{i}, \vec{j},$ and $\vec{k}$.

**Solution.**
We have

\[
\vec{i} \times \vec{j} = ||\vec{i}|| \ ||\vec{j}|| \sin \left(\frac{\pi}{2}\right) \vec{k} = \vec{k}
\]
\[
\vec{j} \times \vec{k} = ||\vec{j}|| \ ||\vec{k}|| \sin \left(\frac{\pi}{2}\right) \vec{i} = \vec{i}
\]
\[
\vec{k} \times \vec{i} = ||\vec{k}|| \ ||\vec{i}|| \sin \left(\frac{\pi}{2}\right) \vec{j} = \vec{j}
\]
\[
\vec{j} \times \vec{i} = ||\vec{j}|| \ ||\vec{i}|| \sin \left(\frac{\pi}{2}\right) \vec{k} = -\vec{k}
\]
\[
\vec{k} \times \vec{j} = ||\vec{k}|| \ ||\vec{j}|| \sin \left(\frac{\pi}{2}\right) \vec{i} = -\vec{i}
\]
\[
\vec{i} \times \vec{k} = ||\vec{i}|| \ ||\vec{k}|| \sin \left(\frac{\pi}{2}\right) \vec{j} = -\vec{j}
\]

There is an easy way to deal with the signs. Consider the diagram in Figure 83.3

---

See Figure 83.2

![Figure 83.2](image_url)

**Figure 83.2**
Any product that moves around the diagram in the direction of the arrows takes a plus sign, and any product that moves around the diagram in the other direction takes a minus sign.

Also, $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$ since the angle between a vector with itself is zero.

So far the cross product has been defined geometrically. How can we find $\vec{u} \times \vec{v}$ if both vectors are given in components? That is, if $\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$ and $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$ then what is $\vec{u} \times \vec{v}$?

Let’s first introduce the determinant of three rows and three columns:

$$\begin{vmatrix} x & y & z \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - v_2 u_3)x - (u_1 v_3 - v_1 u_3)y + (u_1 v_2 - u_2 v_1)z.$$  

By substitution one can easily find that

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - v_2 u_3)u_1 - (u_1 v_3 - v_1 u_3)u_2 + (u_1 v_2 - u_2 v_1)u_3 = 0$$

and

$$\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - v_2 u_3)v_1 - (u_1 v_3 - v_1 u_3)v_2 + (u_1 v_2 - u_2 v_1)v_3 = 0.$$  

Thus, by letting $\vec{w} = (u_2 v_3 - v_2 u_3)\vec{i} - (u_1 v_3 - v_1 u_3)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$ we see that

$$\vec{w} \cdot \vec{u} = 0 \text{ and } \vec{w} \cdot \vec{v} = 0.$$  

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Hence, $\vec{w}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. It follows that $\vec{w}$ and $\vec{u} \times \vec{v}$ are parallel. Hence, $\vec{w} = \lambda (\vec{u} \times \vec{v})$.

On the other hand, we have the following

$$||\vec{w}||^2 = (u_2v_3 - v_2u_3)^2 + (u_1v_3 - v_1u_3)^2 + (u_1v_2 - u_2v_1)^2$$
$$= (u_2^2v_3^2 - 2u_2v_3v_2u_3 + v_2^2u_3^2) + (u_1^2v_3^2 - 2u_1v_3v_1u_3 + v_1^2u_3^2)$$
$$+ (u_1^2v_2^2 - 2u_1v_2v_1u_1 + v_1^2u_2^2)$$
$$= u_2^2v_3^2 + v_2^2u_3^2 + u_1^2v_3^2 + v_1^2u_3^2 + u_1^2v_2^2 + v_1^2u_2^2$$
$$- 2u_2v_3v_2u_3 - 2u_1v_3v_1u_3 - 2u_1v_1v_2u_1$$
$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2$$
$$= ||\vec{u}||^2 ||\vec{v}||^2 - (\vec{u} \cdot \vec{v})^2$$

and

$$||\vec{u} \times \vec{v}||^2 = ||\vec{u}||^2 ||\vec{v}||^2 \sin^2 \theta$$
$$= ||\vec{u}||^2 ||\vec{v}||^2 (1 - \cos^2 \theta)$$
$$= ||\vec{u}||^2 ||\vec{v}||^2 - (\vec{u} \cdot \vec{v})^2.$$

We conclude that $\lambda = \pm 1$. But by checking the formula of $\vec{w}$ when $\vec{u}$ and $\vec{v}$ are the unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ we notice that $\vec{w}$ and $\vec{u} \times \vec{v}$ have the same direction. Hence, $\lambda = 1$ and so $\vec{u} \times \vec{v} = \vec{w}$.

From the above discussion, a convenient way to remember the algebraic cross product is by means of the determinant of the following matrix:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_2 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

Example 83.2
Find the cross product of $\vec{u} = 3\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{v} = 4\hat{i} + 9\hat{j} + 2\hat{k}$ and check that the cross product is perpendicular to both $\vec{u}$ and $\vec{v}$. 

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Solution.
We have
\[\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -3 & 1 \\ 4 & 9 & 2 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 9 & 2 \end{vmatrix} \hat{i} + \begin{vmatrix} 3 & -3 \\ 4 & 9 \end{vmatrix} \hat{j} + \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} \hat{k} = -15 \hat{i} - 2 \hat{j} + 39 \hat{k} \]

Moreover,
\[\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (3)(-15) - 3(-2) + 39 = 0\]
and
\[\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = (4)(-15) + 9(-2) + 2(39) = 0\]
Thus, \(\mathbf{u} \times \mathbf{v}\) is perpendicular to both \(\mathbf{u}\) and \(\mathbf{v}\).

Remark 83.1
The identity \(||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \cdot ||\mathbf{v}|| - (\mathbf{u} \cdot \mathbf{v})^2\) is known as Lagrange’s identity. Since \(||\mathbf{u}|| \cdot ||\mathbf{v}|| - (\mathbf{u} \cdot \mathbf{v})^2 \geq 0\) then we obtain the inequality
\[\mathbf{u} \cdot \mathbf{v} \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||.\]
This is known as Schwartz’s inequality.

The following theorem lists some of the properties of cross product.

Theorem 83.1
For vectors \(\mathbf{u}, \mathbf{v}, \mathbf{w}\) and scalar \(\lambda\) we have
(i) \(\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}\)
(ii) \((\lambda \mathbf{u}) \times \mathbf{v} = \lambda (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (\lambda \mathbf{v})\)
(iii) \(\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}\)
(iv) \(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}\)

Proof.
Let \(\mathbf{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}, \mathbf{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k},\) and \(\mathbf{w} = w_1 \hat{i} + w_2 \hat{j} + w_3 \hat{k}\).

(i) We have
\[\mathbf{u} \times \mathbf{v} = (u_2 v_3 - v_2 u_3) \hat{i} - (u_1 v_3 - u_3 v_1) \hat{j} + (u_1 v_2 - u_2 v_1) \hat{k}\]
and

$$\vec{v} \times \vec{u} = (v_2u_3 - u_2v_3)\vec{i} - (v_1u_3 - v_3u_1)\vec{j} + (v_1u_2 - v_2u_1)\vec{k}$$

Hence, $$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

(ii) We have

$$(\lambda \vec{u}) \times \vec{v} = [(\lambda u_2)v_3 - v_2(\lambda u_3)]\vec{i} - [(\lambda u_1)v_3 - (\lambda u_3)v_1]\vec{j} + [(\lambda u_1)v_2 - (\lambda u_2)v_1]\vec{k}$$

$$= \lambda [(u_2v_3 - v_2u_3)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}]$$

$$= \lambda (\vec{u} \times \vec{v})$$

(iii) We leave this to the reader.

(iv) We have

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (u_1\vec{i} + u_2\vec{j} + u_3\vec{k}) \cdot ((v_2w_3 - w_2v_3)\vec{i} - (v_1w_3 - v_3w_1)\vec{j} + (v_1w_2 - v_2w_1)\vec{k})$$

$$= u_1(v_2w_3 - w_2v_3) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1)$$

$$= (u_2v_3 - v_2u_3)w_1 - (u_1v_3 - u_3v_1)w_2 + (u_1v_2 - u_2v_1)w_3$$

$$= (\vec{u} \times \vec{v}) \cdot \vec{w}$$

Example 83.3 (The Equation of a Plane Through Three Points)
Find the equation for the plane through the points $$P_0 = (0, 1, -7), P_1 = (3, 1, -9),$$ and $$P_2 = (0, -5, -8).$$

Solution.
We have $$\overrightarrow{P_0P_1} = (3 - 0)\vec{i} + (1 - 1)\vec{j} + (-9 + 7)\vec{k} = 3\vec{i} - 2\vec{k}.$$ Similarly, $$\overrightarrow{P_0P_2} = -6\vec{j} - \vec{k}.$$ Thus, the vector $$\vec{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = -12\vec{i} + 3\vec{j} - 18\vec{k}$$ is normal to the plane at $$P_0.$$ Hence, the equation of the plane is

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

where $$P = (x, y, z).$$ Hence,

$$-12x + 3(y - 1) - 18(z + 7) = 0$$

or

$$-12x + 3y - 18z - 129 = 0$$

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Example 83.4
Find the area of the parallelogram with edges \( \vec{u} = 3\vec{i} - 3\vec{j} + \vec{k} \) and \( \vec{v} = 4\vec{i} + 9\vec{j} + 2\vec{k} \).

Solution.
From Example 83.2 we found that

\[
\vec{u} \times \vec{v} = -15\vec{i} - 2\vec{j} + 39\vec{k}.
\]

Thus

\[
\text{Area of Parallelogram} = ||\vec{u} \times \vec{v}|| = \sqrt{(-15)^2 + (-2)^2 + 39^2} = 5\sqrt{70}.
\]

Volume of a Parallelepiped
Consider the parallelepiped spanned by the vectors \( \vec{a}, \vec{b}, \vec{c} \) as shown in Figure 83.4

![Figure 83.4](image)

The volume of this figure is the area of the base times the height. We already know the area of the parallelogram base: \( ||\vec{a} \times \vec{b}|| \). The height is the component of \( \vec{c} \) in the direction normal to the base, i.e., in the direction of \( \vec{a} \times \vec{b} \). Hence the height is \( ||\vec{c}|| \cos \phi \) (the absolute value is needed for if \( \phi > \frac{\pi}{2} \) then \( \cos \phi < 0 \)).

The volume of the parallelepiped is therefore

\[
\text{Volume} = ||\vec{a} \times \vec{b}|| \cdot ||\vec{c}|| \cos \phi = ||(\vec{a} \times \vec{b}) \cdot \vec{c}||.
\]

But recall that the components of \( \vec{a} \times \vec{b} \) are

\[
\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}.
\]
Hence,

\[
(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix}
  a_2 & a_3 & c_1 \\
  b_2 & b_3 & c_2 \\
  a_1 & a_3 & c_3 \\
\end{vmatrix} - \begin{vmatrix}
  a_1 & a_3 & c_2 \\
  b_1 & b_3 & c_2 \\
  a_1 & a_2 & c_3 \\
\end{vmatrix} + \begin{vmatrix}
  a_1 & a_2 & c_1 \\
  b_1 & b_2 & c_1 \\
  a_1 & a_2 & c_3 \\
\end{vmatrix}
\]

and the volume of the parallelepiped is just the absolute value of this determinant.

**Remark 83.2**
A simple algebra shows that

\[
\begin{vmatrix}
  c_1 & c_2 & c_3 \\
  a_1 & a_3 & c_2 \\
  b_1 & b_3 & c_3 \\
\end{vmatrix} = \begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
\]

**Example 83.5**
Find the volume of the parallelepiped spanned by the vectors \( \vec{a} = (-2, 3, 1) \), \( \vec{b} = (0, 4, 0) \), and \( \vec{c} = (-1, 3, 3) \).

**Solution.**
Since

\[
(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix}
  -2 & 3 & 1 \\
  0 & 4 & 0 \\
  -1 & 3 & 3 \\
\end{vmatrix}
\]

\[
= -2(12 - 0) - 3(0 - 0) + 1(0 + 4) = -20
\]

then the volume of the parallelepiped is \( | -20 | = 20 \).
84 Derivatives of Functions of Two Variables: The Partial Derivative

Suppose we have to take the derivative of a function \( f(x, y) \). How does this differ from taking the derivative of a function \( f(x) \)?

You recall that the derivative of a function \( f(x) \) is defined in terms of limit as

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

provided the limit exists. The derivative measures the rate at which \( f(x) \) is changing at \( x \). Geometrically, \( f'(x) \) represents the slope of the tangent line of \( f(x) \) at \( x \) as shown in Figure 84.1.

![Slope of the tangent line](image)

Figure 84.1

Now, let \( z = f(x, y) \) and \( (a, b) \) be a point in its domain. The variable \( z \) is the dependent variable since it depends on both \( x \) and \( y \). The variables \( x \) and \( y \) are the independent variables and they are allowed to vary independently of each other. Thus, we study the influence of \( x \) and \( y \) seperately on the value of the function \( f(x, y) \) by holding one fixed and letting the other vary. In this case, one studies how fast the function is changing in either the \( x \)-direction or the \( y \)-direction.

In the \( x \)-direction we hold \( y \) fixed by setting \( y = b \). In this case, the function \( f(x, b) \) is a function in the variable \( x \), say \( f(x, b) = g(x) \). If \( g(x) \) is differentiable at \( x = a \) then we can write

\[
g'(a) = \lim_{h \to 0} \frac{g(a + h) - g(a)}{h} = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.
\]
Alternative notations for $g'(a)$ are

$$f_x(a, b) = \frac{\partial z}{\partial x}(a, b) = \frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$  

We call $f_x(a, b)$ the **partial derivative** of $f(x, y)$ with respect to $x$ at $(a, b)$. Similarly, the **partial derivative** of $f(x, y)$ with respect to $y$ at $(a, b)$ is defined by

$$f_y(a, b) = \frac{\partial z}{\partial y}(a, b) = \frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$  

**Partial Derivatives: Geometric Interpretation**

Geometrically, $f(x, b)$ is the cross section obtained when we cut the surface $f(x, y)$ by the vertical plane $y = b$. The slope of the tangent line to this cross-section at $x = a$ is the partial derivative $f_x(a, b)$ as shown in Figure 84.2. Likewise, the graph obtained by cutting the surface by the vertical plane $x = a$ has a tangent line at $y = b$ with slope $f_y(a, b)$.

![Figure 84.2](image)

**Estimating Partial Derivatives Using Contour Diagrams**

Numerical estimates of the partial derivatives of $f(x, y)$ at a point $(a, b)$ can be easily made from a contour diagram of the function $f(x, y)$. To estimate $f_x(a, b)$ move from the contour curve through $(a, b)$ in the $x-$direction and make note of the $\Delta x$ required until you hit the next contour curve. Let $\Delta z$ be the change in the $z-$values. Then $f_x(a, b)$ can be estimated using a difference quotient, i.e.,

$$f_x(a, b) \approx \frac{\Delta z}{\Delta x}.$$
Likewise, to estimate $f_x(a,b)$ move from the contour curve through $(a,b)$ in the $y$–direction and make note of the $\Delta y$ required until you hit the next contour curve. Let $\Delta z$ be the change in the $z$–values. Then
\[ f_y(a,b) \approx \frac{\Delta z}{\Delta y}. \]

**Example 84.1**
Consider the function $z = f(x, y) = x^2 + y^2$. A contour diagram is given in Figure 84.3. Estimate $f_x(1,0)$ and $f_y(0,1)$.

![Figure 84.3](image)

**Solution.**
From the figure we see that as $x$ goes from 1 to $\sqrt{2}$, the value of the function changes from 1 to 2. Thus,
\[ f_x(1,0) \approx \frac{\Delta z}{\Delta x} = \frac{1}{\sqrt{2} - 1} \approx 2.4. \]

Similarly, we see that as $y$ goes from 1 to $\sqrt{2}$, the value of the function changes from 1 to 2. Thus,
\[ f_y(0,1) \approx \frac{\Delta z}{\Delta y} = \frac{1}{\sqrt{2} - 1} \approx 2.4 \]

Note that as $x$ increases in the $x$–direction the value of the function increases indicating that $f_x(x,y) > 0$. A similar statement holds if we move in
the y—direction

**Estimating Partial Derivatives from Numeric Data**

Consider a rectangular plate of length 5m and width 3m. For any point 
\((x, y)\) in the rectangular plate we let \(z = T(x, y)\) represent the temperature
(in °C) at \((x, y)\). The following table provides some numeric values of \(T(x, y)\).

<table>
<thead>
<tr>
<th>y/x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>85</td>
<td>90</td>
<td>110</td>
<td>135</td>
<td>155</td>
<td>180</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>110</td>
<td>120</td>
<td>145</td>
<td>190</td>
<td>170</td>
</tr>
<tr>
<td>1</td>
<td>125</td>
<td>128</td>
<td>135</td>
<td>160</td>
<td>175</td>
<td>160</td>
</tr>
<tr>
<td>0</td>
<td>120</td>
<td>135</td>
<td>155</td>
<td>160</td>
<td>160</td>
<td>150</td>
</tr>
</tbody>
</table>

Let us estimate \(T_x(2, 1)\) and \(T_y(2, 1)\). We use difference quotients in the estimation. Thus, we can write

\[
T_x(2, 1) \approx \frac{1}{2} \left( \frac{T(2, 1) - T(1, 1)}{2 - 1} + \frac{T(3, 1) - T(2, 1)}{3 - 2} \right)
\]

\[
= \frac{1}{2} \left( \frac{135 - 128}{2 - 1} + \frac{160 - 135}{3 - 2} \right) = 16°C/m
\]

A similar computation shows that

\[
T_y(2, 1) \approx -17.5°C/m.
\]

We end this section by pointing out that partial derivatives of functions of three or more variables are found by the same method: Differentiate with respect to one variable, regarding the other variables as constants.
85 Algebraic Rules for Computing Partial Derivatives

In the previous section we introduced the definition of a partial derivative in terms of limits. From experience with functions in one variable, taking limits in finding derivatives was cumbersome and time consuming and for these reasons a set of algebraic rules were developed for computing derivatives.

Since the partial derivatives are computed individually by treating the function as if it depended on only one of the variables, with all the other variables considered constant, then one can use all the familiar algebraic rules of derivatives of single variable calculus in computing the partial derivatives of multivariable functions.

Example 85.1
The volume $V$ of a cone depends on the cone’s height $h$ and its radius $r$ according to the formula

$$V = \frac{\pi}{3} r^2 h.$$ 

Find $V_r$ and $V_h$.

Solution.
We have

$$V_r = \frac{\partial V}{\partial r} = \frac{2\pi}{3} rh$$

$$V_h = \frac{\partial V}{\partial h} = \frac{\pi}{3} r^2$$

Example 85.2
Find all the partial derivatives of the functions:

(a) $f(x, y) = x^4 + 6\sqrt{y}$
(b) $f(x, y, z) = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$
(c) $f(s, t) = t^7 \ln(s^2) + \frac{9}{t^3} - \sqrt{s^4}$
(d) $f(x, y) = \cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}$
(e) $f(u, v) = \frac{9u}{u^2+5v}$
(f) $f(x, y, z) = \frac{x\sin y}{z^2}$
(g) $f(x, y) = \sqrt{x^2 + \ln(5x - 3y^2)}$
Solution.
(a) We have
\[ \frac{\partial f}{\partial x} (x, y) = 4x^3 \]
\[ \frac{\partial f}{\partial y} (x, y) = \frac{3}{\sqrt{y}} \]

(b) We have
\[ \frac{\partial f}{\partial x} (x, y) = 2xy + 43 \]
\[ \frac{\partial f}{\partial y} (x, y) = x^2 - 20yz^3 - \frac{28}{1 + 16y^2} \]
\[ \frac{\partial f}{\partial z} (x, y) = -30y^2z^2 \]

(c) We have
\[ \frac{\partial f}{\partial s} (s, t) = \frac{2t^7}{s} - \frac{4}{t} s^{-\frac{3}{2}} \]
\[ \frac{\partial f}{\partial t} (s, t) = 7t^6 \ln (s^2) - \frac{27}{t^4} \]

(d) We have
\[ \frac{\partial f}{\partial x} (x, y) = \frac{4}{x^2} \sin \left( \frac{4}{x} \right) e^{x^2y-5y^3} + \cos \left( \frac{4}{x} \right) e^{x^2y-5y^3}(2xy) \]
\[ \frac{\partial f}{\partial y} (x, y) = \cos \left( \frac{4}{x} \right) e^{x^2y-5y^3}(x^2 - 15y^2) \]

(e) We have
\[ \frac{\partial f}{\partial u} (u, v) = \frac{9(u^2 + 5v) - 9u(2u)}{(u^2 + 5v)^2} = \frac{-9u^2 + 45v}{(u^2 + 5v)^2} \]
\[ \frac{\partial f}{\partial v} (u, v) = \frac{-45u}{(u^2 + 5v)^2} \]
(f) We have
\[
\frac{\partial f}{\partial x}(x, y, z) = \sin y \frac{1}{z^2}
\]
\[
\frac{\partial f}{\partial y}(x, y, z) = \frac{x \cos y}{z^2}
\]
\[
\frac{\partial f}{\partial z}(x, y, z) = -\frac{2x \sin y}{z^3}
\]

(g) We have
\[
\frac{\partial f}{\partial x}(x, y) = \frac{1}{2} \left( 2x + \frac{5}{5x - 3y^2} \right) \left( x^2 + \ln (5x - 3y^2) \right)^{-\frac{1}{2}}
\]
\[
\frac{\partial f}{\partial y}(x, y) = -\frac{3y}{5x - 3y^2} \left( x^2 + \ln (5x - 3y^2) \right)^{-\frac{1}{2}}
\]

Example 85.3
Let \( f(x, y) = e^{3x} \cos y \). Compute \( f_x(0, 2\pi) \).

Solution.
Since
\[
f_x(x, y) = 3e^{3x} \cos y
\]
then
\[
f_x(0, 2\pi) = 3 \cos (2\pi) = 3
\]
Local Linearity of Multivariable Functions

For single variable functions, local linearity is a property of differentiable functions that roughly says that if you zoom in on a point on the graph of the function (with equal scaling horizontally and vertically), the graph will eventually look like a straight line with a slope equal to the derivative of the function at the point. The advantage of this property is that function values at nearby points can be estimated using the tangent line, i.e., a linear function.

The same is true for functions in two variables. A function in two variables is differentiable at \((a,b)\) if whenever we zoom in over sufficiently small region around a point \((a,b)\) on the surface of the function we see that the surface is almost a plane containing \((a,b)\). Thus, a differentiable function of two variables is well approximated locally by a tangent plane, i.e., a linear function in the variables \(x\) and \(y\).

What is the equation of the tangent plane? Let \(z = L(x,y)\) be the equation of this plane. Then \(z = \alpha x + \beta y + \gamma\) with \(\alpha\) being the \(x\)-slope, i.e. \(\alpha = f_x(a,b)\) and \(\beta\) is the \(y\)-slope, i.e. \(\beta = f_y(a,b)\). But \(\Delta z = z - f(a,b) = \alpha \Delta x + \beta \Delta y\).

Hence, from this we conclude that

\[
z = f_x(a,b)(x - a) + f_y(a,b)(y - b) + f(a,b).
\]

Figure 86.1 shows the graph of a function with a tangent plane.

![Figure 86.1](image_url)
Find the equation of the tangent plane to the surface \( z = 4x^3y^2 + 2y \) at the point \((1, -2, 12)\).

**Solution.**
Since \( f_x(x, y) = 12x^2y^2 \) and \( f_y(x, y) = 8x^3y + 2 \) then \( f_x(1, -2) = 48 \) and \( f_y(1, -2) = -14 \). Also, \( f(1, -2) = 12 \). Hence, the equation of the tangent plane is given by

\[
z = 48(x - 1) - 14(y + 2) + 12
\]

or

\[
z = 48x - 14y - 64
\]

Now as we mentioned above, the tangent plane is a "good" approximation of \( f(x, y) \) locally at the point \((a, b)\). Hence we can write:

\[
f(x, y) \approx f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).
\]

This equation is sometimes referred to as a **local linearization** of \( f(x, y) \) at the point \((a, b)\).

**Example 86.2**

Find the local linearization of \( f(x, y) = 4x^3y^2 + 2y \) at the point \((1, -2, 12)\). Estimate \( f(0.9, -2.1) \) using this linearization and compare your answer to the true value.

**Solution.**

By the previous example we have

\[
f(x, y) \approx 48x - 14y - 64.
\]

Thus,

\[
f(0.9, -2.1) \approx 48(0.9) - 14(-2.1) - 64 = 8.6.
\]

The true value is

\[
f(0.9, -2.1) = 4(0.9)^3(-2.1)^2 + 2(-2.1) = 8.65956
\]

Although we have confined our discussion and examples to functions of two variables, the above arguments are easily extended to functions having three or more variables. In particular, for a function of three variables \( f(x, y, z) \), the local linearization at \((a, b, c)\) becomes:

\[
f(x, y, z) \approx f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) + f(a, b, c).
\]
The Differential of $z = f(x, y)$

The formula for the local linearization can be used to estimate the change in the value of the function locally:

$$f(x, y) \approx f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

or

$$f(x, y) - f(a, b) \approx f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

By letting $\Delta f = f(x, y) - f(a, b)$, $\Delta x = x - a$, and $\Delta y = y - b$ we can write

$$\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y.$$  

For arbitrary $\Delta x$ and $\Delta y$ we can introduce the function

$$df = f_x(a, b)dx + f_y(a, b)dy.$$  

This function is called the **differential** of $f$ at $(a, b)$. Note that the differential of $f$ is a linear function of the variables $dx$ and $dy$. $df$ represents the change in $f$ locally due to changes in both $x$ and $y$.

**Example 86.3**

Compute the differential of $f(x, y) = 4x^3y^2 + 2y$ at the point $(1, -2, 12)$.

**Solution.**

We have

$$df = f_x(x, y)dx + f_y(x, y)dy = 12x^2y^2dx + (8x^3y + 2)dy.$$  

The differential at $(1, -2)$ is

$$df = 48dx - 14dy.$$  

Thus if we were to move by $dx = -0.1$ and $dy = -0.1$ i.e to $(0.9, -2.1)$, the change in $f$ is:

$$df = 48(-0.1) - 14(-0.1) = -3.2.$$  

Note that the actual change is

$$f(0.9, -2.1) - f(1, -2) = 8.65956 - 12 = -3.34044.$$
Directional Derivatives and Gradients of Functions of Two Variables

For a function \( z = f(x, y) \), the partial derivative at the point \((a, b)\) with respect to \(x\) gives the rate of change of \(f\) in the \(x\) direction (i.e., in the direction of \(\vec{i}\)) and the partial derivative with respect to \(y\) gives the rate of change of \(f\) in the \(y\) direction (or the direction of \(\vec{j}\)). How do we compute the (instantaneous) rate of change of \(f\) in an arbitrary direction?

In the case of \(f_x(a, b)\) for example, one starts with the point \(P = (a, b)\) and moves to the point \(Q = (a + h, b)\) so that \(\overrightarrow{PQ} = h\vec{i}\). Thus, \(f_x(a, b)\) is the limit as \(h\) approaches zero of the change of \(z\)–values from \(P\) to \(Q\) divided by the distance from \(P\) to \(Q\). We will apply the same idea for an arbitrary direction. Suppose we want to find the derivative of \(f(x, y)\) at \((a, b)\) in the direction of the unit vector \(\vec{u} = u_1\vec{i} + u_2\vec{j}\). For \(h > 0\) we consider the point on the support of \(\vec{u}\) given by \(Q = (a + hu_1, b + hu_2)\) as shown in Figure 87.1. Thus, as in the case of functions of one variable we have

\[
\text{Average rate of change in } f \text{ from } P \text{ to } Q = \frac{\text{change in } f}{\text{Distance from } P \text{ to } Q} = \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}
\]

Figure 87.1

Taking \(h \to 0\) we obtain the instantaneous rate of change at \((a, b)\) in the direction of \(\vec{u}\) which we call the **directional derivative** of \(f\) at \((a, b)\) in the direction of \(\vec{u}\) and is given by

\[
f_{\vec{u}}(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}
\]
provided the limit exists. Evidently the directional derivatives in the directions \( \vec{i} \) and \( \vec{j} \) are the partial derivatives \( f_x(a, b) \) and \( f_y(a, b) \). For example, if \( \vec{u} = \vec{i} \) then \( u_1 = 1 \) and \( u_2 = 0 \) so that

\[
f_{\vec{u}}(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h} = f_x(a, b).
\]

We note that for a general vector \( \vec{v} \), i.e. that is not a unit vector, we can still find the directional derivative in the direction specified by \( \vec{v} \) by constructing the unit vector \( \vec{u} = \frac{\vec{v}}{||\vec{v}||} \) and using this unit vector to compute the directional derivative.

The following example illustrates how to calculate the directional derivative from its definition.

**Example 87.1**

Find the directional derivative of \( f(x, y) = x^2 + y^2 \) at \((1, 0)\) in the direction of \( \vec{v} = \vec{i} + \vec{j} \).

**Solution.**

First we consider the unit vector

\[
\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{1}{\sqrt{2}} \vec{i} + \frac{1}{\sqrt{2}} \vec{j}.
\]

Then

\[
f_{\vec{u}}(1, 0) = \lim_{h \to 0} \frac{f(1 + h/\sqrt{2}, h/\sqrt{2}) - f(1, 0)}{h} = \lim_{h \to 0} \frac{(1 + h/\sqrt{2})^2 + (h/\sqrt{2})^2 - 1}{h}
= \lim_{h \to 0} \frac{h\sqrt{2} + h^2}{h} = \lim_{h \to 0} (\sqrt{2} + h) = \sqrt{2} \]

**Geometric Interpretation of the Directonal Derivative**

Let \( \vec{u} \) be a unit vector with tail \((a, b)\). The plane which contains \( \vec{u} \) and is perpendicular to the \( xy \)-plane intersects the surface \( z = f(x, y) \) in a curve \( C \). The directional derivative \( f_{\vec{u}}(a, b) \) is the slope of the line tangent to \( C \) at the point \((a, b, f(a, b))\) as shown in Figure 87.2.
Estimating Directional Derivatives From a Contour Diagram

The rule for estimating a directional derivative at a point in a certain direction from a contour diagram is similar to that for estimating partial derivatives from a contour diagram.

To estimate the directional derivative $f_{\vec{u}}(a, b)$ move from the contour line through $P = (a, b)$ in the $\vec{u}$ direction until you hit the next contour line, say at a point $Q$. Then the directional derivative in the direction $\vec{u}$ is approximately

$$f_{\vec{u}}(a, b) \approx \frac{\Delta \text{contour}}{\text{distance from P to Q}}.$$  

Example 87.2

Figure 87.3 shows the contour diagram of a function $f(x, y)$. Estimate the directional derivative of $f$ at the point $P$ in the direction of $\vec{v}$ where $||\vec{v}|| = 0.8$.  

Solution.
We have
\[ f_{\vec{v}}(P) \approx \frac{1}{0.8} = 1.25 \]

As with regular derivatives, we would like to have an algebraic procedure for computing the directional derivative as opposed to computing the limit. We can find such a procedure by applying the local linearity property at \((a, b)\).

Thus, for small \(h\) we can write
\[
 f(a + hu_1, b + hu_2) - f(a, b) \approx f_x(a, b)hu_1 + f_y(a, b)hu_2
\]
or
\[
 \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} \approx f_x(a, b)u_1 + f_y(a, b)u_2.
\]
The above approximation becomes exact as \(h \to 0\) to obtain
\[
 f_{\vec{u}}(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2.
\]

Example 87.3
Use the above formula to find the directional derivative of \(f(x, y) = x^2 + y^2\) at \((1, 0)\) in the direction of \(\vec{v} = \vec{i} + \vec{j}\).

Solution.
From Example 87.1, \(\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\). Thus,
\[
 f_{\vec{u}}(1, 0) = \frac{1}{\sqrt{2}}(f_x(1, 0) + f_y(1, 0)) = \frac{2}{\sqrt{2}} = \sqrt{2}
\]

Now, notice that formula (1) can be written as the dot product of two vectors
\[
 f_{\vec{u}}(a, b) = (f_x(a, b)\vec{i} + f_y(a, b)\vec{j}) \cdot (u_1\vec{i} + u_2\vec{j}).
\]

We call the vector
\[
 \nabla f(a, b) = f_x(a, b)\vec{i} + f_y(a, b)\vec{j}
\]
the **gradient** of \(f\) at \((a, b)\). Hence, with this definition we can write
\[
 f_{\vec{u}}(a, b) = \nabla f(a, b) \cdot \vec{u}.
\]
Example 87.4
Find $\nabla f(3, 2)$ where $f(x, y) = x^2y$.

Solution.
We have

\[
\nabla f(x, y) = f_x(x, y)\hat{i} + f_y(x, y)\hat{j} = 2xy\hat{i} + x^2\hat{j}
\]

Hence,

\[
\nabla f(3, 2) = 12\hat{i} + 9\hat{j} \quad \blacksquare
\]

Interpretations of the Gradient
Now, since $f(a, b) = \nabla f(a, b) \cdot \vec{u}$, the definition of the dot product allows us to write

\[
f(a, b) = ||\nabla f(a, b)|| \cdot ||\vec{u}|| \cdot \cos \theta = ||\nabla f(a, b)|| \cdot \cos \theta
\]

where $\theta$ is the angle between $\nabla f(a, b)$ and $\vec{u}$. It follows that $f(a, b)$ is maximum when $\cos \theta = 1$, that is when $\theta = 0$. This means that the maximum rate of change of $f(x, y)$ occurs in the direction of the gradient and the maximum value of $f(a, b)$ is $||\nabla f(a, b)||$. In other words, the gradient points in the direction of increasing $f$.

Similarly, $f(a, b)$ is minimum when $\cos \theta = -1$ that is when $\theta = \pi$. This means that the minimum rate of change occurs in the direction opposite to $\nabla f(a, b)$ with minimum value equals to $-||\nabla f(a, b)||$.

Notes that $f(a, b) = 0$ when the vectors $\vec{u}$ and $\nabla f(a, b)$ are orthogonal.

Example 87.5
Find the maximum value of the directional derivative at the point $(1, 1, 7)$ for the function $f(x, y) = 9 - x^2 - y^2$.

Solution.
Since $\nabla f(x, y) = -2x\hat{i} - 2y\hat{j}$ and $\nabla f(1, 1) = -2\hat{i} - 2\hat{j}$ then the maximum value for the directional derivative at the point $(1, 1, 7)$ will be in the direction of the vector $-2\hat{i} - 2\hat{j}$ and with maximum value equals $||\nabla f(1, 1)|| = 2 \quad \blacksquare$

Example 87.6
Use the gradient to find the directional derivative of $f(x, y) = x^3y^4$ at the point $(6, -1)$ in the direction of the vector $\vec{v} = 2\hat{i} + 5\hat{j}$.
Solution.
We first compute the gradient vector at \((6, -1)\):

\[
\nabla f(x, y) = 3x^2y^4\mathbf{i} + 4x^3y^3\mathbf{j}
\]

and

\[
\nabla f(6, -1) = 108\mathbf{i} - 864\mathbf{j}.
\]

Since \(||\vec{v}|| = \sqrt{29}\) then \(\vec{v}\) is not a unit vector, the unit vector in the direction of \(\vec{v}\) is

\[
\vec{u} = \frac{2}{\sqrt{29}} \mathbf{i} + \frac{5}{\sqrt{29}} \mathbf{j}.
\]

Therefore, we have

\[
f_{\vec{u}}(6, -1) = \nabla f(-6, 1) \cdot \vec{u} = \frac{216}{\sqrt{29}} - \frac{4320}{\sqrt{29}} = -\frac{4104}{\sqrt{29}} \]}

The derivative in the direction tangent to a contour curve is always zero as the following theorem shows.

**Theorem 87.1 (Normal Property of the Gradient)**

Let \(f(x, y)\) be a differentiable function at \((a, b)\) such that \(\nabla f(a, b) \neq \vec{0}\). Then \(\nabla f(a, b)\) is perpendicular to the contour curve of \(f\) through \((a, b)\).

**Remark 87.1**

We know that the rate of change is the largest in the direction of the gradient. According to the previous theorem, the shortest path from a point on a contour curve to the next contour curve gives the greatest rate of change.

**Example 87.7**

Sketch the contour curve corresponding to \(c = 1\) for the function \(f(x, y) = x^2 - y^2\) and find a normal vector at the point \(P(2, \sqrt{3})\).

**Solution.**

The contour curve is a hyperbola given by the equation \(x^2 - y^2 = 1\) as shown in Figure 87.4. The gradient vector is perpendicular to the contour curve. Since

\[
\nabla f(x, y) = 2x\mathbf{i} - 2y\mathbf{j}
\]

then

\[
\nabla f(2, \sqrt{3}) = 4\mathbf{i} - 2\sqrt{3}\mathbf{j}
\]
is the required normal vector

![Figure 87.4](image)

**Example 87.8**

Find the tangent line to $x^2 + 2y^2 = 22$ at the point $(2, 3)$.

**Solution.**

Let $f(x, y) = x^2 + 2y^2$. Then $\nabla f(x, y) = 2xi + 4yj$ and $\nabla f(2, 3) = 4i - 12j$.

Let $(x, y)$ be a point on the tangent line to contour curve $f(x, y) = 22$ at $(2, 3)$. Then $\nabla f(2, 3)$ is perpendicular to the vector $\vec{v} = (x - 2)i + (y - 3)j$.

That is,

$$\nabla f(2, 3) \cdot \vec{v} = 0.$$  

This implies

$$4(x - 2) + 12(y - 3) = 0.$$  

Simplifying this last equation to obtain

$$x + 3y = 11.$$
88 Directional Derivatives and Gradients of Functions of Three Variables

In the previous section we defined the gradient and directional derivatives of functions of two variables. The results of that section also apply to functions having three or more variables. If \( f(x, y, z) \) is differentiable at \((a, b, c)\) then the directional derivative of \( f(x, y, z) \) at \((a, b, c)\) in the direction of the unit vector \( \vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k} \) is given by

\[
f_{\vec{u}}(a, b, c) = f_x(a, b, c)u_1 + f_y(a, b, c)u_2 + f_z(a, b, c)u_3.
\]

Example 88.1
Find the directional derivative of \( f(x, y, z) = x^2z + y^3z^2 - xyz \) in the direction of \( \vec{v} = -\vec{i} + 3\vec{k} \).

Solution.
Since \( ||\vec{v}|| = \sqrt{10} \) then \( \vec{v} \) is not a unit vector. Thus, we let

\[
\vec{u} = -\frac{1}{\sqrt{10}} \vec{i} + \frac{3}{\sqrt{10}} \vec{k}.
\]

Hence,

\[
f_{\vec{u}}(x, y) = f_x(x, y, z)u_1 + f_y(x, y, z)u_2 + f_z(x, y, z)u_3
\]

\[
= -\frac{1}{\sqrt{10}}(2xz - yz) + \frac{3}{\sqrt{10}}(x^2 + 2y^3z - xy)
\]

As in the case of functions of two variables, we can write

\[
f_{\vec{u}}(a, b, c) = \nabla f(a, b, c) \cdot \vec{u}
\]

where we define the gradient of \( f(x, y, z) \) at \((a, b, c)\) to be the vector

\[
\nabla f(a, b, c) = f_x(a, b, c)\vec{i} + f_y(a, b, c)\vec{j} + f_z(a, b, c)\vec{k}.
\]

Using the same arguments as we did in the previous lecture we see that the gradient has the following properties in space

**Theorem 88.1**
If \( f \) is differentiable at \((a, b, c)\) with \( \nabla f(a, b, c) \neq 0 \), then

(i) \( f \) increases the most in the direction of \( \nabla f(a, b, c) \)
(ii) \( \nabla f(a, b, c) \) is perpendicular to the level surface of \( f \) at \((a, b, c)\)
(iii) \( ||\nabla f(a, b, c)|| \) is the maximum rate of change of \( f \) at \((a, b, c)\).
Example 88.2
Let \( f(x, y, z) = x^2 + y^2 \). Describe the directions of the vectors \( \nabla f(0, 1, 1) \) and \( \nabla f(1, 0, 1) \).

Solution.
We note that both points \((0, 1, 1)\) and \((1, 0, 1)\) satisfy the equation \( x^2 + y^2 = 1 \) which is the level surface of \( f \) corresponding to the value 1. This is the equation of a cylinder and since \( f \) does not depend on \( z \), the gradients will only have components in the \( x \) and \( y \) directions i.e. are horizontal. The gradients are perpendicular to the surface of the cylinder and point outward because the value of \( f \) increases in the outward direction as shown in Figure 88.1(a).

![Figure 88.1](image)

Example 88.3
Let \( f(x, y, z) = -(x^2 + y^2 + z^2) \). Describe the directions of the vectors \( \nabla f(0, 1, 1) \) and \( \nabla f(1, 0, 1) \).

Solution.
Both points belong to the level surface with equation \( -(x^2 + y^2 + z^2) = -2 \)
which is the sphere $x^2 + y^2 + z^2 = 2$. The gradient is again perpendicular to
the surface and points inward since the negative signs mean that the function
increases (from negative numbers with large absolute values to negative
values numbers with small absolute values) in the inward direction as shown
in Figure 88.1(b)

**Example 88.4**

Let $f(x, y) = 4 - x^2 - 2y^2$. Compute the vector perpendicular to
(a) the level curve at $(1, 1)$
(b) the surface $z = f(x, y)$ at the point $(1, 1, 1)$

**Solution.**

(a) Since $f(1, 1) = 1$ then $(1, 1)$ belongs to the level curve $x^2 + 2y^2 = 3$ which
is an ellipse. The gradient is

$$\nabla f(1, 1) = -2\vec{i} - 4\vec{j}$$

which is the required normal vector.

(b) Letting $f(x, y, z) = 4 - x^2 - 2y^2 - z$. Then $f(1, 1, 1) = 0$ so that $(1, 1, 1)$
belongs to the level surface $f(x, y, z) = 0$. Thus,

$$\nabla f(1, 1, 1) = -2\vec{i} - 4\vec{j} - \vec{k}$$

is the required normal vector

**Tangent Plane to a Level Surface**

We now take a look at tangent planes for functions of three variables. Suppose
that $f(x, y, z)$ is differentiable at $(a, b, c)$. Then the vector

$$\nabla f(a, b, c) = f_x(a, b, c)\vec{i} + f_y(a, b, c)\vec{j} + f_z(a, b, c)\vec{k}$$

is perpendicular to the tangent plane to the level surface of $f(x, y, z)$ at
$(a, b, c)$. If $(x, y, z)$ is any point in the tangent plane then $\nabla f(a, b, c)$ is per-
pendicular to the vector from $(a, b, c)$ to $(x, y, z)$. This implies that

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0$$

which is the equation of the tangent plane to the level surface $f$ at the point
$(a, b, c)$. 81
Example 88.5
Find the equation to the tangent plane to the sphere \( x^2 + y^2 + z^2 = 14 \) at the point \((1, 2, 3)\).

Solution.
Let \( f(x, y, z) = x^2 + y^2 + z^2 \). Then the given sphere is the level surface \( f(x, y, z) = 14 \). Since

\[
\nabla f(x, y, z) = 2x \vec{i} + 2y \vec{j} + 2z \vec{k}
\]

the normal vector to the tangent plane at \((1, 2, 3)\) is

\[
\nabla f(1, 2, 3) = 2\vec{i} + 4\vec{j} + 6\vec{k}.
\]

Hence, the equation of the tangent plane is

\[
2(x - 1) + 4(y - 2) + 6(z - 3) = 0
\]

or

\[
x + 2y + 3z = 14 \quad \blacksquare
\]
89 Chain Rule for Functions of Two Variables

From single variable calculus we recall that the chain rule is used to differentiate composite functions of the form $f(g(x))$. Is there a version of the chain rule for functions in two variables?

For functions of several variables, composite functions can be generated in a variety of ways. Consider the following two examples:

- Let $z = f(x, y), x = x(t), y = y(t)$ then we can form the composite function $z = f(x(t), y(t))$.
- Let $z = f(x, y), x = x(s, t), y = y(s, t)$ then we can form the composite function $z = f(x(s, t), y(s, t))$.

In the first case, using local linearity we can write

$$\Delta z \approx f_x(x, y) \Delta x + f_y(x, y) \Delta y$$

Divide through by $h \neq 0$ to obtain

$$\frac{\Delta z}{h} \approx f_x(x, y) \frac{\Delta x}{h} + f_y(x, y) \frac{\Delta y}{h}.$$  

As $h \to 0$ the above approximation becomes exact obtaining

$$\frac{dz}{dt} = f_x(x, y) \frac{dx}{dt} + f_y(x, y) \frac{dy}{dt}.$$  

This formula is known as the chain rule.

**Example 89.1**

Compute $\frac{dz}{dt}$ for $z = f(x, y) = xe^{xy}$ with $x(t) = t^2$ and $y(t) = t^{-1}$.

**Solution.**

Using the formula for the chain rule we find

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$$= (e^{xy} + xye^{xy})(2t) + (x^2 e^{xy})(-t^{-2})$$

$$= 2t(e^{xy} + xye^{xy}) - t^{-2} x^2 e^{xy}$$

$$= 2te^t(1 + t) - t^2 e^t = te^t(2 + t) \blacklozenge$$
Next, we consider a function \( z = f(x,y) \) where \( x = x(s,t) \) and \( y = y(s,t) \). Thus, \( z \) is a function of the variables \( s \) and \( t \). What are \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \)?

First, let us fix \( t \). Then \( z = f(x(s,t), y(s,t)) \) depends on \( s \) alone, and so we apply the chain rule above to write

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}.
\]

Similarly, we can write

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}.
\]

**Example 89.2**

If \( z = e^x \sin y \), \( x = st^2 \), and \( y = s^2 t \), find \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \).

**Solution.**

Applying the chain rule we obtain

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (e^x \sin y)(2st^2) + (e^x \cos y)(2st) = t^2e^{st^2} \sin (s^2 t) + 2ste^{st^2} \cos (s^2 t)
\]

Similarly,

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2) = 2ste^{st^2} \sin (s^2 t) + s^2e^{st^2} \cos (s^2 t)
\]
Second Order Partial Derivatives

Just as we had higher order derivatives with functions of one variable we will also have higher order derivatives of functions of more than one variable. Consider the case of a function of two variables \( z = f(x, y) \), since both of the first order partial derivatives are also functions of \( x \) and \( y \) we could in turn differentiate each with respect to \( x \) or \( y \). This means that for the case of a function of two variables there will be a total of four possible second order derivatives. The second order partial derivatives are

\[
(f_x)_x = f_{xx} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}
\]
\[
(f_x)_y = f_{xy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}
\]
\[
(f_y)_x = f_{yx} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}
\]
\[
(f_y)_y = f_{yy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}
\]

The partial derivatives \( f_{xy} \) and \( f_{yx} \) are called mixed partial derivatives.

**Example 90.1**
Find \( f_{xx}, f_{yy}, f_{xy} \) and \( f_{yx} \) given that \( f(x, y) = \sin xy \).

**Solution.**
We have

\[
f(x, y) = y \cos xy
\]
\[
f_{xx}(x, y) = -y^2 \sin xy
\]
\[
f_{xy}(x, y) = \cos xy - xy \sin xy
\]
\[
f_{yx}(x, y) = x \cos xy
\]
\[
f_{yy}(x, y) = -x^2 \sin xy
\]
\[
f_{yx}(x, y) = \cos xy - xy \sin xy
\]

**Example 90.2**
Find \( f_{xx}, f_{yy}, f_{xy} \) and \( f_{yx} \) given that \( f(x, y) = x^3 + 2xy \).
Solution.
We have

\[ f_x(x, y) = 3x^2 + 2y \]
\[ f_{xx}(x, y) = 6x \]
\[ f_{xy}(x, y) = 2 \]
\[ f_y(x, y) = 2x \]
\[ f_{yy}(x, y) = 0 \]
\[ f_{yx}(x, y) = 2 \]

Example 90.3
Use the contour diagram shown in Figure 90.1 to decide the sign (positive, negative, or zero) of each of the following partial derivatives at the point \( P \). Assume the \( x \)- and \( y \)-axes are in the usual positions.
(a) \( f_x(P) \)  (b) \( f_y(P) \)  (c) \( f_{xx}(P) \)  (d) \( f_{yy}(P) \)  (e) \( f_{xy}(P) \).

![Contour diagram](image)

Solution.
(a) \( f_x(P) < 0 \) since \( f \) decreases as we go to the right.
(b) \( f_y(P) > 0 \) since \( f \) increases as we go up.
(c) \( f_{xx}(P) > 0 \) : Since the level curves are further apart to the right, \( f_x \) is changing from a negative number with larger magnitude to a negative number with smaller magnitude, that is, \( f_x \) is increasing.
(d) \( f_{yy}(P) > 0 \) : Since the level curves are closer as we move up, \( f_y \) is changing from a smaller positive number to a larger positive number, that is, \( f_y \) is increasing.
(e) \( f_{xy}(P) < 0 \) : As we move up \( f_x \) changes from a negative number with
smaller magnitude to a negative number with larger magnitude, that is, \( f_x \) is decreasing. ■

Observe that in the first two examples of this section the mixed partials \( f_{xy} \) and \( f_{yx} \) are equal, i.e. \( f_{xy} = f_{yx} \). This is a general result as given by the theorem below.

**Theorem 90.1**
If the mixed partial derivatives \( f_{xy} \) and \( f_{yx} \) are continuous at a point \((a, b)\) then

\[
f_{xy}(a, b) = f_{yx}(a, b).
\]

**Proof.**
For \( h \neq 0 \) small we define the function

\[
F(h) = [f(a + h, b + h) - f(a, b + h)] - [f(a, b + h) - f(a, b)].
\]

If we let \( g(x) = f(x, b + h) - f(x, b) \) then the previous equality can be expressed in terms of \( g \) as follows

\[
F(h) = g(a + h) - g(a).
\]

By the Mean Value Theorem, there is a number \( c \) between \( a \) and \( a + h \) such that

\[
g(a + h) - g(a) = g'(c)h = h[f_x(c, b + h) - f_x(c, b)].
\]

Applying the Mean Value Theorem again, this time to \( f_x \) we get a number \( d \) between \( b \) and \( b + h \) such that

\[
f_x(c, b + h) - f_x(c, b) = f_{xy}(c, d)h.
\]

Combining these equations, we obtain

\[
F(h) = h^2 f_{xy}(c, d).
\]

If \( h \to 0 \) then \((c, d) \to (a, b)\), so the continuity of \( f_{xy} \) at \((a, b)\) gives

\[
\lim_{h \to 0} \frac{F(h)}{h^2} = \lim_{(c, d) \to (a, b)} f_{xy}(c, d) = f_{xy}(a, b).
\]

Similarly, by writing

\[
F(h) = [f(a + h, b + h) - f(a, b+h)] - [f(a + h, b) - f(a, b)]
\]

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and using the Mean Value Theorem twice and the continuity of $f_{yx}$ at $(a, b)$ we obtain
\[ \lim_{h \to 0} \frac{F(h)}{h^2} = \lim_{(c,d) \to (a,b)} f_{yx}(c,d) = f_{yx}(a,b). \]

It follows that $f_{xy}(a, b) = f_{yx}(a, b)$ as desired.

**Quadratic Approximation: Taylor Polynomials for Functions of Two Variables**

In Section 86, we saw how to approximate $f(x, y)$ by a linear function, i.e. its local linearization. This approximation can be improved by using second order partial derivatives to construct a quadratic Taylor approximation for the function.

Recall that for functions of a single variable, we can approximate the function $f(x)$ using the Taylor expansion:

\[ f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots \]

for $x$ near $a$. In particular the first order Taylor approximation gives:

\[ f(x) \approx f(a) + f'(a)(x - a), \]

for $x$ near $a$, which is equivalent to the tangent line to $f$ at the point $a$. The second order approximation:

\[ f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 \]

for $x$ near $a$ which gives an improved approximation to $f$ near $a$ when compared to the first order or linear approximation.

For functions of two variables we already discussed the linear approximation or local linearization of $f$:

\[ f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \]

which is a tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$. This linearization is also called the **first degree Taylor polynomial** of $f$ at $(a, b)$.

Just as in the single variable case, we can use the second order partial derivatives to create an improved second-order approximation for $f$ at $(a, b)$ using
a polynomial of degree 2 in $x$ and $y$ around $(a, b)$, i.e. an expression of the form

$$p(x, y) = a_0 + a_1(x-a) + a_2(y-b) + a_3(x-a)(y-b) + a_4(x-a)^2 + a_5(y-b)^2.$$  

A simple algebra similar to the single variable case (see Section 61) leads to

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2} f_{xx}(a, b)(x - a)^2 + \frac{1}{2} f_{yy}(a, b)(y - b)^2.$$  

The right-hand polynomial is called the **second degree Taylor polynomial** of $f$ at $(a, b)$.

**Example 90.4**

Find the second order Taylor approximation for $f(x, y) = \frac{1}{xy}$ at the point $(1, 2)$.

**Solution.**

First we must compute the partial derivatives and evaluate them at $(1, 2)$:

$$f(x, y) = \frac{1}{xy}, \quad f(1, 2) = \frac{1}{2}$$

$$f_x(x, y) = -\frac{1}{x^2y}, \quad f_x(1, 2) = -\frac{1}{2}$$

$$f_y(x, y) = -\frac{1}{xy^2}, \quad f_y(1, 2) = -\frac{1}{4}$$

$$f_{xx}(x, y) = \frac{2}{x^3y}, \quad f_{xx}(1, 2) = 1$$

$$f_{xy}(x, y) = \frac{1}{x^2y^2}, \quad f_{xy}(1, 2) = \frac{1}{4}$$

$$f_{yy}(x, y) = \frac{2}{xy^3}, \quad f_{yy}(1, 2) = \frac{1}{4}$$

So the quadratic Taylor polynomial for $f$ near $(1, 2)$ is given by

$$f(x, y) \approx \frac{1}{2} - \frac{x - 1}{2} - \frac{y - 2}{4} + \frac{(x - 1)^2}{2} + \frac{(x - 1)(y - 2)}{4} + \frac{(y - 2)^2}{8}$$
91  Differentiability and Some of its Properties

A function of one variable is differentiable at a point if its derivative has a value at that point. However, the term differentiable is not used the same for functions of several variables; in particular, a function of several variables is not necessarily differentiable if its partial derivatives have values at a given point.

Recall from Section 86 that a function of two variables having a tangent plane \((a,b)\) is equivalent to a function being differentiable there. More formally we define differentiability as follows:

A function \(z = f(x,y)\) is said to be differentiable at \((a,b)\) if \(f(x,y)\) can be expressed in the form

\[
f(x,y) = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b) + E(x,y)
\]

with the relative error \(E\) satisfying

\[
\lim_{(x,y) \to (a,b)} \frac{E(x,y)}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.
\]

The function \(f\) is said to be differentiable in the region \(R\) of the plane if \(f\) is differentiable at each point of \(R\).

Remark 91.1

Compare the above definition with the single variable calculus: A function \(f(x)\) is differentiable at \(x = a\) if and only if

\[
f(x) = f(a) + f'(a)h + \epsilon h, \quad h = x - a
\]

where

\[
\lim_{x \to a} \epsilon = \lim_{h \to 0} \frac{\epsilon h}{h} = 0.
\]

It follows from the definition, that differentiability implies existence of partial derivatives. Therefore if any of the partial derivative fails to exist then the function cannot be differentiable.
Example 91.1
Show that the function \( f(x, y) = \sqrt{x^2 + y^2} \) is not differentiable at \((0, 0)\).

Solution. Recall that the surface is a cone opening up with a vertex at \((0, 0, 0)\). Let us compute \( f_x(0, 0) \). We have

\[
f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \frac{|h|}{h} = \pm 1
\]

Thus, \( f_x(0, 0) \) does not exist and as a consequence the function is not differentiable at \((0, 0)\). ■

Now suppose we have a function that is continuous and its partial derivatives exist at a point. Is it differentiable? The answer is no, as the following example illustrates.

Example 91.2
Show that the function

\[
f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}
\]

is not differentiable at \((0, 0)\) even though \( f_x(0, 0) \) and \( f_y(0, 0) \) both exist and \( f(x, y) \) is continuous at \((0, 0)\).

Solution.
The graph is pictured in Figure 91.1. It follows that \( f(x, y) \) is continuous at \((0, 0)\). Moreover,

\[
f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} 0 = 0
\]

\[
f_y(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} 0 = 0
\]

That is, the partial derivatives exist at \((0, 0)\). So if \( f \) were differentiable at \((0, 0)\), we would have

\[
\lim_{(x,y) \to (0,0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} = 0
\]
But along \( y = x \) the limit is
\[
\lim_{(x,y)\to(0,0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,x)\to0} \frac{x \cdot x}{x^2 + x^2} = \frac{1}{2}.
\]
Hence, \( f(x, y) \) is not differentiable at \((0, 0)\). □

![Figure 91.1](image)

As in the case for functions of one variable, multivariable functions that are differentiable are also continuous.

**Theorem 91.1**

Let \( z = f(x, y) \) be a function of two variables with \((a, b)\) in the domain of \( f \). If \( f(x, y) \) is differentiable at \((a, b)\) then \( f(x, y) \) is continuous there.

**Proof.**

We wish to show that
\[
\lim_{(x,y)\to(a,b)} [f(x, y) - f(a, b)] = 0.
\]

Since \( f(x, y) \) is differentiable at \((a, b)\) we can write
\[
f(x, y) - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + E(x, y)
\]
Thus,
\[
\lim_{(x,y) \to (a,b)} [f(x,y) - f(a,b)] = \lim_{(x,y) \to (a,b)} [f_x(a,b)(x-a) + f_y(a,b)(y-b)] \\
+ \frac{E(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} \sqrt{(x-a)^2 + (y-b)^2}
\]
\[
= f_x(a,b) \cdot 0 + f_y(a,b) \cdot 0 + 0 = 0\]

The converse is not true in general. Example 91.1 shows that \( f(x,y) = \sqrt{x^2 + y^2} \) is continuous at \((0,0)\) but is not differentiable there. The mere existence of partial derivatives at a point does not necessarily imply continuity and therefore differentiability as illustrated in the following example.

**Example 91.3**

Show that the function

\[
f(x,y) = \begin{cases} 
\frac{xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\
0 & \text{otherwise}
\end{cases}
\]

is not continuous at \((0,0)\) (and therefore not differentiable there) even though \(f_x(0,0)\) and \(f_y(0,0)\) exist.

**Solution.**

Continuity implies that

\[
\lim_{(x,y) \to (0,0)} f(x,y) = f(0,0) = 0
\]

along any curve through \((0,0)\). If we approach the origin along the line \(y = x\) then:

\[
\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{x \to 0} \frac{x \cdot x}{x^2 + x^2} = \frac{1}{2} \neq f(0,0).
\]

Hence, \(f(x,y)\) is not continuous at \((0,0)\). However, the partial derivatives do exist at \((0,0)\):

\[
f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} 0 = 0
\]

\[
f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} 0 = 0\]

So the natural question to ask then is under what conditions can we conclude that \(f\) is differentiable at \((a,b)\). The answer is contained in the following theorem whose proof is beyond the scope of a calculus course.
Theorem 91.2
If \( f(x, y), f_x(x, y), \) and \( f_y(x, y) \) are continuous for all \((x, y)\) in the disk \((x - a)^2 + (y - b)^2 < \delta, \) for some \( \delta > 0 \) then \( f(x, y) \) is differentiable at \((a, b)\).

Example 91.4
In what region is the function \( f(x, y) = \sqrt{y^2 - x^2} \) differentiable?

Solution.
Since \( f_x(x, y) = -x(y^2 - x^2)^{-\frac{1}{2}} \) and \( f_y(x, y) = y(y^2 - x^2)^{-\frac{1}{2}} \), according to the above theorem the function is differentiable everywhere in the region above \( y = |x| \) and below \( y = -|x| \) as shown in Figure 91.2.
92 Local Extrema for Functions in Two Variables

Just like functions of a single variable, functions of several variables can have local and global extrema i.e. local and global maxima and minima.

We say that $f(x, y)$ has a **global maximum** at a point $(a, b)$ of its domain $D_f$ if $f(x, y) \leq f(a, b)$ for all points $(x, y)$ in $D_f$. That is, $f(a, b)$ is the largest value of $f$ in $D_f$.

We say that $f(x, y)$ has a **global minimum** at a point $(a, b)$ of its domain $D_f$ if $f(a, b) \leq f(x, y)$ for all points $(x, y)$ in $D_f$. That is, $f(a, b)$ is the smallest value of $f$ in $D_f$.

We say that $f(x, y)$ has a **local maximum** at a point $(a, b)$ of its domain $D_f$ if there is an $R > 0$ such that $f(x, y) \leq f(a, b)$ for all points $(x, y)$ satisfying $(x - a)^2 + (y - b)^2 < R^2$.

We say that $f(x, y)$ has a **local minimum** at a point $(a, b)$ of its domain $D_f$ if there is an $R > 0$ such that $f(a, b) \leq f(x, y)$ for all points $(x, y)$ satisfying $(x - a)^2 + (y - b)^2 < R^2$.

Collectively, local maxima and local minima are called local extrema. Figure 92.1 provides an example of a function with local and global extrema.

![Figure 92.1](image)

Recall for single-variable functions $y = f(x)$ the recipe for finding a maximum or a minimum point is that we first locate "critical points" $x = c$ such that $f'(c) = 0$ or $f'(c)$ does not exist; furthermore, if $x = c$ is a local maximum or minimum point, then either $f'(c) = 0$ or $f'(c)$ does not exist. Something similar happens for functions of two variables.

Points where the gradient is either zero or undefined are called **critical**
points of the function.
We next show that local extrema are critical points.

**Theorem 92.1**
If \( f \) has a local maximum or a local minimum (which is not a boundary point of its domain) at a point \( P_0 \) then \( \nabla f(P_0) = 0 \). That is, \( P_0 \) is a critical point.

**Proof.**
Suppose \( f \) has a local maximum at a point \( P_0 \) (which is not a boundary point of the domain), then if the vector \( \nabla f(P_0) \) is defined and nonzero, we could find a larger value of \( f \) by moving in the direction \( \nabla f(P_0) \). However, since \( f \) is a local maximum at \( P_0 \) there is no direction in which \( f \) is increasing and we must therefore have \( \nabla f(P_0) = 0 \).

Likewise, if \( f \) has a local minimum at a point \( P_0 \) (which is not a boundary point of the domain) and if the vector \( \nabla f(P_0) \) is defined and nonzero, then we could find a smaller value of \( f \) by moving in the direction of \(-\nabla f(P_0)\). However, since \( f \) is a local minimum at \( P_0 \) there is no direction in which \( f \) is decreasing and we must therefore have \( \nabla f(P_0) = 0 \).

**Remark 92.1**
It is possible to have local extrema at points that are not critical points. For example, the function \( f(x, y) = (x^2 + y^2)^{-1} \) defined on \( 0 < x^2 + y^2 \leq 1 \) has local minima at all points on the boundary \( x^2 + y^2 = 1 \) even though the gradient is not zero there.

For functions of two variables, one can identify local extrema by looking at a contour diagram for the function. Near a local maximum the gradient
vectors will all point inward toward the maximum, perpendicular to the contours, while near a local minimum the gradient vectors will all point outward, perpendicular to the contours. At the local extrema the gradient is either zero or undefined.

From the discussion above, critical points are candidates for local extrema. To find critical points algebraically we set \( \nabla f(x, y) = f_x(x, y)\hat{i} + f_y(x, y)\hat{j} = \vec{0} \), which is equivalent to setting all first order partial derivatives to zero. We must also look for points where the partial derivatives are undefined.

**Example 92.1**

Locate and classify the critical points of \( f(x, y) = -x^2 - 5y^2 + 8x - 10y - 21 \).

**Solution.**

The gradient of \( f \) is given by \( \nabla f(x, y) = -2(x-4)\hat{i} - 10(y+1)\hat{j} \). Thus, \( \nabla f(x, y) = \vec{0} \) implies \(-2(x-4) = 0 \) and \(-10(y+1) = 0 \). Hence, \( x = 4 \) and \( y = -1 \). Since \( f_x(x, y) = -2(x-4) \) and \( f_y(x, y) = -10(y+1) \) are defined everywhere, \((4, -1)\) is the only critical point of \( f(x, y) \). By completing the square we can write

\[
   f(x, y) = -(x-4)^2 - 5(y+1)^2.
\]

We note that \( f(4, -1) = 0 \) and that \( f \) decreases as we move away from this point. Therefore \((4, -1)\) is a local and global maximum. Figure 92.2 shows the graph of \( f(x, y) \) with a local and global maximum at \((4, -1)\) .

![Figure 92.2](image)

**Example 92.2**

Locate and classify the critical points of \( f(x, y) = \sqrt{x^2 + y^2} \).
Solution.
The gradient of \( f \) is given by \( \nabla f(x,y) = \frac{x}{\sqrt{x^2+y^2}} \mathbf{i} + \frac{y}{\sqrt{x^2+y^2}} \mathbf{j} \). We see that \( f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}} \) and \( f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}} \) are never simultaneously zero; however they are both undefined at \((0,0)\). Therefore, \((0,0)\) is a critical point and a possible extrema. The graph of \( f(x,y) \) is a cone opening upwards with vertex at the origin. Therefore \((0,0)\) is a local and a global minimum.

Example 92.3
Locate and classify the critical points of \( f(x,y) = x^2 - y^2 \).

Solution.
The gradient of \( f \) is given by \( \nabla f(x,y) = 2x \mathbf{i} - 2y \mathbf{j} \). We see that \( f_x(x,y) = 2x \) and \( f_y(x,y) = -2y \) are simultaneously zero at \((0,0)\). Therefore, \((0,0)\) is a critical point and a possible extrema. The graph of \( f(x,y) \) shown in Figure 92.3 indicates that \((0,0)\) is neither a local maximum nor a local minimum. Such a point will be called a saddle point.

Notice that just as the vanishing of the first derivative of a function in one variable does not guarantee a maximum or a minimum, the vanishing of the gradient does not guarantee a local extrema either (See previous example). But once again, the second derivative comes to the rescue.

Second Derivative Test for Functions of two Variables
Let \( z = f(x,y) \) be a function of two variables such that \( f_{xx}(x,y), f_{yy}(x,y), \)
and \( f_{xy}(x, y) \) all exist in a surrounding of a point \((a, b)\). Now we recall that near any point \((a, b)\), the quadratic Taylor expansion is a good approximation to the function:

\[
f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + f_{xy}(a, b)(x - a)(y - b) + \frac{1}{2} f_{xx}(a, b)(x - a)^2 + \frac{1}{2} f_{yy}(a, b)(y - b)^2.
\]

If \((a, b)\) is a critical point then \(f_x(a, b) = f_y(a, b) = 0\) and the last equation becomes

\[
f(x, y) \approx f(a, b) + \frac{1}{2} (f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2).
\]

Let \(A = f_{xx}(a, b), \ B = f_{xy}(a, b), \) and \(C = f_{yy}(a, b)\). Then

\[
2[f(x, y) - f(a, b)] \approx A(x - a)^2 + 2B(x - a)(y - b) + C(y - b)^2
\]

\[
= A \left[ (x - a)^2 + \frac{2B}{A}(x - a)(y - b) + \frac{B^2}{A^2}(y - b)^2 \right]
\]

\[
- \frac{B^2}{A}(y - b)^2 + C(y - b)^2
\]

\[
= A \left[ (x - a) + \frac{B}{A}(y - b) \right]^2 + \left( \frac{AC - B^2}{A} \right)(y - b)^2
\]

\[
= A \left[ (x - a) + \frac{B}{A}(y - b) \right]^2 + \left( \frac{D}{A} \right)(y - b)^2
\]

Where \(D = AC - B^2\) is called the **discriminant**. Since the first term in the brackets, \([ (x - a) + \left( \frac{B}{A} \right)(y - b) ]^2\), is always positive, the shape of \(f(x, y)\) will be determined by \(D\). In particular we note:

1. If \(D > 0\), (i.e. \(AC > B^2\) so that \(A\) and \(C\) have the same sign) and \(f_{xx}(a, b) = A > 0\), then the right-hand side in the last equality is positive. Hence, the function has a local minimum at \((a, b)\), since \(f(x, y) \geq f(a, b)\) in a surrounding of \((a, b)\).
2. If \(D > 0\) and \(f_{xx}(a, b) = A < 0\), then the right-hand side of the last equality is negative. Hence, the function has a local maximum at \((a, b)\), since the graph \(f(x, y) \leq f(a, b)\) in a surrounding of \((a, b)\).
3. If \(D < 0\), then the coefficients of the two squared terms have opposite signs, so by going out in two different directions, the quadratic may be made either to increase or to decrease. (Think of \(z = x^2 - y^2\), and Example 92.3).
In this case, $f(x, y)$ has a saddle point at $(a, b)$.

Thus, we have established the following result

**The Second Derivative Test**

Let $(a, b)$ be a point in the domain of $f$ such that $f_x(a, b) = f_y(a, b) = 0$. Furthermore, let

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(x, y)$ has a relative minimum at $(a, b)$.
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(x, y)$ has a relative maximum at $(a, b)$.
3. If $D < 0$ then $f(x, y)$ has a saddle point at $(a, b)$.

**Example 92.4**

Find the local extrema and saddle points of the function

$$f(x, y) = \frac{1}{3}x^3 - 3x^2 + \frac{y^2}{4} + xy + 13x - y + 2.$$ 

**Solution.**

we first find the critical points for this function. This gives us:

$$f_x(x, y) = x^2 - 6x + y + 13 = 0$$
$$f_y(x, y) = \frac{y}{2} + x - 1 = 0$$

From the second equation we find $y = 2 - 2x$. Substituting this into the first equation we find $x^2 - 8x + 15 = (x - 3)(x - 5) = 0$. Thus, $x = 3$ and $x = 5$ so that the critical points are $(3, -4)$ and $(5, -8)$.

On the other hand, we have $f_{xx}(x, y) = 2x - 6$, $f_{yy}(x, y) = \frac{1}{2}$, and $f_{xy}(x, y) = 1$. Hence, $D(3, -4) = -1 < 0$ so $(3, -4)$ is a saddle point. Similarly, $D(5, -8) = 2 - 1 = 1 > 0$ and $f_{xx}(5, -8) = 4 > 0$ so that $(5, -8)$ is a local minimum.

**Example 92.5**

Find the local extrema and saddle points of the function

$$f(x, y) = x^3 - y^5 - 3x - 10y + 4.$$
Solution.
The partial derivatives give
\[ f_x(x, y) = 3x^2 - 3 = 0 \]
\[ f_y(x, y) = -5y^4 - 10 = 0 \]

Solving each equation we find \( x = \pm 1 \) and \( y = \pm \sqrt[4]{2} \). Thus, the critical points are \((1, \sqrt[4]{2}), (1, -\sqrt[4]{2}), (-1, \sqrt[4]{2}), (-1, -\sqrt[4]{2})\). The discriminant is
\[ D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 = -120xy^3. \]

Since \( D(1, \sqrt[4]{2}) = 120\sqrt[4]{8} \) and \( f_{xx}(1, \sqrt[4]{2}) = 6 > 0 \), \((1, \sqrt[4]{2})\) is a local minimum. Since \( D(1, -\sqrt[4]{2}) = -120\sqrt[4]{8} < 0 \), \((1, -\sqrt[4]{2})\) is a saddle point. Since \( D(-1, \sqrt[4]{2}) = 120\sqrt[4]{8} > 0 \) and \( f_{xx}(-1, \sqrt[4]{2}) = -6 < 0 \), \((-1, \sqrt[4]{2})\) is a local maximum.

The second derivative test discussed above, did not cover the case \( D = 0 \). As illustrated in the example below, the second derivative test in inconclusive in this case. That is one cannot classify the critical point. It can be either a local maximum, a local minimum, a saddle point. or none of these.

Example 92.6
Let \( f(x, y) = x^4 + y^4 \), \( g(x, y) = -x^4 - y^4 \), and \( h(x, y) = x^4 - y^4 \). Show that \( D(0, 0) = 0 \) for each function. Classify the critical point \((0, 0)\) for each function.

Solution.
Note that \( f_x(0, 0) = f_y(0, 0) = 0 \) so that \( f(x, y) \) has a critical point at \((0, 0)\). Since \( f_{xx}(x, y) = 12x^2, f_{yy}(x, y) = 12y^2 \) and \( f_{xy}(x, y) = 0 \), we have \( D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0 \). But the smallest value of \( f(x, y) \) occurs at \((0, 0)\) so that \( f(x, y) \) has a local and global minimum at \((0, 0)\) with \( D(0, 0) = 0 \).

Similarly, \( g_x(0, 0) = g_y(0, 0) = 0 \) so that \((0, 0)\) is a critical point of \( g \). Moreover, \( g_{xx}(x, y) = -12x^2, g_{yy}(x, y) = -12y^2 \) and \( g_{xy}(x, y) = 0 \), we have \( D(0, 0) = g_{xx}(0, 0)g_{yy}(0, 0) - [g_{xy}(0, 0)]^2 = 0 \). Since \( g(x, y) \leq 0 \) then the largest value occurs at \((0, 0)\). That is, \( g \) has a local and global maximum at \((0, 0)\) with \( D(0, 0) = 0 \).

Finally, we have \( h_x(0, 0) = h_y(0, 0) = 0 \) so that \((0, 0)\) is a critical point of
Since \( h_{xx}(x,y) = 12x^2 \), \( h_{yy}(x,y) = -12y^2 \) and \( h_{xy}(x,y) = 0 \), we have \( D(0,0) = h_{xx}(0,0)h_{yy}(0,0) - h_{xy}(0,0) = 0 \). However, \( h(0,0) = 0 \), \( z = h(x,0) = x^4 > 0 \) and \( z = h(0,y) = -y^4 < 0 \). Hence, \((0,0)\) is a saddle point with \( D(0,0) = 0 \).
93 Optimization: Finding Global Extrema

In real life, one is most likely interested in finding the places at which the largest and smallest values of a function $f$ occur.

We recall the reader that a point $(a,b)$ in the domain of $f(x,y)$ is called an absolute maximum if $f(x,y) \leq f(a,b)$ for all points in the domain of $f$. If $f(a,b) \leq f(x,y)$ for all points in the domain of $f$ then $f(x,y)$ has a global minimum at $(a,b)$.

Optimization typically refers to finding the global maximum or minimum of a function. If the domain of $f$ is the entire $xy$–plane then we have an unconstrained optimization; if the domain of $f$ is not the entire $xy$–plane then we have a constrained optimization.

The next question that we look at is the question of finding the global extrema in the unconstrained case. Since global extrema are also local extrema, by Theorem 92.1 global extrema are critical points. Thus, if $f$ is a continuous function whose domain is the entire $xy$–plane and no restrictions are placed on $x$ or $y$, then the optimal value of $f$ can be found by:

1. Finding the critical points of $f$.
2. Investigating where the critical points give global maxima or minima.

Example 93.1
Consider the function $f(x,y) = x^2(y+1)^3 + y^2$. Find the global extrema of $f$, if they exist.

Solution.
The first partials give
\[
\begin{align*}
 f_x(x, y) &= 2x(y + 1)^3 = 0 \\
 f_y(x, y) &= 3x^2(y + 1)^2 + 2y = 0
\end{align*}
\]
This implies that the only critical point is $(0, 0)$. Finding second partials we have
\[
\begin{align*}
 f_{xx}(x, y) &= (y + 1)^3 \\
 f_{yy}(x, y) &= 6x^2(y + 1) + 2 \\
 f_{xy}(x, y) &= 6x(y + 1)^2 \\
 f_{yy}(0, 0) &= 1 \\
 f_{yy}(0, 0) &= 2 \\
 f_{xy}(0, 0) &= 0
\end{align*}
\]
Since $D = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 2 > 0$ and $f_{xx}(0, 0) = 2 > 0$ the point $(0, 0)$ is a local minimum. Since $f(-3, -2) = -5 < f(0, 0) = 0$ the point $(0, 0)$ is not a global minimum. Thus, $f$ has no global extrema.\[\blacksquare\]
**Example 93.2**

A trucker plans to purchase an open top rectangular container for his truck to hall twenty cubic meters of gravel to a work site. Several trips, each costing $2 per trip will be required. The manufacturer of the container requires that its height be 0.5 m. The trucker is free to specify the length and width of the container. The cost of the container is $20 per square meters for the vertical front and back and $10 per square meters for the bottom and vertical sides. What size box should the trucker order to minimize the cost?

**Solution.**

Let the length of the box be \( x \) meters and the width \( y \) meters. Then the cost, \( B \), of the box is:

\[
B(x, y) = 10xy + 10x + 20y
\]

But to hall 20 \( m^3 \) of gravel requires \( \frac{20}{0.5xy} \) trips which cost \( \frac{40}{0.5xy} = \frac{80}{xy} \) dollars. Hence, the total cost is

\[
C(x, y) = 10xy + \frac{80}{xy} + 10x + 20y
\]

Since

\[
C_x(x, y) = 10y - \frac{80}{x^2y} + 10 = 0
\]

\[
C_y(x, y) = 10x - \frac{80}{xy^2} + 20 = 0
\]

We must therefore solve the following equations to find the critical points:

\[
10y - \frac{80}{x^2y} + 10 = 0
\]

\[
10x - \frac{80}{xy^2} + 20 = 0
\]

Multiplying the first equation by \( x \) and the second equation by \( y \) we get:

\[
10xy - \frac{80}{xy} + 10x = 0
\]

\[
10xy - \frac{80}{xy} + 20y = 0
\]
Subtracting we find $20y - 10x = 0$ or $x = 2y$. Now substituting into $10xy - \frac{80}{xy} + 20y = 0$ we find $20y^2 - \frac{40}{y^2} + 20y = 0$ or $y^4 + y^3 - 2 = 0$. This equation has only one positive real root at $y = 1$. Hence, the only critical point is $(2, 1)$.

The second order partial derivatives are

\[
C_{xx}(x, y) = \frac{160}{x^3y} \quad C_{xx}(2, 1) = 20
\]

\[
C_{yy}(x, y) = \frac{160}{xy^3} \quad C_{yy}(2, 1) = 80
\]

\[
C_{xy}(x, y) = 10 + \frac{80}{x^2y^2} \quad C_{xy}(2, 1) = 30
\]

Hence,

\[
D(2, 1) = C_{xx}(2, 1)C_{yy}(2, 1) - C_{xy}(2, 1)^2 = 700.
\]

Since $D > 0$ and $C_{xx}(2, 1) = 20 > 0$, the point $(2, 1)$ is a local minimum. A computer graphic of $f(x, y)$ shows that $(2, 1)$ is the global minimum. Thus, the optimal box is 2 meters long and 1 meter wide. 

Not all functions have a global maximum or a global minimum. For example, the function $f(x, y) = x + y - 1$ does not have either a global maximum or a global minimum.

In general, the fact that a function has a single local maximum or minimum does not guarantee that the point is the global maximum or minimum as shown in Example 93.1.

Like functions in one variable, a function $f(x, y)$ can have both a global maximum and minimum; a local maximum but no global minimum; a global minimum but no global maximum; or none. So are there conditions that guarantee that a function has a global maximum and global minimum? In single variable calculus we saw that a function $f(x)$ continuous on a closed (i.e. including the endpoints) and bounded (i.e. of finite length) interval has both a global maximum and a global minimum. A similar result is true for functions of two variables. However, we need to define what we mean by ”bounded” and ”closed” in 2D case. A closed region is one which contains its boundary and with no holes in its interior. For example, the disk $x^2 + y^2 \leq 1$ is a closed region whereas $x^2 + y^2 < 1$ is not since the boundary, which is the circumference of the circle, is not included. Similarly, $0 < x^2 + y^2 \leq 1$ is not closed since it has a
hole at the origin.

A bounded region is one that does not stretch to infinity in any direction. Using these definitions, we have the following theorem for multivariable functions:

**Extreme Value Theorem for Multivariable Functions**

If $f$ is a continuous function on a closed and bounded region $R$, then $f$ has a global maximum and a global minimum in $R$.

We note that if $f$ is not continuous or the region $R$ is not closed or bounded, then there is no guarantee that $f$ will have a global maximum or minimum. For example, the plane $f(x, y) = x + y - 1$ is continuous in the entire plane but does not have global extrema since the region is not bounded.

**Example 93.3**

Does the function $f(x, y) = \frac{1}{x^2 + y^2}$ have a global minimum or maximum on the region $R$ given by $0 < x^2 + y^2 \leq 1$?

**Solution.**

We note that $R$ is bounded but not closed since it excludes the point $(0, 0)$. From the graph of $f$ we see that $f$ has a global minimum on the circle $x^2 + y^2 = 1$ but no global maximum since $f(x, y) \to \infty$ as $x \to 0$, $y \to 0$. Note that even though a point on the circle $x^2 + y^2 = 1$ is a global minimum, the gradient is not zero there since the point is a boundary point. That is, Theorem 92.1 holds only for points not on the boundary.

![Graph of the function $f(x, y) = \frac{1}{x^2 + y^2}$](image)
Constraint Optimization: Lagrange’s Multipliers

Most optimization problems encountered in the real world are constrained by external circumstances. For example, a city wanting to improve its roads has only limited number of tax dollars it can spend on the project.

**Constrained optimization** is the maximization or minimization of an objective function subject to constraints on the possible values of the independent variable. Constraints can be either equality constraints or inequality constraints. In this section, we see how to find an optimum value of a function of two variables subject to some constraints using a graphical approach and an analytical approach employing Lagrange Multipliers.

**Graphical Approach**

We consider the following example. Let \( f(x, y) = x^{\frac{2}{3}} y^{\frac{1}{3}} \) represent the production of a product as a function of the quantities of two raw materials specified by \( x \) and \( y \). The quantities of these raw materials are constrained by the budget available to purchase them. If \( x \) and \( y \) cost $1000 per unit and $3780 is the budget available to purchase them, then what is the maximum production that can be obtained under these circumstances?

Mathematically, we are asked to maximize the function \( f \) subject to the constraint \( x + y \leq 3.78 \).

Graphically, the line \( x + y = 3.78 \) represents all the combinations of raw materials that just exhaust the budget but are still affordable. Points below this line, do not exhaust the budget but are still affordable. Points above the line are unaffordable.

The maximum production can be located graphically by plotting contours of \( f \) on a plot containing the line \( x + y = 3.78 \) as shown in Figure 94.1.

To maximize \( f \) we find the point which lies on the level curve with the largest possible value of \( f \) and which lies on (or below) the line \( x + y = 3.78 \). This figure shows that at the maximum, \( f \) must be tangent to the constraint line since if we are on a contour to the left of the point of tangency, we can increase the value of \( f \) by moving to the right along the budget constraint curve until we reach the point of tangency. Likewise if we are to the right of the point of tangency, we can move left to the point of tangency and increase \( f \).

In general, the maximum of \( f(x, y) \) is located at a point where the constraint
curve $g(x, y)$ is tangent to a level curve of $f$.

![Figure 94.1](image)

**Analytical Approach: Lagrange Multipliers**

As noted above, the maximum production is achieved at the point where the constraint is tangent to a level curve of the production function. The method of Lagrange Multipliers uses this fact in algebraic form to calculate the maximum.

From Figure 94.1, we see that $\nabla f(x, y)$ and $\nabla g(x, y)$ are parallel so that $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some $\lambda$ which we call the **Lagrange multiplier**. We therefore have

$$\frac{2}{3} \sqrt[3]{\frac{y}{x}} + \frac{1}{3} \sqrt[3]{\left(\frac{x}{y}\right)^2} \vec{j} = \lambda \vec{i} + \lambda \vec{j}.$$  

Hence,

$$\frac{2}{3} \sqrt[3]{\frac{y}{x}} = \lambda \text{ and } \frac{1}{3} \sqrt[3]{\left(\frac{x}{y}\right)^2} = \lambda$$

Eliminating $\lambda$ gives

$$\frac{2}{3} \sqrt[3]{\frac{y}{x}} = \frac{1}{3} \sqrt[3]{\left(\frac{x}{y}\right)^2} \text{ which leads to } 2y = x$$
Substituting this into the equation $x + y = 3.78$ we find $x = 2.52$ and $y = 1.26$. Hence,

$$f(2.52, 1.26) = (2.52)^{\frac{2}{3}}(1.26)^{\frac{1}{3}} \approx 2.$$ 

As before, we see that the maximum value of $f$ is approximately 2. Also, note that $\lambda \approx 0.53$

**Generalization**

We are given a constraint equation

$$g(x, y) = c$$

and an objective function $f(x, y)$. The goal is to find the maximum and minimum values of $f$ among the values taken on by $f$ along the constraint curve; i.e., the set of points for which $g(x, y) = c$. Moreover, we would like to find all of the points $(x, y)$ at which these maxima and minima are attained. This is provided by the following theorem.

**Theorem 94.1**

If there is a maximum or a minimum of the values that the function $f(x, y)$ assumes on the constraint curve $g(x, y) = c$, then it occurs at a point at which

$$\nabla f(x, y) = \lambda \nabla g(x, y) \text{ and } g(x, y) = c$$

Again, geometrically this means that the extrema occur at points where the contour curves of $f(x, y)$ and $g(x, y)$ are tangent.

We summarize the two steps for finding the extrema:

1. Solve the following system of equations.

$$\nabla f(x, y) = \lambda \nabla g(x, y) \text{ and } g(x, y) = c$$

2. Plug in all solutions, $(x, y)$, from the first step into $f(x, y)$ and identify the minimum and maximum values, provided they exist.

**Example 94.1**

Find the maximum and minimum of $f(x, y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$. 

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Solution.
Since $\nabla f(x, y) = 5\vec{i} - 3\vec{j}$ and $\nabla g(x, y) = 2x\vec{i} + 2y\vec{j}$ then we must have $2\lambda x = 5$ and $2\lambda y = -3$. Eliminating $\lambda$ we find $\frac{5}{2x} = -\frac{3}{2y}$ and this leads to $y = -\frac{3}{5}x$.
Substituting into the constraint equation we find $x^2 + \frac{9}{25}x^2 = 136$. Solving this equation for $x$ we find $x = \pm 10$. If $x = -10$ then $y = 6$ and if $x = 10$ then $y = -6$. Since, $f(-10, 6) = -68$ and $f(10, -6) = 68$, the maximum of $f$ occurs at the point $(10, -6)$ and the minimum occurs at the point $(-10, 6)$.

Optimization with Inequality Constraints
To this point we have only looked at constraints that were equations. We can also have constraints that are inequalities. The process for these types of problems is nearly identical to what we have been doing in this section to this point. The main difference between the two types of problems is that we will also need to find all the critical points that satisfy the inequality in the constraint and check these in the function when we check the values we found using Lagrange Multipliers.

Let’s work an example to see how these kinds of problems work.

Example 94.2
Find the maximum and minimum values of $f(x, y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$.

Solution.
The first step is to find all the critical points that are in the disk (i.e. satisfy the constraint). We have

$$f_x(x, y) = 2x = 0$$
$$f_y(x, y) = 2y = 0$$

So $(0, 0)$ is the only critical point satisfying the constraint.

Next, we proceed with Lagrange Multipliers and we treat the constraint as an equality instead of the inequality. Since $\nabla f(x, y) = 8x\vec{i} + 20y\vec{j}$ and $\nabla g(x, y) = 2x\vec{i} + 2y\vec{j}$ we must have $2\lambda x = 8x$ and $2\lambda y = 20y$ These equations imply

$$2x(\lambda - 4) = 0$$
$$2y(\lambda - 10) = 0.$$
If \( x = 0 \) we find \( y = \pm 2 \). If \( x \neq 0 \) then \( \lambda = 4 \) and so the second equation gives \( y = 0 \) and so \( x = \pm 2 \). If we had performed a similar analysis on the second equation we would arrive at the same points. So, Lagrange Multipliers gives us four points to check: \((0, -2), (0, 2), (-2, 0), (2, 0)\).

Now, since

\[
\begin{align*}
f(0, 0) &= 0 \\
f(2, 0) &= f(-2, 0) = 16 \\
f(0, -2) &= f(0, 2) = 40
\end{align*}
\]

the maximum of \( f \) occur at the points \((0, -2)\) and \((0, 2)\) and the minimum occurs at \((0, 0)\) ■

**Practical Interpretation of \( \lambda \)**

Let \((x_0, y_0)\) be an optimum value. Then its location depends on \( c \) where \( g(x, y) = c \). Thus, \( x_0 = x_0(c) \) and \( y_0 = y_0(c) \). Using the chain rule we can write

\[
\frac{df}{dc} = \frac{\partial f}{\partial x} \frac{dx_0}{dc} + \frac{\partial f}{\partial y} \frac{dy_0}{dc}
\]

However, we have

\[
\frac{\partial f}{\partial x}(x_0, y_0) = \lambda \frac{\partial g}{\partial x}(x_0, y_0) \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = \lambda \frac{\partial g}{\partial y}(x_0, y_0)
\]

so that

\[
\frac{df}{dc} = \lambda \frac{\partial g}{\partial x} \frac{dx_0}{dc} + \lambda \frac{\partial g}{\partial y} \frac{dy_0}{dc} = \lambda \frac{dg}{dc}.
\]

Since \( g(x_0(c), y_0(c)) = c \) we must have \( \frac{dg}{dc} = 1 \). Thus,

\[
\frac{df}{dc} = \lambda
\]

This says that \( \lambda \) is the rate of change of the optimum value of \( f \) as \( c \) increases. For example, in the budget function discussed earlier, an increase of \$1000\ in the budget will lead to an increase of about 0.53 unit in production.
95 The Definite Integral of $f(x, y)$

In this section, we introduce the concept of definite integral of a function of two variables over a rectangular region. By a rectangular region we mean a region $R$ as shown in Figure 95.1(I).

![Figure 95.1](image)

Let $f(x, y)$ be a continuous function on $R$. Our definition of the definite integral of $f$ over the rectangle $R$ will follow the definition from one variable calculus. Partition the interval $a \leq x \leq b$ into $n$ equal subintervals using the mesh points $a \leq x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ with $\Delta x = \frac{b-a}{n}$ denoting the length of each subinterval. Similarly, partition $c \leq y \leq d$ into $m$ subintervals using the mesh points $c = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = d$ with $\Delta y = \frac{d-c}{m}$ denoting the length of each subinterval. This way, the rectangle $R$ is partitioned into $mn$ subrectangles each of area equals to $\Delta x \Delta y$ as shown in Figure 95.1(II).

Let $D_{ij}$ be a typical rectangle. Let $m_{ij}$ be the smallest value of $f$ in $D_{ij}$ and $M_{ij}$ be the largest value in $D_{ij}$. Pick a point $(x^*_i, y^*_j)$ in this rectangle. Then we can write

$$m_{ij} \Delta x \Delta y \leq f(x^*_i, y^*_j) \Delta x \Delta y \leq M_{ij} \Delta x \Delta y.$$

Sum over all $i$ and $j$ to obtain

$$\sum_{j=1}^{m} \sum_{i=1}^{n} m_{ij} \Delta x \Delta y \leq \sum_{j=1}^{m} \sum_{i=1}^{n} f(x^*_i, y^*_j) \Delta x \Delta y \leq \sum_{j=1}^{m} \sum_{i=1}^{n} M_{ij} \Delta x \Delta y.$$
We call
\[ L = \sum_{j=1}^{m} \sum_{i=1}^{n} m_{ij} \Delta x \Delta y \]
the lower Riemann sum and
\[ U = \sum_{j=1}^{m} \sum_{i=1}^{n} M_{ij} \Delta x \Delta y \]
the upper Riemann sum. If
\[ \lim_{m,n \to \infty} L = \lim_{m,n \to \infty} U \]
then we write
\[ \int_{R} f(x,y) \, dx \, dy = \lim_{m,n \to \infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*) \Delta x \Delta y \]
and we call \( \int_{R} f(x,y) \, dx \, dy \) the double integral of \( f \) over the rectangle \( R \). The use of the word "double" will be justified in Section 96.

**Double Integral as Volume Under a Surface**

Just as the definite integral of a positive one-variable function can be interpreted as area, so the double integral of a positive two-variable function can be interpreted as a volume.

Let \( f(x,y) > 0 \) with surface \( S \) shown in Figure 95.2(I). Partition the rectangle \( R \) as above. Over each rectangle \( D_{ij} \) we will construct a box whose height is given by \( f(x_i^*, y_j^*) \) as shown in Figure 95.2 (II). Each of the boxes has a base area of \( \Delta x \Delta y \) and a height of \( f(x_i^*, y_j^*) \) so the volume of each of these boxes is \( f(x_i^*, y_j^*) \Delta x \Delta y \). So the volume under the surface \( S \) is then approximately,
\[ V \approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*) \Delta x \Delta y \]
As the number of subdivisions grows, the tops of the boxes approximate the surface better, and the volume of the boxes gets closer and closer to the volume under the graph of the function. Thus, we have the following result,

If \( f(x,y) > 0 \) then the volume under the graph of \( f \) above the region \( R \) is
\[ \int_{R} f(x,y) \, dx \, dy \]
Double Integral as Area
If we choose a function $f(x, y) = 1$ everywhere in $R$ then our integral becomes:

$$\text{Area of } R = \int_R 1 \, dx \, dy$$

That is, when $f(x, y) = 1$, the integral gives us the area of the region we are integrating over.

Example 95.1
Use the Riemann sum with $n = 3, m = 2$ and sample point the upper right corner of each subrectangle to estimate the volume under $z = xy$ and above the rectangle $0 \leq x \leq 6, \ 0 \leq y \leq 4$.

Solution.
The interval on the $x$–axis is to be divided into $n = 3$ subintervals of equal length, so $\Delta x = \frac{6-0}{3} = 2$. Likewise, the interval on the $y$–axis is to be divided into $m = 2$ subintervals, also with width $\Delta y = 2$; and the rectangle is divided into six squares with sides 2.

Next, the upper right corners are at (2, 2), (4, 2) and (6, 2), for the lower three squares, and (2, 4), (4, 4) and (6, 4), for the upper three squares. The approximation is then

$$[f(2, 2) + f(4, 2) + f(6, 2)) + f(2, 4) + f(4, 4) + f(6, 4)] \cdot 2 \cdot 2$$

$$= [4 + 8 + 12 + 8 + 16 + 24] \cdot 4 = 72 \cdot 4 = 288$$

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Example 95.2
Values of \(f(x, y)\) are given in the table below. Let \(R\) be the rectangle \(0 \leq x \leq 1.2,\ 2 \leq y \leq 2.4\). Find Riemann sums which are reasonable over- and under-estimates for \(\int_R f(x, y) \, dx \, dy\) with \(\Delta x = 0.1\) and \(\Delta y = 0.2\)

<table>
<thead>
<tr>
<th>y (\backslash) x</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>5</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>2.2</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>2.4</td>
<td>3</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

Solution.
We mark the values of the function on the plane, as shown in Figure 95.3, so that we can guess respectively at the smallest and largest value the function takes on each subrectangle.

Lower sum = \((4 + 6 + 3 + 4) \Delta x \Delta y = (17)(0.1)(0.2) = 0.34\)

Upper sum = \((7 + 10 + 6 + 8) \Delta x \Delta y = (31)(0.1)(0.2) = 0.62\)

Integral Over Bounded Regions That Are Not Rectangles
The region of integration \(R\) can be of any bounded shape not just rectangles. In our presentation above we chose a rectangular shaped region for convenience since it makes the summation limits and partitioning of the \(xy\)-plane into squares or rectangles simpler. However, this need not be the case. We can instead picture covering an arbitrary shaped region in the \(xy\)-plane with rectangles so that either all the rectangles lie just inside the region or the rectangles extend just outside the region (so that the region is contained inside our rectangles) as shown in Figure 95.4. We can then compute either the
minimum or maximum value of the function on each rectangle and compute the volume of the boxes, and sum.

Figure 95.4

The Average of \( f(x, y) \)
As in the case of single variable calculus, the **average value** of \( f(x, y) \) over a region \( R \) is defined by

\[
\frac{1}{\text{Area of } R} \int_R f(x, y) \, dx \, dy.
\]
96 Iterated Integrals

In the previous section we used Riemann sums to approximate a double integral. In this section, we see how to compute double integrals exactly using one-variable integrals.

Going back to our definition of the integral over a region as the limit of a double Riemann sum:

\[
\int_R f(x,y) \, dx \, dy = \lim_{m,n \to \infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*) \Delta x \Delta y
\]

\[
= \lim_{m,n \to \infty} \sum_{j=1}^{m} \left( \sum_{i=1}^{n} f(x_i^*, y_j^*) \Delta x \right) \Delta y
\]

\[
= \lim_{m,n \to \infty} \sum_{j=1}^{m} \Delta y \left( \sum_{i=1}^{n} f(x_i^*, y_j^*) \Delta x \right)
\]

\[
= \lim_{m \to \infty} \sum_{j=1}^{m} \Delta y \int_{a}^{b} f(x, y_j^*) \, dx
\]

We now let

\[
F(y_j^*) = \int_{a}^{b} f(x, y_j^*) \, dx
\]

and, substituting into the expression above, we obtain

\[
\int_R f(x,y) \, dx \, dy = \lim_{m \to \infty} \sum_{j=1}^{m} F(y_j^*) \Delta y = \int_{c}^{d} F(y) \, dy = \int_{c}^{d} \int_{a}^{b} f(x, y) \, dx \, dy.
\]

Thus, if \( f \) is continuous over a rectangle \( R \) then the integral of \( f \) over \( R \) can be expressed as an \textbf{iterated integral}. To evaluate this iterated integral, first perform the inside integral with respect to \( x \), holding \( y \) constant, then integrate the result with respect to \( y \).

Example 96.1

Compute \( \int_{0}^{16} \int_{0}^{8} (12 - \frac{x}{4} - \frac{y}{8}) \, dx \, dy \).
Solution.
We have
\[
\int_0^8 \int_0^{16} \left( 12 - \frac{x}{4} - \frac{y}{8} \right) \, dx \, dy = \int_0^{16} \left( \int_0^8 \left( 12 - \frac{x}{4} - \frac{y}{8} \right) \, dx \right) \, dy \\
= \int_0^{16} \left[ 12x - \frac{x^2}{8} - \frac{xy}{8} \right]_0^8 \, dy \\
= \int_0^{16} (88 - y) \, dy = 88y - \frac{y^2}{2} \bigg|_0^{16} = 1280
\]

We note, that we can repeat the argument above for establishing the iterated integral, reversing the order of the summation so that we sum over \( j \) first and \( i \) second (i.e. integrate over \( y \) first and \( x \) second) so the result has the order of integration reversed. That is we can show that
\[
\int_R f(x, y) \, dx \, dy = \int_a^b \int_c^d f(x, y) \, dy \, dx.
\]

Example 96.2
Compute \( \int_0^8 \int_0^{16} \left( 12 - \frac{x}{4} - \frac{y}{8} \right) \, dy \, dx \).

Solution.
We have
\[
\int_0^8 \int_0^{16} \left( 12 - \frac{x}{4} - \frac{y}{8} \right) \, dy \, dx = \int_0^8 \left( \int_0^{16} \left( 12 - \frac{x}{4} - \frac{y}{8} \right) \, dy \right) \, dx \\
= \int_0^8 \left[ 12y - \frac{xy}{4} - \frac{y^2}{16} \right]_0^{16} \, dx \\
= \int_0^8 (176 - 4x) \, dx = 176x - \frac{2x^2}{4} \bigg|_0^8 = 1280
\]

Iterated Integrals Over Non-Rectangular Regions
So far we looked at double integrals over rectangular regions. The problem with this is that most of the regions are not rectangular so we need to now look at the following double integral,
\[
\int_R f(x, y) \, dx \, dy
\]
where \( R \) is any region. We consider the two types of regions shown in Figure 96.1.

In Case 1, the iterated integral of \( f \) over \( R \) is defined by

\[
\int_{R} f(x,y) \, dx \, dy = \int_{a}^{b} \int_{g_{1}(x)}^{g_{2}(x)} f(x,y) \, dy \, dx
\]

This means, that we are integrating using vertical strips from \( g_{1}(x) \) to \( g_{2}(x) \) and moving these strips from \( x = a \) to \( x = b \).

In case 2, we have

\[
\int_{R} f(x,y) \, dx \, dy = \int_{c}^{d} \int_{h_{1}(y)}^{h_{2}(y)} f(x,y) \, dx \, dy
\]

so we use horizontal strips from \( h_{1}(y) \) to \( h_{2}(y) \). Note that in both cases, the limits on the outer integral must always be constants.

**Remark 96.1**

Choosing the order of integration will depend on the problem and is usually determined by the function being integrated and the shape of the region \( R \). The order of integration which results in the "simplest" evaluation of the integrals is the one that is preferred.

**Example 96.3**

Let \( f(x,y) = xy \). Integrate \( f(x,y) \) for the triangular region bounded by the \( x \)–axis, the \( y \)–axis, and the line \( y = 2 - 2x \).
Solution.
Figure 96.2 shows the region of integration for this example.

Graphically integrating over $y$ first is equivalent to moving along the $x$ axis from 0 to 1 and integrating from $y = 0$ to $y = 2 - 2x$. That is, summing up the vertical strips as shown in Figure 96.3(I).

$$\int_{R} xy \, dy \, dx = \int_{0}^{1} \int_{0}^{2-2x} xy \, dy \, dx$$

$$= \int_{0}^{1} \left[ \frac{xy^2}{2} \right]_{0}^{2-2x} \, dx = \frac{1}{2} \int_{0}^{1} \left[ x(2-2x)^2 \right] \, dx$$

$$= 2 \int_{0}^{1} (x - 2x^2 + x^3) \, dx = 2 \left( \frac{x^2}{2} - \frac{2}{3} x^3 + \frac{x^4}{4} \right)_{0}^{1} = \frac{1}{6}$$

If we choose to do the integral in the opposite order, then we need to invert the $y = 2 - 2x$ i.e. express $x$ as function of $y$. In this case we get $x = 1 - \frac{1}{2}y$. Integrating in this order corresponds to integrating from $y = 0$ to $y = 2$ along horizontal strips ranging from $x = 0$ to $x = 1 - \frac{1}{2}y$, as shown in Figure
\[ \int_R xy \, dx \, dy = \int_0^2 \int_0^{1-\frac{1}{2}y} xy \, dx \, dy \]
\[ = \int_0^2 \frac{x^2 y}{2} \bigg|_0^{1-\frac{1}{2}y} \, dy = \frac{1}{2} \int_0^2 y(1-\frac{1}{2}y)^2 \, dy \]
\[ = \frac{1}{2} \int_0^2 (y - y^2 + \frac{y^3}{4}) \, dy = \frac{y^2}{4} - \frac{y^3}{6} + \frac{y^4}{32} \bigg|_0^2 = \frac{1}{6} \]

**Figure 96.3**

**Example 96.4**

Find \( \int_R (4xy - y^3) \, dx \, dy \) where \( R \) is the region bounded by the curves \( y = \sqrt{x} \) and \( y = x^3 \).

**Solution.**

A sketch of \( R \) is given in Figure 96.4. Using horizontal strips we can write

\[ \int_R (4xy - y^3) \, dx \, dy = \int_0^1 \int_{y^2}^{\sqrt[4]{y}} (4xy - y^3) \, dx \, dy \]
\[ = \int_0^1 2x^2 y - xy^3 \bigg|_{y^2}^{\sqrt[4]{y}} \, dy = \int_0^1 \left( 2y^{\frac{5}{4}} - y^{\frac{13}{4}} - y^5 \right) \, dy \]
\[ = \frac{3}{4} y^{\frac{6}{4}} - \frac{3}{13} y^{\frac{15}{4}} - \frac{1}{6} y^6 \bigg|_0^1 = \frac{55}{156} \]
Example 96.5
Sketch the region of integration of $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} xy \, dy \, dx$

Solution.
A sketch of the region is given in Figure 96.5.
97 Triple Integrals

In the previous lecture we showed how a function of two variables can be integrated over a region in 2-space i.e. an area and how integration over a region is equivalent to an iterated or double integral over two intervals. This concept can be extended to integration over a solid region or volume of 3-space using triple integrals.

Let \( f(x, y, z) \) be a continuous function on \( a \leq x \leq b, c \leq y \leq d, e \leq z \leq f \). Partition the interval \( a \leq x \leq b \) into \( n \) equal subintervals using the mesh points \( a = x_0 < x_2 < x_3 < \cdots < x_n = b \) with \( \Delta x = \frac{b-a}{n} \) denoting the length of each subinterval. Similarly, partition \( c \leq y \leq d \) into \( m \) subintervals using the mesh points \( c = y_0 < y_1 < y_2 < \cdots < y_m = d \) with \( \Delta y = \frac{d-c}{m} \) denoting the length of each subinterval. Finally, partition \( e \leq z \leq f \) into \( l \) subintervals using the mesh points \( e = z_0 < z_1 < z_2 < \cdots < z_l = f \) with \( \Delta z = \frac{f-e}{l} \). This way, the box \( R \) is partitioned into \( mnl \) smaller boxes each of volume equals to \( \Delta x \Delta y \Delta z \) as shown in Figure 97.1.

Let \( R_{ijk} \) be a typical box. Let \( m_{ijk} \) be the smallest value of \( f \) on \( R_{ijk} \) and \( M_{ijk} \) be the largest value in \( R_{ijk} \). Pick a point \( (x_i^*, y_j^*, z_k^*) \) in this box. Then we can write

\[
m_{ijk} \Delta x \Delta y \Delta z \leq f(x_i^*, y_j^*, z_k^*) \Delta x \Delta y \Delta z \leq M_{ijk} \Delta x \Delta y \Delta z.
\]

Sum over all \( i, j \) and \( k \) to obtain

\[
\sum_{k=1}^{l} \sum_{j=1}^{m} \sum_{i=1}^{n} m_{ijk} \Delta x \Delta y \Delta z \leq \sum_{k=1}^{l} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*, z_k^*) \Delta x \Delta y \Delta z \leq \sum_{k=1}^{l} \sum_{j=1}^{m} \sum_{i=1}^{n} M_{ijk} \Delta x \Delta y \Delta z.
\]

We call

\[
L = \sum_{k=1}^{l} \sum_{j=1}^{m} \sum_{i=1}^{n} m_{ijk} \Delta x \Delta y \Delta z
\]

the lower Riemann sum and

\[
U = \sum_{k=1}^{l} \sum_{j=1}^{m} \sum_{i=1}^{n} M_{ijk} \Delta x \Delta y \Delta z
\]

the upper Riemann sum. If

\[
\lim_{l,m,n \to \infty} L = \lim_{l,m,n \to \infty} U
\]

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then we write

\[
\int_{R} f(x, y, z) \, dx \, dy \, dz = \lim_{l,m,n \to \infty} \sum_{k=1}^{l} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*, z_k^*) \Delta x \Delta y \Delta z
\]

and we call \( \int_{R} f(x, y, z) \, dx \, dy \, dz \) the **triple integral** of \( f \) over the box \( R \).

![Figure 97.1](image)

We can show that the integral over the solid region \( R \) is equivalent to a triple integral over three intervals. The argument is similar to the argument we used for the double integral and is therefore not repeated here. We state formally that:

\[
\int_{R} f(x, y, z) \, dx \, dy \, dz = \int_{e}^{f} \int_{c}^{d} \int_{a}^{b} f(x, y, z) \, dx \, dy \, dz.
\]

**Example 97.1**

Integrate \( f(x, y, z) = 1 + xyz \) over the cube of length 4.

**Solution.**

We have

\[
\int_{R} f(x, y, z) \, dx \, dy \, dz = \int_{0}^{4} \int_{0}^{4} \int_{0}^{4} (1 + xyz) \, dx \, dy \, dz = \int_{0}^{4} \int_{0}^{4} x + \frac{x^2 y z}{2} \bigg|_{0}^{4} \, dy \, dz
\]

\[
= \int_{0}^{4} \int_{0}^{4} (4 + 8yz) \, dy \, dz = \int_{0}^{4} 4y + 4y^2 z \bigg|_{0}^{4} \, dz
\]

\[
= \int_{0}^{4} (16 + 64z) \, dz = 16z + 32z^2 \bigg|_{0}^{4} = 576
\]

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We note that if \( f(x, y, z) = 1 \), the triple integral is just the volume of the solid region \( R \).

**Example 97.2**

Find the volume under the plane \( z = 12 - \frac{x}{4} - \frac{y}{8} \) and above the rectangle \( 0 \leq x \leq 8, \ 0 \leq y \leq 16 \).

**Solution.**

The region is shown in Figure 97.2. Thus, we see that as \( x \) goes from 0 to 8 and \( y \) from 0 to 16, \( z \) ranges from 0 to the plane \( z = 12 - \frac{x}{4} - \frac{y}{8} \). Thus the limits of integration are \( a = 0, b = 8, c = 0, d = 16, e = 0, \) and \( f = 12 - \frac{x}{4} - \frac{y}{8} \).

![Figure 97.2](image)

The volume is

\[
C = \int_0^8 \int_0^{16} \int_{12 - \frac{x}{4} - \frac{y}{8}}^0 dz \, dy \, dx = \int_0^8 \int_0^{16} (12 - \frac{x}{4} - \frac{y}{8}) \, dy \, dz
\]

\[
= \int_0^8 \left[ 12y - \frac{xy}{4} - \frac{y^2}{16} \right]_0^{16} \, dx
\]

\[
= \int_0^8 (176 - 4x) \, dx = 176x - 2x^2 \bigg|_0^8 = 1280
\]

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Example 97.3
Set up the integral representing the volume of the solid "ice cream cone" bounded by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $z = \sqrt{1 - x^2 - y^2}$

Solution. The solid is shown in Figure 97.3

The "ice cream cone" is between these two surfaces:

$\sqrt{x^2 + y^2} \leq z \leq \sqrt{1 - x^2 - y^2}$

This gives the range of $z$ as a function of $x$ and $y$. Now we need to find the maximal range of $x$ and $y$. Inside the ice cream cone, the maximal range of $x$ and $y$ occurs where the two surfaces meet, i.e., where the "ice cream" (the sphere) meets the cone. From the figure, you can see that the surfaces meet in a circle, and the range of $x$ and $y$ is the disk that is the interior of that circle.

The surfaces meet when $\sqrt{x^2 + y^2} = \sqrt{1 - x^2 - y^2}$, which means $x^2 + y^2 = 1 - x^2 - y^2$ or

$x^2 + y^2 = \frac{1}{2}$

We have shown that in the "ice cream cone"

$x^2 + y^2 \leq \frac{1}{2}$

This gives that

$-\sqrt{\frac{1}{2} - x^2} \leq y \leq \sqrt{\frac{1}{2} - x^2}$

and

$-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$
Hence, the volume of the "ice cream cone" is

\[\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dzdydx\]

We summarize our findings from these examples:

- The outer limits have to be constant. They cannot depend on any of the variables.
- The middle limits can depend on the variable from the outer integral only. They cannot depend on the variable from the inner integral.
- The inner limits can depend on the variable from the outer integral and the variable from the middle integral.
98 Double Integrals in Polar Coordinates

There are regions in the plane that are not easily used as domains of iterated integrals in rectangular coordinates. For instance, regions such as a disk, ring, or a portion of a disk or ring.

We start by recalling from Section 50 the relationship between Cartesian and polar coordinates.

The Cartesian system consists of two rectangular axes. A point $P$ in this system is uniquely determined by two points $x$ and $y$ as shown in Figure 98.1(a). The polar coordinate system consists of a point $O$, called the pole, and a half-axis starting at $O$ and pointing to the right, known as the polar axis. A point $P$ in this system is determined by two numbers: the distance $r$ between $P$ and $O$ and an angle $\theta$ between the ray $OP$ and the polar axis as shown in Figure 98.1(b).

![Figure 98.1](image)

The Cartesian and polar coordinates can be combined into one figure as shown in Figure 98.2.

Figure 98.2 reveals the relationship between the Cartesian and polar coordinates:

\[
\begin{align*}
    r &= \sqrt{x^2 + y^2} \\
    x &= r \cos \theta \\
    y &= r \sin \theta \\
    \tan \theta &= \frac{y}{x}
\end{align*}
\]
A double integral in polar coordinates can be defined as follows. Suppose we have a region

\[ R = \{(r, \theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta\} \]

as shown in Figure 98.3(a).

Partition the interval \( \alpha \leq \theta \leq \beta \) into \( m \) equal subintervals, and the interval \( a \leq r \leq b \) into \( n \) equal subintervals, thus obtaining \( mn \) subrectangles as shown in Figure 98.3(b). Choose a sample point \((r_{ij}, \theta_{ij})\) in the subrectangle \( R_{ij} \) defined by \( r_{i-1} \leq r \leq r_i \) and \( \theta_{j-1} \leq \theta \leq \theta_j \). Then

\[
\int_R f(x, y) \, dx \, dy \approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(r_{ij}, \theta_{ij}) \Delta R_{ij}
\]
where $\Delta R_{ij}$ is the area of the subrectangle $R_{ij}$.

To calculate the area of $R_{ij}$, look at Figure 98.4. If $\Delta r$ and $\Delta \theta$ are small then $R_{ij}$ is approximately a rectangle with area $r_{ij}\Delta r\Delta \theta$ so

$$\Delta R_{ij} \approx r_{ij}\Delta r\Delta \theta.$$ 

Thus, the double integral can be approximated by a Riemann sum

$$\int_R f(x, y)\,dxdy \approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(r_{ij}, \theta_{ij})r_{ij}\Delta r\Delta \theta$$

Taking the limit as $m, n \to \infty$ we obtain

$$\int_R f(x, y)\,dxdy = \int_{\alpha}^{\beta} \int_{a}^{b} f(r, \theta)\,rdrd\theta.$$

Example 98.1
Evaluate $\int_R e^{x^2+y^2}\,dxdy$ where $R: x^2 + y^2 \leq 1$. 

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Solution.
We have
\[
\int_R e^{x^2+y^2} \, dx \, dy = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{x^2+y^2} \, dy \, dx
\]
\[
= \int_{0}^{2\pi} \int_{0}^{1} r^2 e^r \, r \, dr \, d\theta
\]
\[
= \int_{0}^{2\pi} \frac{1}{2} (e-1) \, d\theta = \pi (e-1)
\]

Example 98.2
Compute the area of a circle of radius 1.

Solution.
The area is given by
\[
\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \, dy \, dx
\]
\[
= \int_{0}^{2\pi} \int_{0}^{1} r \, dr \, d\theta
\]
\[
= \frac{1}{2} \int_{0}^{2\pi} \, d\theta = \pi
\]

Example 98.3
Evaluate \( f(x, y) = \frac{1}{x^2+y^2} \) over the region \( D \) shown below.

Solution.
We have
\[
\int_{0}^{\frac{\pi}{4}} \int_{0}^{2} \frac{1}{r^2} r \, dr \, d\theta
\]
\[
= \int_{0}^{\frac{\pi}{4}} \ln 2 \, d\theta = \frac{\pi}{4} \ln 2
\]
Example 98.4
For each of the regions shown below, decide whether to integrate using rectangular or polar coordinates. In each case write an iterated integral of an arbitrary function $f(x, y)$ over the region.

Solution.
(a) $\int_0^{2\pi} \int_0^3 f(r, \theta) r dr d\theta$
(b) $\int_1^3 \int_{-1}^2 f(x, y) dy dx$
(c) $\int_0^2 \int_{1/2}^{3-x} f(x, y) dy dx$
99.1 Triple Integrals in Cylindrical Coordinates

When we were working with double integrals, we saw that it was often easier to convert to polar coordinates. For triple integrals we have been introduced to three coordinate systems. The Cartesian coordinate system \((x, y, z)\) is the system that we are used to. The other two systems, cylindrical coordinates \((r, \theta, z)\) and spherical coordinates \((r, \theta, \phi)\) are the topic of this and the next sections.

**Cylindrical Coordinates**

Consider a point \(P = (x, y, z)\) in the Cartesian 3-space. Let \(Q = (x, y, 0)\) be the orthogonal projection of \(P\) into the \(xy\)-plane. An alternative representation of the point \(P\) is the ordered triples \((r, \theta, z)\) where \(r\) is the distance from the \(z\)-axis to \(P\), and \(\theta\) is the angle between \(\overrightarrow{OQ}\) and the positive \(x\)-axis. See Figure 99.1.

![Figure 99.1](image)

Thus, the transformation from the Cartesian coordinates to the cylindrical coordinates is given by

\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}
\]

where \(0 \leq r < \infty\), \(0 \leq \theta \leq 2\pi\), \(-\infty < z < \infty\). Note that
\[ r = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1} \left( \frac{y}{x} \right) \]

**Example 99.5**

Convert the point \((-1, 1, \sqrt{2})\) from Cartesian to cylindrical coordinates.

**Solution.**

We have

\[ r = \sqrt{x^2 + y^2} = \sqrt{1 + 1} = \sqrt{2} \]
\[ \theta = \tan^{-1} (-1) = \frac{3\pi}{4} \]
\[ z = \sqrt{2} \]

Thus \((-1, 1, \sqrt{2}) = (\sqrt{2}, \frac{3\pi}{4}, \sqrt{2})\) ■

**Example 99.6**

Identify the surface for each of the following equations.

(a) \(r = 2\)
(b) \(r^2 + z^2 = 9\)
(c) \(z = r\)

**Solution.**

(a) \(r = 2\) is equivalent to \(x^2 + y^2 = 4\) with \(z\) arbitrary. Thus, \(r = 2\) is a cylinder with axis of symmetry the \(z\)-axis and with radius 2.

(b) \(r^2 + z^2 = 9\) is equivalent to \(x^2 + y^2 + z^2 = 9\). This is the equation of a sphere centered at the origin and with radius 3.

(c) \(z = r\) is equivalent to \(z = \sqrt{x^2 + y^2}\). This is a cone with vertex at the origin and that opens up ■

**Integration in Cylindrical Coordinates**

If \(\Delta r, \Delta \theta, \text{ and } \Delta z\) are sufficiently small we can view the cylindrical elemental volume as a box of length \(\Delta r\), width \(r \Delta \theta\) and height \(\Delta z\) as shown in Figure 99.2. Thus, \(\Delta x \Delta y \Delta z \approx r \Delta r \Delta \theta \Delta z\). Assuming \(a \leq r \leq b, \alpha \leq \theta \leq \beta, \ c \leq \ z \leq d\), the triple integral in cylindrical coordinates can be expressed as an iterated integral

\[ \int_S f(x, y, z) dxdydz = \int_c^d \int_a^b f(r, \theta, z) rdrd\theta dz. \]
Example 99.7
Find the volume of the upper hemisphere centered at the origin and with radius $a$.

Solution.
The upper hemisphere is shown in Figure 99.3

The equation of the upper hemisphere in cylindrical coordinates is $r = \sqrt{a^2 - z^2}$ since $z = \sqrt{a^2 - x^2 - y^2}$. Thus, $r$ varies from 0 to $\sqrt{a^2 - z^2}$, $\theta$
varies from 0 to $2\pi$, and $z$ varies from 0 to $a$. Hence, the volume of the upper hemisphere is

$$V = \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-z^2}} r dr dz d\theta = \int_0^{2\pi} \int_0^a \frac{r^2}{2} \sqrt{a^2-z^2} dz d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^a (a^2 - z^2) dz d\theta = \frac{1}{2} \int_0^{2\pi} \left[ a^2 z - \frac{z^3}{3} \right]_0^a d\theta$$

$$= \frac{a^3}{3} \int_0^{2\pi} d\theta = \frac{2\pi}{3} a^3$$
99.2 Triple Integrals in Spherical Coordinates

In spherical coordinates, a point \( P = (x, y, z) \) in the Cartesian 3-space can be represented by the ordered triple \((\rho, \theta, \phi)\) where \( \rho \) is the distance from the origin to the point, \( \theta \) is the angle between the positive \( x \)-axis and the line connecting the origin to the point \( Q = (x, y, 0) \), and \( \phi \) is the angle between the positive \( z \)-axis and the line connecting the origin to the point \( P \). Figure 99.4 shows the location of a point in both spherical and Cartesian coordinates.

![Figure 99.4](image)

From Figure 99.4 we see that the relationship between the spherical and Cartesian coordinates is as follows:

\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2 + z^2} \\
r &= \rho \sin \phi \\
x &= \rho \cos \theta \sin \phi \\
y &= \rho \sin \theta \sin \phi \\
z &= \rho \cos \phi \\
\rho^2 &= x^2 + y^2 + z^2
\end{align*}
\]

where \( 0 \leq \rho < \infty \), \( 0 \leq \phi \leq \pi \), \( 0 \leq \theta \leq 2\pi \). Simple trigonometry yields

\[
\begin{align*}
\theta &= \tan^{-1}\left(\frac{y}{x}\right) \\
\phi &= \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)
\end{align*}
\]
Example 99.8
Convert the point \((-1, 1, \sqrt{2})\) from Cartesian to spherical coordinates.

Solution.
We have
\[
\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 1 + 2} = 2
\]
\[
\theta = \tan^{-1} (-1) = \frac{3\pi}{4}
\]
\[
\phi = \tan^{-1} (-1) = \frac{3\pi}{4}
\]

Example 99.9
Identify the surface for each of the following equations.
(a) \(\rho = 5\)
(b) \(\phi = \frac{\pi}{3}\)
(c) \(\theta = \frac{2\pi}{3}\)
(d) \(\rho \sin \phi = 2\).

Solution.
(a) \(\rho = 5\) implies \(x^2 + y^2 + z^2 = 25\). Thus, the surface is a sphere centered at the origin and with radius 5.

(b) This equation says that no matter how far away from the origin that we move and no matter how much we rotate around the \(z\)-axis the point must always be at an angle of \(\frac{\pi}{3}\) from the \(z\)-axis.
This is exactly what happens in a cone. All of the points on a cone are a fixed angle from the \(z\)-axis. So, we have a cone whose points are all at an angle of \(\frac{\pi}{3}\) from the \(z\)-axis. That is, a cone with vertex at the origin and that opens up. The reflection of this cone about the \(xy\)-plane has equation \(\phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}\).

(c) This equation says that no matter how far from the origin we get or how much we rotate down from the positive \(z\)-axis the points must always form an angle of \(\frac{2\pi}{3}\) with the \(x\)-axis.
Points in a vertical plane will do this. So, we have a vertical plane that forms an angle of \(\frac{2\pi}{3}\) with the \(x\)-axis.

(d) The given equation is equivalent to \(r = 2\) where \(r\) as defined in Figure 99.4. Thus, \(x^2 + y^2 = 4\). This is a cylinder with radius 2 and with axis of symmetry the \(z\)-axis.
Integration in Spherical Coordinates

As in the case of cylindrical coordinates we want to express the elemental volume $\Delta V = \Delta x \Delta y \Delta z$ in terms of $\Delta \rho$, $\Delta \theta$, and $\Delta \phi$. The elemental volume in spherical coordinates is shown in Figure 99.5.

If $\Delta \rho$, $\Delta \theta$ and $\Delta \phi$ are small enough, then we can regard this volume as approximately a box-shaped. The height of the box is $\Delta \rho$, the width $\rho \sin \phi \Delta \theta$, and the length $\rho \Delta \phi$. The elemental volume is therefore:

$$\Delta V = \rho^2 \sin \phi \Delta \rho \Delta \phi \Delta \theta$$

The integral over the solid region $S$ in spherical coordinates is:

$$\int_S f(x, y, z) dV = \int_0^\beta \int_\alpha^\gamma \int_a^b f(\rho, \theta, \phi) \rho^2 \sin \phi d\rho d\phi d\theta$$

**Example 99.10**

Use spherical coordinates to derive the formula for the volume of a sphere centered at the origin and with radius $a$. 

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Solution.
For a sphere, \(0 \leq \rho \leq a, \ 0 \leq \theta \leq 2\pi, \ 0 \leq \phi \leq \pi\). We therefore have
\[
\int_S dV = \int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^2 \sin \phi d\rho d\phi d\theta \\
= \int_0^{2\pi} \int_0^{\pi} \frac{a^3}{3} \sin \phi d\phi d\theta \\
= \frac{a^3}{3} \int_0^{2\pi} -\cos \phi |_0^{\pi} d\theta \\
= \frac{2}{3} a^3 \int_0^{2\pi} d\theta = \frac{4}{3} \pi a^3
\]

Example 99.11
Find the volume that lies inside the sphere
\[
x^2 + y^2 + z^2 = 2
\]
and outside the cone
\[
z^2 = x^2 + y^2
\]

Solution.
We convert to spherical coordinates. The sphere becomes
\[
\rho = \sqrt{2}
\]
To convert the cone, we add \(z^2\) to both sides of the equation \(z^2 = x^2 + y^2\) to obtain
\[
2z^2 = x^2 + y^2 + z^2
\]
and this implies
\[
2\rho^2 \cos^2 \phi = \rho^2
\]
Thus, \( \phi = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \) or \( \frac{3\pi}{4} \).

To find the volume we compute

\[
V = \int_{0}^{2\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{0}^{\sqrt{2}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

\[
= \frac{2\sqrt{2}}{3} \int_{0}^{2\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin \phi \, d\phi \, d\theta
\]

\[
= \frac{2\sqrt{2}}{3} \int_{0}^{2\pi} -\cos \phi \bigg|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \, d\theta
\]

\[
= \frac{4}{3} \int_{0}^{2\pi} \, d\theta = \frac{8}{3} \pi
\]

\[\blacksquare\]
Applications of Double Integrals to Probability

Probability is one of the major topics in mathematics. It is used widely across many fields: physics, medicine, insurance, and finance, to name just a few. The medical and insurance industries for example, collect data on disease (cardiovascular, cancer, etc.) and mortality (due to accidents, suicide, murder, etc.) in the general population and use this information to predict the likely outcomes of these "events" e.g. having a heart attack, being in an automobile accident, etc.

The medical industry may use this information to concentrate its efforts on development of new drugs, education, or treatment of disease in the general population, while the insurance industry may use this information to set policy rates.

Some basic definitions follow. An experiment is any operation whose outcome cannot predicted with certainty. The sample space $S$ of an experiment is the set of all possible outcomes for the experiment. For example, if you roll a die one time then the experiment is the roll of the die. A sample space for this experiment could be $S = \{1, 2, 3, 4, 5, 6\}$ where each digit represents a face of the die. An event is any subset of the sample space.

Probability is the measure of occurrence of an event. It is a number between 0 and 1. If the event is impossible to occur then its probability is 0. If the occurrence is certain then the probability is 1. The closer to 1 the probability is, the more likely the event is to occur.

If $E$ is any event of a sample space $S$, then the probability of occurrence of $E$ is given by the formula

$$P(E) = \frac{|E|}{|S|}$$

where $|E|$ denotes the number of outcomes in the set $E$.

A random variable $X$ is a numerical valued function defined on a sample space. For example, in rolling two dice $X$ might represent the sum of the points on the two dice. Similarly, in taking samples of college students $X$ might represent the number of hours per week a student studies, or a student’s GPA.

Random variables may be divided into two types: discrete random variables and continuous random variables. A discrete random variable is one that can assume only a countable number of values. It is usually the result
of counting. A continuous random variable can assume any value in one or more intervals on a line. It is usually the result of a measurement.

**Example 100.1**

State whether the random variables are discrete or continuous:

(a) The height of a student in your class.
(b) The number of left-handed students in your class.

**Solution.**

(a) The random variable in this case is a result of measurement and so it is a continuous random variable.
(b) The random variable takes whole positive integers as values and so is a discrete random variable.

Two random variables $X$ and $Y$ are said to be **jointly continuous** if there is a function $p(x, y)$ that satisfies the following properties:

\[
p(x, y) \geq 0
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) \, dx \, dy = 1
\]

For any two events $A$ and $B$ we have

\[
P(X \in A, Y \in B) = \int_B \int_A p(x, y) \, dx \, dy
\]

In particular if $A$ is the closed interval $[a, b]$ and $B$ is the closed interval $[c, d]$ then

\[
P(a \leq X \leq b, c \leq Y \leq d) = \int_c^d \int_a^b p(x, y) \, dx \, dy
\]

The function $p(x, y)$ is called the **joint probability density function** of $X$ and $Y$. If $X$ and $Y$ are discrete random variables then we call $p(x, y)$ the **joint probability mass function**.

**Example 100.2**

Let $p(x, y) = x + y$ on the unit square $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and 0 elsewhere. Verify that $p(x, y)$ is a joint density function.
Solution.
To show that \( p \) is a joint probability density function we must show that \( p(x, y) \geq 0 \) for all \( x \) and \( y \) and \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) \, dx \, dy = 1 \). Since \( x \geq 0 \) and \( y \geq 0 \) then \( p(x, y) = x + y \geq 0 \). Furthermore,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y) \, dx \, dy = \int_{0}^{1} \int_{0}^{1} (x + y) \, dx \, dy \\
= \int_{0}^{1} \left[ \frac{x^2}{2} + xy \right]_{0}^{1} \, dy = \int_{0}^{1} \left( \frac{1}{2} + y \right) \, dy \\
= \left[ \frac{1}{2} y + \frac{1}{2} y^2 \right]_{0}^{1} = 1 \]

Example 100.3
A supermarket has two express lines. Let \( X \) and \( Y \) denote the number of customers in the first and second line at any given time. The joint probability function of \( X \) and \( Y, p(x, y) \), is summarized by the following table

<table>
<thead>
<tr>
<th>( X ) ( \backslash ) ( Y )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>( p_X(.) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.25</td>
<td>0.05</td>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.05</td>
<td>0.05</td>
<td>0.025</td>
<td>0.125</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0.025</td>
<td>0.05</td>
<td>0.075</td>
</tr>
<tr>
<td>( p_Y(.) )</td>
<td>0.3</td>
<td>0.5</td>
<td>0.125</td>
<td>0.075</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) Verify that \( p(x, y) \) is a joint probability mass function.
(b) Find the probability that more than two customers are in line.
(c) Find \( P(|X - Y| = 1) \), the probability that \( X \) and \( Y \) differ by exactly 1.

Solution.
(a) From the table we see that the sum of all the entries is 1.
(b) \( p(0, 3) + p(1, 2) + p(1, 3) + p(2, 1) + p(2, 2) + p(2, 3) + p(3, 0) + p(3, 1) + p(3, 2) + p(3, 3) = 0.25 \).
(c) \( P(|X - Y| = 1) = p(0, 1) + p(1, 0) + p(1, 2) + p(2, 1) + p(2, 3) + p(3, 2) = 0.55 \).

Example 100.4
Let \( X \) and \( Y \) be random variables with joint pdf
\[
p(x, y) = \begin{cases} 
\frac{1}{4} & -1 \leq x, y \leq 1 \\
0 & \text{Otherwise}
\end{cases}
\]
Determine $P(X^2 + Y^2 < 1)$

**Solution.**

\[ P(X^2 + Y^2 < 1) = \int_0^{2\pi} \int_0^1 \frac{1}{4} r dr d\theta = \frac{\pi}{4} \]

**Example 100.5**

An insurance company insures a large number of drivers. Let $X$ be the random variable representing the company’s losses under collision insurance, and let $Y$ represent the company’s losses under liability insurance. $X$ and $Y$ have joint density function

\[ p(x, y) = \begin{cases} \frac{2x + 2 - y}{4} & 0 < x < 1, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases} \]

What is the probability that the total loss is at least 1?

**Solution.**

We want to find $P(X + Y > 1)$. The region representing $X + Y > 1$ is shown in red.

Therefore

\[ P(X + Y > 1) = \int_0^1 \int_0^2 \left[ \frac{2x + 2 - y}{4} \right] dy dx = \int_0^1 \left[ \frac{1}{2} xy + \frac{1}{2} y - \frac{y^2}{8} \right]_{1-x}^{2} dx \]

\[ = \int_0^1 \left( \frac{5}{8} x^2 + \frac{3}{4} x + \frac{1}{8} \right) dx = \left[ \frac{5}{24} x^3 + \frac{3}{8} x^2 + \frac{1}{8} x \right]_0^{1} = \frac{17}{24} \]

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101 Change of Variables in a Multiple Integral

In single variable calculus, it is often the case that changing the variable of integration will simplify the evaluation of an integral. In this section, we will extend the change of variable process to multivariable functions.

Suppose \( x = x(u, v) \) and \( y = y(u, v) \) are two differentiable functions of \( u \) and \( v \). We assume that the functions \( x \) and \( y \) take a point in the \( uv \)-plane to exactly one point in the \( xy \)-plane.

Let us see what happens to a small rectangle \( T \) in the \( uv \)-plane with sides of lengths \( \Delta u \) and \( \Delta v \) as shown in Figure 101.1. Since the side-lengths are small, by local linearity each side of the rectangle in the \( uv \)-plane is transformed into a line segment in the \( xy \)-plane. The result is that the rectangle in the \( uv \)-plane is transformed into a parallelogram \( R \) in the \( xy \)-plane with sides in vector form are

\[
\vec{a} = [x(u + \Delta u, v) - x(u, v)]\vec{i} + [y(u + \Delta u, v) - y(u, v)]\vec{j} \approx \frac{\partial x}{\partial u} \Delta u \vec{i} + \frac{\partial y}{\partial u} \Delta u \vec{j}
\]

and

\[
\vec{b} = [x(u, v + \Delta v) - x(u, v)]\vec{i} + [y(u, v + \Delta v) - y(u, v)]\vec{j} \approx \frac{\partial x}{\partial v} \Delta v \vec{i} + \frac{\partial y}{\partial v} \Delta v \vec{j}
\]

Now, the area of \( R \) is

\[
\text{Area } R \approx ||\vec{a} \times \vec{b}|| = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right| \Delta u \Delta v
\]
Using determinant notation, we define the Jacobian, \( \frac{\partial(x,y)}{\partial(u,v)} \), as follows

\[
\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix}
\]

Thus, we can write

\[
\text{Area } R \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v
\]

Now, suppose we are integrating \( f(x,y) \) over a region \( R \). Partition \( R \) into \( mn \) small parallelograms. Then using Riemann sums we can write

\[
\int_{R} f(x,y) \, dx \, dy \approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_{ij}, y_{ij}) \cdot \text{Area of } R_{ij}
\]

\[
\approx \sum_{j=1}^{m} \sum_{i=1}^{n} f(u_{ij}, v_{ij}) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v
\]

where \((x_{ij}, y_{ij})\) in \( R_{ij} \) corresponds to a point \((u_{ij}, v_{ij})\) in \( T_{ij} \). Now, letting \( m, n \rightarrow \infty \) we obtain

\[
\int_{R} f(x,y) \, dx \, dy = \int_{T} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, dudv.
\]

In summary, to convert from \( x, y \) to \( u, v \) coordinates we make three changes:
1. Substitute for \( x \) and \( y \) in the integral in terms of \( u \) and \( v \);
2. Compute the Jacobian and use its absolute value to make the appropriate change in the differential area i.e. \( dx \, dy \rightarrow \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \);
3. Change the region of integration to reflect the equivalent limits for the new variables.

**Example 101.1**

Suppose that \( x = r \cos \theta \) and \( y = r \sin \theta \) transforms a region \( R \) onto a region \( T \). Use the change of variables formula to express the integral \( \int_{R} f(x,y) \, dx \, dy \) in terms of the integral of \( f \) over \( T \).

**Solution.**

The Jacobian is given by

\[
\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r
\]
Thus,\[
\int_R f(x, y) \, dx \, dy = \int_T f(r, \theta) r \, dr \, d\theta \]

**Example 101.2**
Find the Jacobian for the change of variables: \(x = 5u + 2v, \ y = 3u + v\).

**Solution.**
\[
\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix} = -1
\]

**Change of Variables in Triple Integrals**
Suppose that the transformations \(x = x(u, v, w), \ y = y(u, v, w), \ z = z(u, v, w)\)
map a region \(S\) in the \(uvw\)-space to a region \(W\) in the \(xyz\)-plane then the Jacobian of this change of variables is given by the determinant
\[
\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}
\]
and
\[
\int_W f(x, y, z) \, dx \, dy \, dz = \int_S f(u, v, w) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw.
\]

**Example 101.3**
Find the volume of the ellipsoid \(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\).

**Solution.**
Let \(x = au, \ y = bv, \ z = cw\). Then ellipsoid in transformed to the unit sphere \(u^2 + v^2 + w^2 = 1\). The Jacobian is
\[
\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.
\]
Thus,
\[
\int_S dx \, dy \, dz = \int_W abc \, du \, dv \, dw = abc \int_W du \, dv \, dw = \frac{4}{3} \pi abc
\]
where \(\int_W du \, dv \, dw\) is the volume of the sphere.
102 Parametric Curves in Three Dimensions

Let $x, y$ and $z$ be continuous functions of a variable $t$:

$$x = x(t), \quad y = y(t), \quad z = z(t).$$

The parameter $t$ is the independent variable, and for each value of $t$ we obtain a point which we can plot on a coordinate plane. The trace of points as $t$ varies is called a **parametric curve**. Sets of equations that are defined by a parameter are called **parametric equations**. The process of describing a curve by parametric equations is referred to as **parameterization**.

**Example 102.1**
Find parametric equations for the curve $y = x^2$ in the $xy$–plane.

**Solution.**
A possible parameterization of the curve is

$$x = t, \quad y = t^2, \quad z = 0$$

If the domain of $t$ is a closed interval $[a, b]$ then the parametric curve is an oriented curve with initial point $(x(a), y(a), z(a))$ and terminal point $(x(b), y(b), z(b))$. In this case, the curve is called a **path**. For example, ”the unit circle, traversed counter-clockwise” specifies a path. Also, ”the unit circle, traversed clockwise” specifies another. They are not the same because their orientations differ.

There are many ways to parameterize a given path. For example, here are several parameterizations of the unit circle, traversed counter clockwise:

$$
\begin{align*}
x(t) &= \cos (2\pi t), & y(t) &= \sin (2\pi t), & z(t) &= 0, & 0 \leq t \leq 1 \\
x(t) &= \cos (t^2), & y(t) &= \sin (t^2), & z(t) &= 0, & 0 \leq t \leq 2\pi \\
x(t) &= \cos t, & y(t) &= \sin t, & z(t) &= 0, & 0 \leq t \leq 2\pi
\end{align*}
$$

**Example 102.2**
Find a parameterization for each of the paths shown in Figure 102.1
Solution.
A parameterization of the path (a) is
\[ x = 2 \cos t, \quad y = 3 \sin t, \quad z = 0, \quad 0 \leq t \leq 2\pi. \]

For (b) we note that when \( t = 0 \) then \((x, y, 0) = (0, 3, 0)\) and when \( t = \frac{\pi}{2} \) we have \((x, y, z) = (0, 0, 3)\). Thus, a parameterization of the path (b) is
\[ x = 0, \quad y = 3 \cos t, \quad z = -3 \sin t, \quad 0 \leq t \leq 2\pi. \]

Example 102.3
Describe the curve given parametrically by
\[ x = \cos t, \quad y = \sin t, \quad z = t \]

Solution.
A point moving on the curve follows a rising circular path around the \( z \)-axis. This curve is known as a **circular helix** and is shown in Figure 102.2.
Parameterization of a Line in 3-D

Let \( \vec{v} = a\vec{i} + b\vec{j} + c\vec{k} \) be a vector in space. Let \( P = (x_0, y_0, z_0) \) be a point not on the support of \( \vec{v} \). We would like to find a parameterization of the line through \( P \) that is parallel to \( \vec{v} \). For that, we pick an arbitrary point \( M = (x, y, z) \) on the line. Then the vector \( \overrightarrow{AM} \) is parallel to \( \vec{v} \). Thus, there is a \( t \) such that \( \overrightarrow{AM} = t\vec{v} \). That is,

\[
(x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} = (ta)\vec{i} + (tb)\vec{j} + (ct)\vec{k}
\]

Equating components we find the parameterization

\[
x = x_0 + at, \quad y = y_0 + at, \quad z = z_0 + at.
\]

Example 102.4

(a) Describe in words the curve given by the parameterization

\[
x = 3 + t, \quad y = 2t, \quad z = 1 - t.
\]

(b) Find a parameterization for the line through the points \( A = (1, 2, -1) \) and \( B = (3, 3, 4) \).

Solution.

(a) The curve represents the line passing through the point \( (1, 0, 1) \) and parallel to the vector \( \vec{v} = \vec{i} + 2\vec{j} - \vec{k} \).

(b) Let \( \vec{v} = \overrightarrow{AB} = 2\vec{i} + \vec{j} + 5\vec{k} \). Thus, using the point \( A \), a parameterization is

\[
x = 1 + 2t, \quad y = 2 + t, \quad z = -1 + 5t
\]

Representation of a Parametric Curve as a Vector-Valued Function

We have seen that any point \( P = (x, y, z) \) in the Cartesian plane is defined by its position vector:

\[
\vec{r} = \overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}.
\]

Now, consider a path \( C \) with a parameterization \( x = x(t), y = y(t), \) and \( z = z(t) \). We can write these as a single vector

\[
\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}.
\]
As the parameter $t$ varies, the vector $\vec{r}(t)$ traces out the path $C$ in 3-space. See Figure 102.3.

![Figure 102.3](image)

For example, a parameterization of a line passing through $A = (x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ is given by

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

since

$$(x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} = a\vec{i} + b\vec{j} + c\vec{k}$$

**Example 102.5**

Write the parameterization of the circular helix of Example 102.2 in vector form.

**Solution.**

Since a circular helix has the parametric equations $x = \cos t$, $y = \sin t$, $z = t$, the corresponding parameterization is

$$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$$

**Example 102.6**

Find the parameterization in vector form of the line segment passing through the points $(2, -1, 3)$ and $(-1, 5, 4)$.

**Solution.**

The parameterization of the line segment with endpoints the given points is

$$\vec{r}(t) = 2\vec{i} - \vec{j} + 3\vec{k} + t(-3\vec{i} + 6\vec{j} + \vec{k}), \quad 0 \leq t \leq 1$$
Intersection of Curves and Surfaces
We can use the parametric equations for a curve to find where a curve intersects a surface or another curve. We illustrate these questions in the following examples.

Example 102.7
Find the points where the line \( x = t, \ y = 2t, \ z = 1 + t \) intersects the surface of the sphere \( x^2 + y^2 + z^2 = 100 \).

Solution.
We are looking for the values of \( t \) where
\[
6t^2 + 2t - 99 = 0.
\]
The two solutions are found by the quadratic formula
\[
t_1 = \frac{-2-\sqrt{2380}}{12} = \frac{-1-\sqrt{595}}{6} \approx -4.23 \text{ and } t_2 = \frac{-2+\sqrt{2380}}{12} = \frac{-1+\sqrt{595}}{6} \approx 3.90
\]
Thus, the points of intersection are \((-4.23, 2(-4.23), 1+(-4.23)) = (-4.23, -8.46, -3.23)\) and \((3.90, 7.80, 4.90) \)

Example 102.8
Do the lines
\[
\begin{align*}
x & = t - 1, \\
y & = 2t + 1, \\
z & = 5 - t
\end{align*}
\]
and
\[
\begin{align*}
x & = 2t + 2, \\
y & = t + 4, \\
z & = t + 3
\end{align*}
\]
intersect?

Solution.
For the lines to intersect we must find \( t_1 \) and \( t_2 \) such that:
\[
t_1 - 1 = 2t_2 + 2, \quad 2t_1 + 1 = t_2 + 4, \quad 5 - t_1 = t_2 + 3.
\]
Multiplying the first equation by 2 and subtracting the resulting equation from the second we get: \( 3 = -3t_2 \) or \( t_2 = -1 \). Substituting this value in the first equation we find \( t_1 = 1 \). Since \( t_1 \) and \( t_2 \) do not satisfy the third equation, the two lines do not intersect.
Example 102.9
Show that the lines

\[ x = 3 - 2t, \quad y = 5 + t, \quad z = -4 + 2t \]

and

\[ x = 7 + 6t, \quad y = 1 - 3t, \quad z = 5 - 6t \]

are parallel.

Solution.
The first line passes the point \((3, 5, -4)\) and parallel to the vector \(-2\vec{i} + \vec{j} + 2\vec{k}\).
The second line passes through the point \((7, 1, 5)\) and parallel to the vector \(6\vec{i} - 3\vec{j} - 6\vec{k}\). Since \(6\vec{i} - 3\vec{j} - 6\vec{k} = -3(-2\vec{i} + \vec{j} + 2\vec{k})\), the two lines are parallel.
103 Two and Three-Dimmmensional Motion: Velocity and Acceleration

In Section 81, we saw that the objects of velocity and acceleration are modeled by vectors. That is, they both had a magnitude and a direction associated with them. In particular the velocity of a moving object e.g. a particle has the following properties:

- The magnitude of the vector $\vec{v}$ is the speed of the object.
- The direction of $\vec{v}$ is the direction of motion.

In this section we discuss the two dimensional velocity an acceleration.

Let $\vec{r}(t)$ be the position vector at time $t$ and $\vec{r}(t + \Delta t)$ the position at time $t + \Delta t$. Then the displacement vector between these two positions is $\Delta \vec{r}(t) = \vec{r}(t + \Delta t) - \vec{r}(t)$ as shown in Figure 103.1. The average velocity over the interval $[t, t + \Delta t]$ is defined by

$$\text{Average velocity} = \frac{\Delta \vec{r}(t)}{\Delta t}$$

and the instantaneous velocity is

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

whenever the limit exists. We write $\vec{v}(t) = \vec{r}'(t)$. Note that $\vec{v}(t)$ is tangent to the object’s path of motion.

Figure 103.1

$$\Delta \vec{r}(t) = \vec{r}(t + \Delta t) - \vec{r}(t)$$

In a similar way, we define the acceleration vector to be

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t}$$
whenever the limit exists. We write \( \vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) \)

**Components of \( \vec{v}(t) \) and \( \vec{a}(t) \)**

Next, we will express \( \vec{v}(t) \) and \( \vec{a}(t) \) in terms of their components. To start with, let \( x = x(t), \ y = y(t), \ z = z(t) \) be a parameterization of the path of the object. Then we can express the position at time \( t \) by the position vector \( \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \). From the definition of the velocity vector we have

\[
\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t)\hat{i} + y(t + \Delta t)\hat{j} + z(t + \Delta t)\hat{k} - (x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k})}{\Delta t} = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}
\]

In a similar argument, we find

\[
\vec{a}(t) = x''(t)\hat{i} + y''(t)\hat{j} + z''(t)\hat{k}
\]

**Example 103.1**

Find the velocity and acceleration vectors given the parameterization \( x(t) = 3 \cos (t^2), \ y(t) = 3 \sin (t^2), \ z(t) = t^2 \).

**Solution.**

The velocity vector is given by

\[
\vec{v}(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k} = -6t \sin (t^2)\hat{i} + 6t \cos (t^2)\hat{j} + 2t\hat{k}
\]

and the acceleration is the vector

\[
\vec{a}(t) = x''(t)\hat{i} + y''(t)\hat{j} + z''(t)\hat{k} = -6(\sin (t^2) + 2t \cos (t^2))\hat{i} + 6(\cos (t^2) - 2t \sin (t^2))\hat{j} + 2\hat{k}
\]

As noted above, the velocity vector is tangent to the path of motion. It can be used to find a parameterization of the tangent line as illustrated in the following example.
Example 103.2
Find the tangent line at the point $(1, 1, 2)$ to the path with vector parameterization
\[ \vec{r}(t) = t^2 \vec{i} + t^3 \vec{j} + 2t \vec{k} \]

Solution.
We have \( \vec{r}_0 = \vec{r}(1) = \vec{i} + \vec{j} + 2\vec{k} \). The vector parameterization of the tangent line is
\[ \vec{r}(t) = \vec{r}_0 + t \vec{v}(1) = \vec{i} + \vec{j} + 2\vec{k} + t(2\vec{i} + 3\vec{j} + 2\vec{k}) \]

Uniform Circular Motion
A circular motion is the motion of an object along a path of radius \( R \) as shown in Figure 103.2.

There are two ways to describe the motion of the object—linear and angular speed. The linear speed \( v \) of the object is the rate at which the distance traveled is changing. It is defined by the formula
\[ v = \frac{s}{t} \]
where \( s \) is the distance traveled on the circle.
The angular speed \( \omega \) is the rate at which the central angle is changing. It is given by
\[ \omega = \frac{\theta}{t} \]
Its units are radians per seconds.
Thus, a parameterization of the motion is given by
\[ \vec{r}(t) = R \cos \theta \vec{i} + R \sin \theta \vec{j} = R \cos (\omega t) \vec{i} + R \sin (\omega t) \vec{j} \]
where \( t \) is the time in seconds.
The time it takes for the object to make one complete revolution, also known as the period, is \( \frac{2\pi}{\omega} \). This is due to the fact that both functions \( \cos(\omega t) \) and \( \sin(\omega t) \) have the same period \( \frac{2\pi}{\omega} \).
The linear velocity of the object is always tangent to the circle. The velocity is
\[
\vec{v}(t) = -R\omega \sin(\omega t) \hat{i} + R\omega \cos(\omega t) \hat{j}
\]
Also, the speed of the object is
\[
||\vec{v}(t)|| = \sqrt{(R\omega \sin(\omega t))^2 + (R\omega \cos(\omega t))^2} = R\omega.
\]
Hence, the speed of the object is constant and thus the name uniform.
We note that
\[
\vec{v} \cdot \vec{r} = (-R\omega \sin(\omega t) \hat{i} + R\omega \cos(\omega t) \hat{j}) \cdot (R \cos(\omega t) \hat{i} + R \sin(\omega t) \hat{j}) = -\omega R^2 \sin(\omega t) \cos(\omega t) + \omega R^2 \sin(\omega t) \cos(\omega t) = 0
\]
Thus the velocity and the position are perpendicular.
Next, the acceleration vector is given by
\[
\vec{a}(t) = \vec{v}'(t) = -\omega^2 (R \cos(\omega t) \hat{i} + R \sin(\omega t) \hat{j}) = -\omega^2 \vec{r}(t)
\]
Since the acceleration is a multiple of the position, these vectors are parallel.
Note the minus sign in the above expression means that the acceleration points in the opposite direction of \( \vec{r}(t) \) that is towards the center of the circle. Since \( \vec{v} \) is perpendicular to \( \vec{r} \), the acceleration is also perpendicular to \( \vec{v} \).
The magnitude of the acceleration is:
\[
||\vec{a}|| = \omega^2 ||\vec{r}|| = \omega^2 R = \frac{||\vec{r}||^2}{R}.
\]
Example 103.3
Find the velocity and the acceleration vectors of the uniform circular motion given by
\[
x = 3 \cos(2\pi t), \quad y = 3 \sin(2\pi t), \quad z = 0.
\]
Check that \( \vec{v} \) and \( \vec{a} \) are perpendicular. Also, check that the speed and the magnitude of the acceleration are constant.
Solution.
The velocity is
\[ \vec{v}(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k} = -6\pi \sin(2\pi t)\vec{i} + 6\pi \cos(2\pi t)\vec{j} \]
and the acceleration is
\[ \vec{a}(t) = x''(t)\vec{i} + y''(t)\vec{j} + z''(t)\vec{k} = -12\pi^2 \cos(2\pi t)\vec{i} - 12\pi^2 \sin(2\pi t)\vec{j} \]
Now,
\[ \vec{v} \cdot \vec{a} = (-6\pi \sin(2\pi t)\vec{i} + 6\pi \cos(2\pi t)\vec{j}) \cdot (-12\pi^2 \cos(2\pi t)\vec{i} - 12\pi^2 \sin(2\pi t)\vec{j}) \]
= \[72\pi^3 \sin(2\pi t) \cos(2\pi t) - 72\pi^3 \sin(2\pi t) \cos(2\pi t) = 0 \]
Hence, \( \vec{v} \) and \( \vec{a} \) are perpendicular. Finally, we have
\[ ||\vec{v}|| = \sqrt{(-6\pi \sin(2\pi t))^2 + (6\pi \cos(2\pi t))^2} = 6\pi \]
and
\[ ||\vec{a}|| = \sqrt{(-12\pi^2 \cos(2\pi t))^2 + (-12\pi^2 \sin(2\pi t))^2} = 12\pi^2 \]

Linear Motion
In uniform circular motion the magnitude of the velocity is constant but its direction is always changing during the motion. For linear motion (i.e. motion in a straight line), the magnitude of the velocity is changing but its direction is constant. The acceleration is parallel to the velocity vector (versus perpendicular in uniform circular motion) and points in the same direction if the particle is speeding up and in the opposite direction if the particle is slowing down.
A linear motion is a motion described by
\[ \vec{r}(t) = \vec{r}_0 + f(t)\vec{v}_0 \]
That is, the motion is along a straight line through the point with position vector \( \vec{r}_0 \) and parallel to \( \vec{v}_0 \).

Example 103.4
Consider the linear motion given by the vector equation
\[ \vec{r}(t) = 2\vec{i} + 6\vec{j} + (t^3 + t)(4\vec{i} + 3\vec{j} + \vec{k}) \]
Compute the velocity and acceleration vectors.
Solution.
We have
\[ \vec{v}(t) = \vec{r}'(t) = (3t^2 + 1)(4\vec{i} + 3\vec{j} + \vec{k}) \]
and
\[ \vec{a}(t) = 6t(4\vec{i} + 3\vec{j} + \vec{k}). \]
We note that the acceleration and velocity vectors are parallel since they are both multiples of the same vector: \(4\vec{i} + 3\vec{j} + \vec{k}\). Also, the vector \( \vec{r} \) is always pointing in the direction of \(4\vec{i} + 3\vec{j} + \vec{k} \) since \(3t^2 + 1 > 0\). Now, for \( t < 0 \), the acceleration is pointing in the opposite direction of the vector \(4\vec{i} + 3\vec{j} + \vec{k} \) and therefore in the opposite direction of \( \vec{v} \) so the object is slowing down. For \( t > 0 \) the acceleration is pointing in the same direction as the vector \(4\vec{i} + 3\vec{j} + \vec{k} \) and thus in the same direction as \( \vec{v} \) so the object is speeding up.

The Length of a Curve
From the definition of velocity we have:
\[ \vec{v}(t) = \frac{dx}{dt} \vec{i} + \frac{dy}{dt} \vec{j} + \frac{dz}{dt} \vec{k} \]
And the speed is therefore:
\[ ||\vec{v}|| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}. \]
As in one dimension, we have
\[ \text{Distance traveled} = \int_a^b ||\vec{v}|| dt \]

Example 103.5
Find the length of the curve given by the parameterization
\[ x = 3 + 5t, \quad y = 1 + 4t, \quad z = 3 - t, \quad 1 \leq t \leq 2. \]

Solution.
\[ \text{Length of curve} = \int_1^2 ||\vec{v}|| dt = \int_1^2 \sqrt{25 + 16 + 1} dt = \sqrt{42} \]
104 Vector Fields

Up to this point we have seen functions that take a point in the Cartesian system to a number. Such functions are called scalar functions. However, we encountered functions such as the gradient that takes a point to a vector. The gradient provides an example of a vector field.

By a vector field we mean a function \( \vec{F} \) that assigns a vector to each point in 2- or 3-space. The standard notation for the function \( \vec{F} \) is,

\[
\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}
\]

\[
\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}
\]

depending on whether or not we are in two or three dimensions.

Vector fields can be visualized by sampling the vector function at various points and drawing arrows with tails at the point in the appropriate directions. The length of the arrows is proportional to the magnitude of the vector function at the point.

Example 104.1

Sketch the vector field in 2-space given by \( \vec{F}(x, y) = -y\vec{i} + x\vec{j} \).

Solution.

First, we note that \( ||\vec{F}(x, y)|| = \sqrt{x^2 + y^2} \). This means that vectors at a fixed distance from the origin have the same magnitude. Moreover, the length of the arrows increasing as the distance from the origin increases. The vector \( \vec{F}(x, y) \) is perpendicular to the position vector \( \vec{r} = x\vec{i} + y\vec{j} \) since

\[
\vec{F}(x, y) \cdot \vec{r} = -xy + xy = 0.
\]

Evaluating \( \vec{F} \) at few points in the Cartesian system and plotting the corresponding vectors we find the vector field shown in Figure 104.1.
Example 104.2
Sketch the vector field in 2-space given by
\[(a) \vec{F}(x,y) = x\hat{j}.\]
\[(b) \vec{G}(x,y) = x\hat{i}.\]

Solution.
The vector $x\hat{j}$ is parallel to the $y$–axis, pointing up when $x > 0$ and down when $x < 0$. The larger the $|x|$ is the longer the vector. The vectors in the field are constant along vertical lines since the vector field does not depend on $y$. See Figure 104.2(a).
The vector $x\hat{i}$ is parallel to the $x$–axis, pointing to the right when $x > 0$ and to the left when $x < 0$. The larger the $|x|$ is the longer the vector. The vectors in the field are constant along horizontal lines since the vector field does not depend on $y$. See Figure 104.2(b).

Example 104.3
Sketch the gradient vector field for $f(x,y) = x^2 + y^2$ as well as several contours for this function.

Solution.
Recall that the contours for a function are nothing more than curves defined...
by the equation
\[ x^2 + y^2 = k, \quad k \geq 0. \]
That is, the contour curves are circles centered at the origin and with radius \( \sqrt{k} \).
The gradient vector field for this function is
\[ \vec{F}(x, y) = 2x\vec{i} + 2y\vec{j}. \]
The vector field and the contour curves are shown in Figure 104.3.
An example of a physical vector field is a force field. An example of a force field is the earth’s magnetic field, shown in Figure 104.4. In this case, the vectors point around the earth starting at the south-pole towards the north-pole.

![Figure 104.4](image)

Sometimes it is convenient to represent $\vec{F}(x, y, z)$ by its position vector $\vec{r} = xi + yj + zk$ and write the vector field as $\vec{F}(\vec{r})$.

**Example 104.4**

Newton’s Law of Gravitation states that the magnitude of the gravitational force exerted by an object of mass $M$ on an object of mass $m$ is proportional to $M$ and $m$ and inversely proportional to the square of the distance between them. The direction of the force is from $m$ to $M$ along the line connecting them. Find a formula for the vector field $\vec{F}(\vec{r})$ that represents the gravitational force, assuming that $M$ is located at the origin and $m$ is located at the point with position vector $\vec{r}$.

**Solution.**

By Newton’s law we have

$$||\vec{F}(\vec{r})|| = \frac{GMm}{||\vec{r}||^2}.$$

A unit vector in the direction of the force is $-\frac{\vec{r}}{||\vec{r}||}$. The negative sign means that the force is attractive. Thus, the expression for the force vector field is

$$\vec{F}(\vec{r}) = -\frac{GMm}{||\vec{r}||^3} \vec{r}.$$

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Flow lines (also known as integral curves or streamlines) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus the vectors in a vector field are tangent to the flow lines. If the vector field is $\vec{F}$ and the path is $\vec{r}(t)$ then a flow line satisfies the equation
$$\vec{r}'(t) = \vec{F}(\vec{r}(t)).$$

Resolving $\vec{F}$ and $\vec{r}(t)$ into components, $\vec{F} = F_1 \vec{i} + F_2 \vec{j}$ and $\vec{r}(t) = x(t) \vec{i} + y(t) \vec{j}$, a flow line is the solution to the system of differential equation
$$x'(t) = F_1(x(t), y(t)) \text{ and } y'(t) = F_2(x(t), y(t)).$$

The flow of a vector field is the collection of all its flow lines.

**Example 105.1**
Find the flow line of the vector field $\vec{F}(x(t), y(t)) = x \vec{i} - y \vec{j}$ that passes through the point $(1, 1)$ at time $t = 0$.

**Solution.**
We must solve the differential equations $x'(t) = x(t)$ and $y'(t) = -y(t)$ with conditions $x(0) = 1$ and $y(0) = 1$. We solve the first equation:
$$x'(t) = x(t)$$
$$\frac{1}{x(t)} x'(t) = 1$$
$$\frac{d}{dt} \ln |x(t)| = 1$$
$$\ln |x(t)| = t + A$$
$$|x(t)| = e^{A+t}$$
$$x(t) = \pm e^{A+t} = Ae^t$$

Similarly, we find $y(t) = Be^{-t}$. But $x(0) = 1$ and $y(0) = 1$ so that $A = B = 1$.

Hence, a parameterization of the flow line is
$$x(t) = e^t, \quad y(t) = e^{-t}$$

Note that the flow line can be represented explicitly by the equation $y = \frac{1}{x}$.

The flow line and the vector field are shown in Figure 105.1(a)
Example 105.2

Find the flow of the vector field $\vec{F} = y\vec{i} - x\vec{j}$.

Solution.

Flow lines are solutions to the differential equations $x'(t) = y(t)$ and $y'(t) = -x(t)$. The solutions are given by $x(t) = A\cos t$ and $y(t) = -A\sin t$. Hence, the flowlines can be defined implicitly by the equation $x^2 + y^2 = A^2$. That is, the flow lines are circles centered at the origin and oriented clockwise. The flow vector is shown in Figure 105.1(b).