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Introductory Notes in Linear Algebra for the Engineers

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Contents

Preface

Linear algebra has evolved as a branch of mathematics with wide range of applications to the natural sciences, to engineering, to computer sciences, to management and social sciences, and more.
This book is addressed primarely to second and third year college engineering students who have already had a course in calculus and analytic geometry. It is the result of lecture notes given by the author at Arkansas Tech University. I have included as many problems as possible of varying degrees of difficulty. Most of the exercises are computational, others are routine and seek to fix some ideas in the reader's mind; yet others are of theoretical nature and have the intention to enhance the reader's mathematical reasoning.

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Linear Systems of Equations

In this chapter we shall develop the theory of general systems of linear equations. The tool we will use to find the solutions is the row-echelon form of a matrix. In fact, the solutions can be read off from the row- echelon form of the augmented matrix of the system. The solution technique, known as **elimination** method, is developed in Section 4.

1. Systems of Linear Equations

Consider the following problem: At a carry-out pizza restaurant, an order of 3 slices of pizza, 4 breadsticks, and 2 soft drinks cost \$13.35. A second order of 5 slices of pizza, 2 breadsticks , and 3 soft drinks cost \$19.50. If four bread sticks and a can of soda cost \$0.30 more than a slice of pizza, what is the cost of each item?
Let x_1 be the cost of a slice of pizza, x_2 the cost of a breadsticks, and x_3 the cost of a soft drink. The assumptions of the problem yield the following three equations:

$$\begin{cases} 3x_1 & + & 4x_2 & + & 2x_3 & = & 13.35 \\ 5x_1 & + & 2x_2 & + & 3x_3 & = & 19.50 \\ & & 4x_2 & + & x_3 & = & 0.30 + x_1 \end{cases}$$

or equivalently

$$\begin{cases} 3x_1 & + & 4x_2 & + & 2x_3 & = & 13.35 \\ 5x_1 & + & 2x_2 & + & 3x_3 & = & 19.50 \\ -x_1 & + & 4x_2 & + & x_3 & = & 0.30. \end{cases}$$

Thus, the problem is to find the values of x_1, x_2, and x_3. A system like the one above is called a linear system.
Many practical problems can be reduced to solving systems of linear equations. The main purpose of linear algebra is to find systematic methods for solving these systems. So it is natural to start our discussion of linear algebra by studying linear equations.

A **linear equation** in n variables is an equation of the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b \tag{1.1}$$

where $x_1, x_2, ..., x_n$ are the **unknowns** (i.e. quantities to be found) and $a_1, \cdots, a_n$ are the **coefficients** (i.e. given numbers). We assume that the a_i's are not all zero. Also given the number b known as the **constant term.** In the special case where $b = 0$, Equation (1.1) is called a **homogeneous linear equation.**
Observe that a linear equation does not involve any products, inverses, or roots of variables. All variables occur only to the first power and do not appear as arguments for trigonometric, logarithmic, or exponential functions.

Example 1.1
Determine whether the given equations are linear or not (i.e., non-linear):
(a) $3x_1 - 4x_2 + 5x_3 = 6.$
(b) $4x_1 - 5x_2 = x_1x_2.$
(c) $x_2 = 2\sqrt{x_1} - 6.$
(d) $x_1 + \sin x_2 + x_3 = 1.$
(e) $x_1 - x_2 + x_3 = \sin 3.$

Solution
(a) The given equation is in the form given by (1.1) and therefore is linear.
(b) The equation is non-linear because the term on the right side of the equation involves a product of the variables x_1 and x_2.
(c) A non-linear equation because the term $2\sqrt{x_1}$ involves a square root of the variable x_1.
(d) Since x_2 is an argument of a trigonometric function, the given equation is non-linear.
(e) The equation is linear according to (1.1) ■

In the case of $n = 2$, sometimes we will drop the subscripts and use instead $x_1 = x$ and $x_2 = y$. For example, $ax + by = c$. Geometrically, this is a straight line in the $xy-$coordinate system. Likewise, for $n = 3$, we will use $x_1 = x, x_2 = y$, and $x_3 = z$ and write $ax + by + cz = d$ which is a plane in the $xyz-$coordinate system.
A **solution** of a linear equation (1.1) in n unknowns is a finite ordered collection of numbers $s_1, s_2, ..., s_n$ which make (1.1) a true equality when $x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$ are substituted in (1.1). The collection of all solutions of a linear equation is called the **solution set** or the **general solution**.

Example 1.2
Show that $(5 + 4s - 7t, s, t)$, where $s, t \in \mathbb{R}$, is a solution to the equation

$$x_1 - 4x_2 + 7x_3 = 5.$$

Solution
$x_1 = 5 + 4s - 7t, x_2 = s$, and $x_3 = t$ is a solution to the given equation because

$$x_1 - 4x_2 + 7x_3 = (5 + 4s - 7t) - 4s + 7t = 5 \ ■$$

Many problems in the sciences lead to solving more than one linear equation. The general situation can be described by a linear system.
A **system of linear equations** or simply a **linear system** is any finite collection of linear equations. A linear system of m equations in n variables has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n & = b_2 \\ \quad\quad .... & .... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n & = b_m \end{cases}$$

Note that the coefficients a_{ij} consist of two subscripts. The subscript i indicates the equation in which the coefficient occurs, and the subscript j indicates which unknown it multiplies.
When a linear system has more equations than unknowns, we call the system **overdetermined.** When the system has more unknowns than equations then we call the system **underdetermined.**
A **solution** of a linear system in n unknowns is a finite ordered collection of numbers $s_1, s_2, ..., s_n$ for which the substitution

$$x_1 = s_1, x_2 = s_2, \cdots, x_n = s_n$$

makes each equation a true statement. In compact form, a solution is an ordered $n-$tuple of the form

$$(s_1, s_2, \cdots, s_n).$$

The collection of all solutions of a linear system is called the **solution set** or the **general solution**. To **solve** a linear system is to find its general solution.
A linear system can have infinitely many solutions (**dependent system**), exactly one solution (**independent system**) or no solutions at all. When a linear system has a solution we say that the system is **consistent**. Otherwise, the system is said to be **inconsistent.** Thus, for the case $n = 2$, a linear system is consistent if the two lines either intersect at one point (independent) or they coincide (dependent). In the case the two lines are parallel, the system is inconsistent. For the case, $n = 3$, replace a line by a plane.

Example 1.3
Find the general solution of the linear system

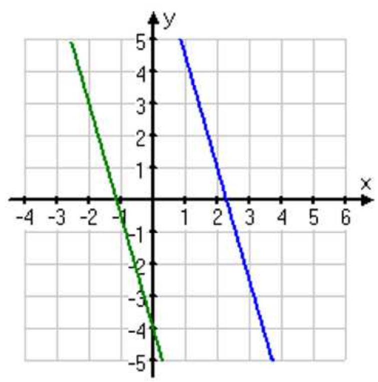
$$\begin{cases} x & + & y & = & 7 \\ 2x & + & 4y & = & 18. \end{cases}$$

Solution.
Multiply the first equation of the system by -2 and then add the resulting equation to the second equation to find $2y = 4$. Solving for y we find $y = 2$. Plugging this value in one of the equations of the given system and then solving for x one finds $x = 5$ ■

Example 1.4
Solve the system

$$\begin{cases} 7x + 2y = 16 \\ -21x - 6y = 24. \end{cases}$$

Solution.
Graphing the two lines we find

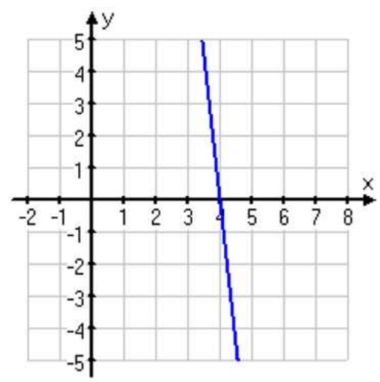


Thus, the system is inconsistent ■

Example 1.5
Solve the system

$$\begin{cases} 9x + y = 36 \\ 3x + \frac{1}{3}y = 12. \end{cases}$$

Solution.
Graphing the two lines we find



Thus, the system is consistent and dependent. Note that the two equations are basically the same: $9x + y = 36$. Letting $y = t$, where t is called a **parameter**, we can solve for x and find $x = \frac{36-t}{9}$. Thus, the general solution is defined by the **parametric equations**

$$x = \frac{36-t}{9}, \ y = t \blacksquare$$

Example 1.6
By letting $x_3 = t$, find the general solution of the linear system

$$\left\{ \begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 7 \\ 2x_1 & + & 4x_2 & + & x_3 & = & 18. \end{array} \right.$$

Solution.
By letting $x_3 = t$ the given system can be rewritten in the form

$$\left\{ \begin{array}{ccccc} x_1 & + & x_2 & = & 7-t \\ 2x_1 & + & 4x_2 & = & 18-t. \end{array} \right.$$

By multiplying the first equation by -2 and adding to the second equation one finds $x_2 = \frac{4+t}{2}$. Substituting this expression in one of the individual equations of the system and then solving for x_1 one finds $x_1 = \frac{10-3t}{2}$ $\blacksquare$

Practice Problems

Problem 1.1
Which of the following equations are not linear and why:
(a) $x_1^2 + 3x_2 - 2x_3 = 5$.
(b) $x_1 + x_1x_2 + 2x_3 = 1$.
(c) $x_1 + \frac{2}{x_2} + x_3 = 5$.

Problem 1.2
Show that $(2s + 12t + 13, s, -s - 3t - 3, t)$ is a solution to the system

$$\left\{\begin{array}{ccccccccc} 2x_1 & + & 5x_2 & + & 9x_3 & + & 3x_4 & = & -1 \\ x_1 & + & 2x_2 & + & 4x_3 & & & = & 1 \end{array}\right.$$

Problem 1.3
Solve each of the following systems graphically:
(a)

$$\left\{\begin{array}{r} 4x_1 - 3x_2 = 0 \\ 2x_1 + 3x_2 = 18 \end{array}\right.$$

(b)

$$\left\{\begin{array}{r} 4x_1 - 6x_2 = 10 \\ 6x_1 - 9x_2 = 15 \end{array}\right.$$

(c)

$$\left\{\begin{array}{r} 2x_1 + x_2 = 3 \\ 2x_1 + x_2 = 1 \end{array}\right.$$

Which of the above systems is consistent and which is inconsistent?

Problem 1.4
Determine whether the system of equations is linear or non-linear.

(a)

$$\left\{\begin{array}{r} \ln x_1 + x_2 + x_3 = 3 \\ 2x_1 + x_2 - 5x_3 = 1 \\ -x_1 + 5x_2 + 3x_3 = -1. \end{array}\right.$$

(b)

$$\left\{\begin{array}{ccccccc} 3x_1 & + & 4x_2 & + & 2x_3 & = & 13.35 \\ 5x_1 & + & 2x_2 & + & 3x_3 & = & 19.50 \\ -x_1 & + & 4x_2 & + & x_3 & = & 0.30. \end{array}\right.$$

Problem 1.5
Find the parametric equations of the solution set to the equation $-x_1+5x_2+3x_3-2x_4=-1$.

Problem 1.6
Write a system of linear equations consisting of three equations in three unknowns with
(a) no solutions.
(b) exactly one solution.
(c) infinitely many solutions.

Problem 1.7
For what values of h and k the system below has (a) no solution, (b) a unique solution, and (c) many solutions.

$$\begin{cases} x_1 + 3x_2 = 2 \\ 3x_1 + hx_2 = k. \end{cases}$$

Problem 1.8
True/False:
(a) A general solution of a linear system is an explicit description of all the solutions of the system.
(b) A linear system with either one solution or infinitely many solutions is said to be inconsistent.
(c) Finding a parametric description of the solution set of a linear system is the same as solving the system.
(d) A linear system with a unique solution is consistent and dependent.

Problem 1.9
Find a linear equation in the variables x and y that has the general solution $x=5+2t$ and $y=t$.

Problem 1.10
Find a relationship between a, b, c so that the following system is consistent.

$$\begin{cases} x_1 + x_2 + 2x_3 = a \\ x_1 + x_3 = b \\ 2x_1 + x_2 + 3x_3 = c. \end{cases}$$

2. Equivalent Systems and Elementary Row Operations: The Elimination Method

Next, we shift our attention for solving linear systems of equations. In this section we introduce the concept of elementary row operations that will be vital for our algebraic method of solving linear systems.
First, we define what we mean by equivalent systems: Two linear systems are said to be **equivalent** if and only if they have the same set of solutions.

Example 2.1
Show that the system

$$\left\{\begin{array}{rcrcr} x_1 & - & 3x_2 & = & -7 \\ 2x_1 & + & x_2 & = & 7 \end{array}\right.$$

is equivalent to the system

$$\left\{\begin{array}{rcrcr} 8x_1 & - & 3x_2 & = & 7 \\ 3x_1 & - & 2x_2 & = & 0 \\ 10x_1 & - & 2x_2 & = & 14. \end{array}\right.$$

Solution.
Solving the first system one finds the solution $x_1 = 2, x_2 = 3$. Similarly, solving the second system one finds the solution $x_1 = 2$ and $x_2 = 3$. Hence, the two systems are equivalent ■

Example 2.2
Show that if $x_1 + kx_2 = c$ and $x_1 + \ell x_2 = d$ are equivalent then $k = l$ and $c = d$.

Solution.
For arbitrary t the ordered pair $(c-kt, t)$ is a solution to the second equation. That is $c - kt + \ell t = d$ for all $t \in \mathbb{R}$. In particular, if $t = 0$ we find $c = d$. Thus, $kt = \ell t$ for all $t \in \mathbb{R}$. Letting $t = 1$ we find $k = \ell$ ■

Our basic algebraic method for solving a linear system is known as the **method of elimination**. The method consists of reducing the original system to an equivalent system that is easier to solve. The reduced system has the shape of an upper (resp. lower) triangle. This new system can

be solved by a technique called **backward-substitution** (resp. **forward-substitution**): The unknowns are found starting from the bottom (resp. the top) of the system.
The three basic operations in the above method, known as the **elementary row operations**, are summarized as follows:
(I) Multiply an equation by a non-zero number.
(II) Replace an equation by the sum of this equation and another equation multiplied by a number.
(III) Interchange two equations.

To indicate which operation is being used in the process one can use the following shorthand notation. For example, $r_3 \leftarrow \frac{1}{2}r_3$ represents the row operation of type (I) where each entry of row 3 is being replaced by $\frac{1}{2}$ that entry. Similar interpretations for types (II) and (III) operations.
The following theorem asserts that the system obtained from the original system by means of elementary row operations has the same set of solutions as the original one.

Theorem 2.1
Suppose that an elementary row operation is performed on a linear system. Then the resulting system is equivalent to the original system.

Example 2.3
Use the elimination method described above to solve the system

$$\left\{\begin{array}{ccccccc} x_1 & + & x_2 & - & x_3 & = & 3 \\ x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 2x_1 & - & 2x_2 & + & x_3 & = & 4. \end{array}\right.$$

Solution.
Step 1: We eliminate x_1 from the second and third equations by performing two operations $r_2 \leftarrow r_2 - r_1$ and $r_3 \leftarrow r_3 - 2r_1$ obtaining

$$\left\{\begin{array}{ccccccc} x_1 & + & x_2 & - & x_3 & = & 3 \\ & - & 4x_2 & + & 3x_3 & = & -2 \\ & - & 4x_2 & + & 3x_3 & = & -2 \end{array}\right.$$

Step 2: The operation $r_3 \leftarrow r_3 - r_2$ leads to the system

$$\left\{\begin{array}{ccccccc} x_1 & + & x_2 & - & x_3 & = & 3 \\ & - & 4x_2 & + & 3x_3 & = & -2 \end{array}\right.$$

By assigning x_3 an arbitrary value t we obtain the general solution $x_1 = \frac{t+10}{4}, x_2 = \frac{2+3t}{4}, x_3 = t$. This means that the linear system has infinitely many solutions (consistent and dependent). Every time we assign a value to t we obtain a different solution ■

Example 2.4
Determine if the following system is consistent or not

$$\left\{\begin{array}{rcrcrcr} 3x_1 & + & 4x_2 & + & x_3 & = & 1 \\ 2x_1 & + & 3x_2 & & & = & 0 \\ 4x_1 & + & 3x_2 & - & x_3 & = & -2. \end{array}\right.$$

Solution.
Step 1: To eliminate the variable x_1 from the second and third equations we perform the operations $r_2 \leftarrow 3r_2 - 2r_1$ and $r_3 \leftarrow 3r_3 - 4r_1$ obtaining the system

$$\left\{\begin{array}{rcrcrcr} 3x_1 & + & 4x_2 & + & x_3 & = & 1 \\ & & x_2 & - & 2x_3 & = & -2 \\ & - & 7x_2 & - & 7x_3 & = & -10. \end{array}\right.$$

Step 2: Now, to eliminate the variable x_3 from the third equation we apply the operation $r_3 \leftarrow r_3 + 7r_2$ to obtain

$$\left\{\begin{array}{rcrcrcr} 3x_1 & + & 4x_2 & + & x_3 & = & 1 \\ & & x_2 & - & 2x_3 & = & -2 \\ & & & - & 21x_3 & = & -24. \end{array}\right.$$

Solving the system by the method of backward substitution we find the unique solution $x_1 = -\frac{3}{7}, x_2 = \frac{2}{7}, x_3 = \frac{8}{7}$. Hence the system is consistent and independent ■

Example 2.5
Determine whether the following system is consistent:

$$\left\{\begin{array}{rcrcr} x_1 & - & 3x_2 & = & 4 \\ -3x_1 & + & 9x_2 & = & 8. \end{array}\right.$$

Solution.
Multiplying the first equation by 3 and adding the resulting equation to the second equation we find $0 = 20$ which is impossible. Hence, the given system is inconsistent ■

Practice Problems

Problem 2.1
Solve each of the following systems using the method of elimination:
(a)

$$\begin{cases} 4x_1 - 3x_2 = 0 \\ 2x_1 + 3x_2 = 18 \end{cases}$$

(b)

$$\begin{cases} 4x_1 - 6x_2 = 10 \\ 6x_1 - 9x_2 = 15 \end{cases}$$

(c)

$$\begin{cases} 2x_1 + x_2 = 3 \\ 2x_1 + x_2 = 1 \end{cases}$$

Which of the above systems is consistent and which is inconsistent?

Problem 2.2
Find the values of A,B,C in the following partial fraction

$$\frac{x^2 - x + 3}{(x^2+2)(2x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{2x-1}.$$

Problem 2.3
Find a quadratic equation of the form $y = ax^2 + bx + c$ that goes through the points $(-2, 20), (1, 5)$, and $(3, 25)$.

Problem 2.4
Solve the following system using the method of elimination.

$$\begin{cases} 5x_1 & - & 5x_2 & - & 15x_3 & = & 40 \\ 4x_1 & - & 2x_2 & - & 6x_3 & = & 19 \\ 3x_1 & - & 6x_2 & - & 17x_3 & = & 41 \end{cases}$$

Problem 2.5
Solve the following system using elimination.

$$\begin{cases} 2x_1 & + & x_2 & + & x_3 & = & -1 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & & & - & 2x_3 & = & 5 \end{cases}$$

Problem 2.6
Find the general solution of the linear system

$$\left\{\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & + & x_4 & = & -3 \\ 2x_1 & - & x_2 & + & 3x_3 & - & x_4 & = & 0 \end{array}\right.$$

Problem 2.7
Find a, b, and c so that the system

$$\left\{\begin{array}{rcrcrcr} x_1 & + & ax_2 & + & cx_3 & = & 0 \\ bx_1 & + & cx_2 & - & 3x_3 & = & 1 \\ ax_1 & + & 2x_2 & + & bx_3 & = & 5 \end{array}\right.$$

has the solution $x_1 = 3, x_2 = -1, x_3 = 2$.

Problem 2.8
Show that the following systems are equivalent.

$$\left\{\begin{array}{rcrcrcr} 7x_1 & + & 2x_2 & + & 2x_3 & = & 21 \\ & - & 2x_2 & + & 3x_3 & = & 1 \\ & & & & 4x_3 & = & 12 \end{array}\right.$$

and

$$\left\{\begin{array}{rcrcrcr} 21x_1 & + & 6x_2 & + & 6x_3 & = & 63 \\ & - & 4x_2 & + & 6x_3 & = & 2 \\ & & & & x_3 & = & 3 \end{array}\right.$$

Problem 2.9
Solve the following system by elimination.

$$\left\{\begin{array}{rcrcrcr} 3x_1 & + & x_2 & + & 2x_3 & = & 13 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 19 \\ x_1 & + & 4x_2 & + & 3x_3 & = & 15 \end{array}\right.$$

Problem 2.10
Solve the following system by elimination.

$$\left\{\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 7 \\ 2x_1 & + & x_2 & + & x_3 & = & 4 \\ -3x_1 & + & 2x_2 & - & 2x_3 & = & -10 \end{array}\right.$$

3. Solving Linear Systems Using Augmented Matrices

In this section we apply the elimination method described in the previous section to the rectangular array consisting of the coefficients of the unknowns and the right-hand side of a given system rather than to the individual equations. To elaborate, consider the linear system

$$\left\{\begin{array}{ll} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n & = b_2 \\ \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots. \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n & = b_m \end{array}\right.$$

We define the **augmented matrix** corresponding to the above system to be the rectangular array

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

We then apply elementary row operations on the augmented matrix and reduces it to a triangular matrix. Then the corresponding system is triangular as well and is equivalent to the original system. Next, use either the backward-substitution or the forward-substitution technique to find the unknowns. We illustrate this technique in the following examples.

Example 3.1
Solve the following linear system using elementary row operations on the augmented matrix:

$$\left\{\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9. \end{array}\right.$$

Solution.
The augmented matrix for the system is

$$\left[\begin{array}{rrrr} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Step 1: The operations $r_2 \leftarrow \frac{1}{2}r_2$ and $r_3 \leftarrow r_3 + 4r_1$ give

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Step 2: The operation $r_3 \leftarrow r_3 + 3r_2$ gives

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The corresponding system of equations is

$$\left\{ \begin{array}{ccccccc} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \right.$$

Using back-substitution we find the unique solution $x_1 = 29, x_2 = 16, x_3 = 3$ ■

Example 3.2
Solve the following linear system using the method described above.

$$\left\{ \begin{array}{ccccccc} & & x_2 & + & 5x_3 & = & -4 \\ x_1 & + & 4x_2 & + & 3x_3 & = & -2 \\ 2x_1 & + & 7x_2 & + & x_3 & = & -1. \end{array} \right.$$

Solution.
The augmented matrix for the system is

$$\begin{bmatrix} 0 & 1 & 5 & -4 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & -1 \end{bmatrix}$$

Step 1:The operation $r_2 \leftrightarrow r_1$ gives

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 2 & 7 & 1 & -1 \end{bmatrix}$$

Step 2: The operation $r_3 \leftarrow r_3 - 2r_1$ gives the system

$$\left[\begin{array}{rrrr} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & -1 & -5 & 3 \end{array}\right]$$

Step 3: The operation $r_3 \leftarrow r_3 + r_2$ gives

$$\left[\begin{array}{rrrr} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The corresponding system of equations is

$$\left\{\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & -2 \\ & & x_2 & + & 5x_3 & = & -4 \\ & & & & 0 & = & -1 \end{array}\right.$$

From the last equation we conclude that the system is inconsistent ■

Example 3.3
Determine if the following system is consistent.

$$\left\{\begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1. \end{array}\right.$$

Solution.
The augmented matrix of the given system is

$$\left[\begin{array}{rrrr} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right]$$

Step 1: The operation $r_3 \leftarrow r_3 - 2r_2$ gives

$$\left[\begin{array}{rrrr} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 1 & -2 & 3 & -1 \end{array}\right]$$

Step 2: The operation $r_3 \leftrightarrow r_1$ leads to

$$\left[\begin{array}{rrrr} 1 & -2 & 3 & -1 \\ 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \end{array}\right]$$

Step 3: Applying $r_2 \leftarrow r_2 - 2r_1$ to obtain

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -4 & 3 \\ 0 & 1 & -4 & 8 \end{bmatrix}$$

Step 4: Finally, the operation $r_3 \leftarrow r_3 - r_2$ gives

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Hence, the equivalent system is

$$\left\{ \begin{array}{ccccccc} x_1 & - & 2x_2 & + & 3x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 3 \\ & & & & 0 & = & 5 \end{array} \right.$$

This last system has no solution (the last equation requires x_1, x_2, and x_3 to satisfy the equation $0x_1 + 0x_2 + 0x_3 = 5$ and no such x_1, x_2, and x_3 exist). Hence the original system is inconsistent ∎

Pay close attention to the last row of the trinagular matrix of the previous exercise. This situation is typical of an inconsistent system.

Practice Problems

Problem 3.1
Solve the following linear system using the elimination method of this section.

$$\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & & & = & 0\\ -x_1 & + & 3x_2 & + & 3x_3 & = & -2\\ & & x_2 & + & x_3 & = & 0. \end{array}\right.$$

Problem 3.2
Find an equation involving g, h, and k that makes the following augmented matrix corresponds to a consistent system.

$$\left[\begin{array}{rrrc} 2 & 5 & -3 & g\\ 4 & 7 & -4 & h\\ -6 & -3 & 1 & k \end{array}\right]$$

Problem 3.3
Solve the following system using elementary row operations on the augmented matrix:

$$\left\{\begin{array}{rcrcrcr} 5x_1 & - & 5x_2 & - & 15x_3 & = & 40\\ 4x_1 & - & 2x_2 & - & 6x_3 & = & 19\\ 3x_1 & - & 6x_2 & - & 17x_3 & = & 41 \end{array}\right.$$

Problem 3.4
Solve the following system using elementary row operations on the augmented matrix:

$$\left\{\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & x_3 & = & -1\\ x_1 & + & 2x_2 & + & x_3 & = & 0\\ 3x_1 & & & - & 2x_3 & = & 5 \end{array}\right.$$

Problem 3.5
Solve the following system using elementary row operations on the augmented matrix:

$$\left\{\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 0\\ 2x_1 & + & 2x_2 & & & - & x_4 & = & 0\\ 3x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 0 \end{array}\right.$$

Problem 3.6
Find the value(s) of a for which the following system has a nontrivial solution. Find the general solution.

$$\begin{cases} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ x_1 & + & 3x_2 & + & 6x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & ax_3 & = & 0 \end{cases}$$

Problem 3.7
Solve the linear system whose augmented matrix is given by

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

Problem 3.8
Solve the linear system whose augmented matrix is reduced to the following triangular form

$$\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 3.9
Solve the linear system whose augmented matrix is reduced to the following triangular form

$$\begin{bmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{bmatrix}$$

Problem 3.10
Reduce the matrix to triangular matrix.

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 4 & 0 & -2 & 1 \\ 3 & -1 & 0 & 4 \end{bmatrix}$$

Problem 3.11
Solve the following system using elementary row operations on the augmented

matrix:

$$\begin{cases} 3x_1 + x_2 + 7x_3 + 2x_4 = 13 \\ 2x_1 - 4x_2 + 14x_3 - x_4 = -10 \\ 5x_1 + 11x_2 - 7x_3 + 8x_4 = 59 \\ 2x_1 + 5x_2 - 4x_3 - 3x_4 = 39 \end{cases}$$

4. Echelon Form and Reduced Echelon Form: Gaussian Elimination

The elimination method introduced in the previous section reduces the augmented matrix to a "nice" matrix (meaning the corresponding equations are easy to solve). Two of the "nice" matrices discussed in this section are matrices in either row-echelon form or reduced row-echelon form, concepts that we discuss next.
By a **leading entry** of a row in a matrix we mean the leftmost non-zero entry in the row.
A rectangular matrix is said to be in **row-echelon form** if it has the following three characterizations:

(1) All rows consisting entirely of zeros are at the bottom.
(2) The leading entry in each non-zero row is 1 and is located in a column to the right of the leading entry of the row above it.
(3) All entries in a column below a leading entry are zero.

The matrix is said to be in **reduced row-echelon form** if in addition to the above, the matrix has the following additional characterization:

(4) Each leading 1 is the only nonzero entry in its column.

Remark 4.1 From the definition above, note that a matrix in row-echelon form has zeros below each leading 1, whereas a matrix in reduced row-echelon form has zeros both above and below each leading 1.

Example 4.1
Determine which matrices are in row-echelon form (but not in reduced row-echelon form) and which are in reduced row-echelon form
(a)

$$\begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution.
(a)The given matrix is in row-echelon form but not in reduced row-echelon form since the $(1,2)-$entry is not zero.
(b) The given matrix satisfies the characterization of a reduced row-echelon form ■

The importance of the row-echelon matrices is indicated in the following theorem.

Theorem 4.1
Every nonzero matrix can be brought to (reduced) row-echelon form by a finite number of elementary row operations.

The process of reducing a matrix to a row-echelon form is known as **Gaussian elimination**. That of reducing a matrix to a reduced row-echelon form is known as **Gauss-Jordan elimination**.

Example 4.2
Use Gauss-Jordan elimination to transform the following matrix first into row-echelon form and then into reduced row-echelon form

$$\left[\begin{array}{ccccc} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array}\right]$$

Solution.
The reduction of the given matrix to row-echelon form is as follows.

Step 1: $r_1 \leftrightarrow r_4$

$$\left[\begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right]$$

Step 2: $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 + 2r_1$

$$\left[\begin{array}{ccccc} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right]$$

Step 3: $r_2 \leftarrow \frac{1}{2}r_2$ and $r_3 \leftarrow \frac{1}{5}r_3$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Step 4: $r_3 \leftarrow r_3 - r_2$ and $r_4 \leftarrow r_4 + 3r_2$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Step 5: $r_3 \leftrightarrow r_4$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 6: $r_5 \leftarrow -\frac{1}{5}r_5$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 7: $r_1 \leftarrow r_1 - 4r_2$

$$\begin{bmatrix} 1 & 0 & -3 & 3 & 5 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 8: $r_1 \leftarrow r_1 - 3r_3$ and $r_2 \leftarrow r_2 + 3r_3$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \blacksquare$$

Example 4.3
Use Gauss-Jordan elimination to transform the following matrix first into row-echelon form and then into reduced row-echelon form

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution.
By following the steps in the Gauss-Jordan algorithm we find

Step 1: $r_3 \leftarrow \frac{1}{3}r_3$

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 1 & -3 & 4 & -3 & 2 & 5 \end{bmatrix}$$

Step 2: $r_1 \leftrightarrow r_3$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 3: $r_2 \leftarrow r_2 - 3r_1$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 4: $r_2 \leftarrow \frac{1}{2}r_2$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Step 5: $r_3 \leftarrow r_3 - 3r_2$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Step 6: $r_1 \leftarrow r_1 + 3r_2$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Step 7: $r_1 \leftarrow r_1 - 5r_3$ and $r_2 \leftarrow r_2 - r_3$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \blacksquare$$

Remark 4.2
It can be shown that no matter how the elementary row operations are varied, one will always arrive at the same reduced row-echelon form; that is the reduced row echelon form is unique. On the contrary row-echelon form is **not** unique. However, the number of leading 1's of two different row-echelon forms is the same. That is, two row-echelon matrices have the same number of nonzero rows. This number is knwon as the **rank** of the matrix.

Example 4.4
Consider the system

$$\begin{cases} ax & + & by & = & k \\ cx & + & dy & = & l. \end{cases}$$

Show that if $ad - bc \neq 0$ then the reduced row-echelon form of the coefficient matrix is the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution.
The coefficient matrix is the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Assume first that $a \neq 0$. Using Gaussian elimination we reduce the above matrix into row-echelon form as follows:

Step 1: $r_2 \leftarrow ar_2 - cr_1$

$$\begin{bmatrix} a & b \\ 0 & ad - bc \end{bmatrix}$$

Step 2: $r_2 \leftarrow \frac{1}{ad-bc} r_2$

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

Step 3: $r_1 \leftarrow r_1 - br_2$

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

Step 4: $r_1 \leftarrow \frac{1}{a}r_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Next, assume that $a = 0$. Then $c \neq 0$ and $b \neq 0$. Following the steps of Gauss-Jordan elimination algorithm we find

Step 1: $r_1 \leftrightarrow r_2$

$$\begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$$

Step 2: $r_1 \leftarrow \frac{1}{c}r_1$ and $r_2 \leftarrow \frac{1}{b}r_2$

$$\begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix}$$

Step 3: $r_1 \leftarrow r_1 - \frac{d}{c}r_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \blacksquare$$

Example 4.5
Find the rank of each of the following matrices
(a)

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} 3 & 1 & 0 & 1 & -9 \\ 0 & -2 & 12 & -8 & -6 \\ 2 & -3 & 22 & -14 & -17 \end{bmatrix}$$

Solution.
(a) We use Gaussian elimination to reduce the given matrix into row-echelon form as follows:

Step 1: $r_2 \leftarrow r_2 - r_1$

$$\begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Step 2: $r_1 \leftrightarrow r_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Step 3: $r_2 \leftarrow r_2 - 2r_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -1 & 1 \end{bmatrix}$$

Step 4: $r_3 \leftarrow r_3 - r_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

Thus, $rank(A) = 3$.
(b) As in (a), we reduce the matrix into row-echelon form as follows:

Step 1: $r_1 \leftarrow r_1 - r_3$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 8 \\ 0 & -2 & 12 & -8 & -6 \\ 2 & -3 & 22 & -14 & -17 \end{bmatrix}$$

Step 2: $r_3 \leftarrow r_3 - 2r_1$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 25 \\ 0 & -2 & 12 & -8 & -6 \\ 0 & -11 & -22 & -44 & -33 \end{bmatrix}$$

Step 3: $r_2 \leftarrow -\frac{1}{2}r_2$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 8 \\ 0 & 1 & -6 & 4 & 3 \\ 0 & -11 & -22 & -44 & -33 \end{bmatrix}$$

Step 4: $r_3 \leftarrow r_3 + 11r_2$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 8 \\ 0 & 1 & -6 & 4 & 3 \\ 0 & 0 & -88 & 0 & 0 \end{bmatrix}$$

Step 5: $r_3 \leftarrow \frac{1}{8}r_3$

$$\begin{bmatrix} 1 & 4 & -22 & 15 & 8 \\ 0 & 1 & -6 & 4 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Hence, $rank(B) = 3$ ■

Practice Problems

Problem 4.1
Use Gaussina elimination to reduce the given matrix to row echelon form.

$$\begin{bmatrix} 1 & -2 & 3 & 1 & -3 \\ 2 & -1 & 3 & -1 & 0 \end{bmatrix}$$

Problem 4.2
Use Gaussina elimination to reduce the given matrix to row echelon form.

$$\begin{bmatrix} -1 & 0 & 2 & -3 \\ 0 & 3 & -1 & 7 \\ 3 & 2 & 0 & 7 \end{bmatrix}$$

Problem 4.3
Use Gaussina elimination to reduce the given matrix to row echelon form.

$$\begin{bmatrix} 5 & -5 & -15 & 40 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{bmatrix}$$

Problem 4.4
Use Gaussina elimination to reduce the given matrix to row echelon form.

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ 3 & 0 & -2 & 5 \end{bmatrix}$$

Problem 4.5
Which of the following matrices are not in reduced row-ehelon form and why?
(a)

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 4.6

Use Gaussian elimination to convert the following matrix into a row-echelon matrix.

$$\begin{bmatrix} 1 & -3 & 1 & -1 & 0 & -1 \\ -1 & 3 & 0 & 3 & 1 & 3 \\ 2 & -6 & 3 & 0 & -1 & 2 \\ -1 & 3 & 1 & 5 & 1 & 6 \end{bmatrix}$$

Problem 4.7

Use Gauss-Jordan elimination to convert the following matrix into reduced row-echelon form.

$$\begin{bmatrix} -2 & 1 & 1 & 15 \\ 6 & -1 & -2 & -36 \\ 1 & -1 & -1 & -11 \\ -5 & -5 & -5 & -14 \end{bmatrix}$$

Problem 4.8

Use Gauss-Jordan elimination to convert the following matrix into reduced row-echelon form.

$$\begin{bmatrix} 3 & 1 & 7 & 2 & 13 \\ 2 & -4 & 14 & -1 & -10 \\ 5 & 11 & -7 & 8 & 59 \\ 2 & 5 & -4 & -3 & 39 \end{bmatrix}.$$

Problem 4.9

Use Gauss elimination to convert the following matrix into row-echelon form.

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 4 & 0 & -2 & 1 \\ 3 & -1 & 0 & 4 \end{bmatrix}.$$

Problem 4.10

Use Gauss elimination to convert the following matrix into row-echelon form.

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}.$$

Problem 4.11
Find the rank of each of the following matrices.
(a)

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 4 & 0 & -2 & 1 \\ 3 & -1 & 0 & 4 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ -1 & -3 & 1 \end{bmatrix}$$

Solution.
(a) We reduce the given matrix to row-echelon form.

Step 1: $r_3 \leftarrow r_3 + 4r_1$ and $r_4 \leftarrow r_4 + 3r_1$

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & -4 & -2 & 1 \\ 0 & -4 & 0 & 4 \end{pmatrix}$$

Step 2: $r_4 \leftarrow r_4 - r_3$

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & -4 & -2 & 1 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

Step 3: $r_1 \leftarrow -r_1$ and $r_2 \leftrightarrow r_3$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

Step 4: $r_4 \leftarrow r_3 - r_4$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 5: $r_2 \leftarrow -\frac{1}{4}r_2$ and $r_3 \leftarrow \frac{1}{2}r_3$

$$\left(\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & .5 & -.25 \\ 0 & 0 & 1 & 1.5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, the rank of the given matrix is 3.
(b) Apply the Gauss algorithm as follows.

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_3 \leftarrow r_3 + r_1$

$$\left(\begin{array}{ccc} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & -4 & 4 \end{array}\right)$$

Step 2: $r_3 \leftarrow r_3 + 2r_2$

$$\left(\begin{array}{ccc} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{array}\right)$$

Step 3: $r_2 \leftarrow \frac{1}{2}r_2$

$$\left(\begin{array}{ccc} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

Hence, the rank is 2 ■

5. Echelon Forms and Solutions to Linear Systems

In this section we give a systematic procedure for solving systems of linear equations; it is based on the idea of reducing the augmented matrix to either the row-echelon form or the reduced row-echelon form. The new system is equivalent to the original system.
Unknowns corresponding to leading entries in the echelon augmented matrix are called **dependent** or **leading variables**. If an unknown is not dependent then it is called **free** or **independent** variable.

Example 5.1
Find the dependent and independent variables of the following system

$$\left\{\begin{array}{rcrcrcrcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1 \\ & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5 \\ 2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6 \end{array}\right.$$

Solution.
The augmented matrix for the system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

Using the Gaussian algorithm we bring the augmented matrix to row-echelon form as follows:

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_4 \leftarrow r_4 - 2r_1$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

Step 2: $r_2 \leftarrow -r_2$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

Step 3: $r_3 \leftarrow r_3 - 5r_2$ and $r_4 \leftarrow r_4 - 4r_2$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Step 4: $r_3 \leftrightarrow r_4$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 5: $r_3 \leftarrow \frac{1}{6}r_3$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading variables are x_1, x_3, and x_6. The free variables are x_2, x_4, and x_5 ■

One way to solve a linear system is to apply the elementary row operations to reduce the augmented matrix to a (reduced) row-echelon form. If the augmented matrix is in reduced row-echelon form then to obtain the general solution one just has to move all independent variables to the right side of the equations and consider them as parameters. The dependent variables are given in terms of these parameters.

Example 5.2
Solve the following linear system.

$$\begin{cases} x_1 + 2x_2 + x_4 = 6 \\ x_3 + 6x_4 = 7 \\ x_5 = 1. \end{cases}$$

Solution.
The augmented matrix is already in row-echelon form. The free variables are x_2 and x_4. So let $x_2 = s$ and $x_4 = t$. Solving the system starting from the bottom we find $x_1 = -2s - t + 6, x_3 = 7 - 6t$, and $x_5 = 1$ ■

If the augmented matrix does not have the reduced row-echelon form but the row-echelon form then the general solution also can be easily found by using the method of backward substitution.

Example 5.3
Solve the following linear system

$$\left\{\begin{array}{rcrcrcrcr} x_1 & - & 3x_2 & + & x_3 & - & x_4 & = & 2 \\ & & x_2 & + & 2x_3 & - & x_4 & = & 3 \\ & & & & x_3 & + & x_4 & = & 1. \end{array}\right.$$

Solution.
The augmented matrix is in row-echelon form. The free variable is $x_4 = t$. Solving for the leading variables we find, $x_1 = 11t + 4, x_2 = 3t + 1$, and $x_3 = 1 - t$ ■

The questions of existence and uniqueness of solutions are fundamental questions in linear algebra. The following theorem provides some relevant information.

Theorem 5.1
A system of m linear equations in n unknowns can have exactly one solution, infinitely many solutions, or no solutions at all.
(1) If the reduced augmented matrix has a row of the form $[0, 0, \cdots, 0, b]$ where b is a nonzero constant, then the system has no solutions.
(2) If the reduced augmented matrix has indepedent variables and no rows of the form $[0, 0, \cdots, 0, b]$ with $b \neq 0$ then the system has infinitely many solutions.
(3) If the reduced augmented matrix has no independent variables and no rows of the form $[0, 0, \cdots, 0, b]$ with $b \neq 0$, then the system has exactly one solution.

Example 5.4
Find the general solution of the system whose augmented matrix is given by

$$\left[\begin{array}{ccc} 1 & 2 & -7 \\ -1 & -1 & 1 \\ 2 & 1 & 5 \end{array}\right]$$

Solution.
We first reduce the system to row-echelon form as follows.

Step 1: $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 - 2r_1$

$$\begin{bmatrix} 1 & 2 & -7 \\ 0 & 1 & -6 \\ 0 & -3 & 19 \end{bmatrix}$$

Step 2: $r_3 \leftarrow r_3 + 3r_2$

$$\begin{bmatrix} 1 & 2 & -7 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

The corresponding system is given by

$$\left\{ \begin{array}{ccccc} x_1 & + & 2x_2 & = & -7 \\ & & x_2 & = & -6 \\ & & 0 & = & 1 \end{array} \right.$$

Because of the last equation the system is inconsistent ■

Example 5.5
Find the general solution of the system whose augmented matrix is given by

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 7 & -3 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution.
By adding two times the second row to the first row we find the reduced row-echelon form of the augmented matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that the free variables are $x_3 = s$ and $x_5 = t$. Solving for the leading variables we find $x_1 = -1 - t, x_2 = 1 + 3t$, and $x_4 = -4 - 5t$ ■

Example 5.6
Determine the value(s) of h such that the following matrix is the augmented matrix of a consistent linear system

$$\begin{bmatrix} 1 & 4 & 2 \\ -3 & h & -1 \end{bmatrix}$$

Solution.
By adding three times the first row to the second row we find

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 12+h & 5 \end{bmatrix}$$

The system is consistent if and only if $12 + h \neq 0$; that is, $h \neq -12$ ■

Example 5.7
Find (if possible) conditions on the numbers a, b, and c such that the following system is consistent

$$\begin{cases} x_1 & + & 3x_2 & + & x_3 & = & a \\ -x_1 & - & 2x_2 & + & x_3 & = & b \\ 3x_1 & + & 7x_2 & - & x_3 & = & c \end{cases}$$

Solution.
The augmented matrix of the system is

$$\begin{bmatrix} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{bmatrix}$$

Now apply Gaussian elimination as follows.

Step 1: $r_2 \leftarrow r_2 + r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 2 & b+a \\ 0 & -2 & -4 & c-3a \end{bmatrix}$$

Step 2: $r_3 \leftarrow r_3 + 2r_2$

$$\begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 2 & b+a \\ 0 & 0 & 0 & c-a+2b \end{bmatrix}$$

The system has no solution if $c - a + 2b \neq 0$. The system has infinitely many solutions if $c - a + 2b = 0$. In this case, the solution is given by $x_1 = 5t - (2a + 3b), x_2 = (a + b) - 2t, x_3 = t$ ■

Practice Problems

Problem 5.1
Using Gaussian elimination, solve the linear system whose augmented matrix is given by

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

Problem 5.2
Solve the linear system whose augmented matrix is reduced to the following reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{bmatrix}$$

Problem 5.3
Solve the linear system whose augmented matrix is reduced to the following row-echelon form

$$\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 5.4
Solve the following system using Gauss-Jordan elimination.

$$\left\{\begin{array}{rcrcrcrcr} 3x_1 & + & x_2 & + & 7x_3 & + & 2x_4 & = & 13 \\ 2x_1 & - & 4x_2 & + & 14x_3 & - & x_4 & = & -10 \\ 5x_1 & + & 11x_2 & - & 7x_3 & + & 8x_4 & = & 59 \\ 2x_1 & + & 5x_2 & - & 4x_3 & - & 3x_4 & = & 39 \end{array}\right.$$

Problem 5.5
Solve the following system.

$$\left\{\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & x_3 & = & -1 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & & & - & 2x_3 & = & 5 \end{array}\right.$$

Problem 5.6
Solve the following system using elementary row operations on the augmented matrix:

$$\left\{\begin{array}{rcrcrcr} 5x_1 & - & 5x_2 & - & 15x_3 & = & 40 \\ 4x_1 & - & 2x_2 & - & 6x_3 & = & 19 \\ 3x_1 & - & 6x_2 & - & 17x_3 & = & 41 \end{array}\right.$$

Problem 5.7
Reduce the following system to row echelon form and then find the solution.

$$\left\{\begin{array}{rcrcrcrcr} 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 5 \\ 4x_1 & + & 5x_2 & - & 3x_3 & + & 6x_4 & = & 9 \\ -2x_1 & + & 5x_2 & - & 2x_3 & + & 6x_4 & = & 4 \\ 4x_1 & + & 11x_2 & - & 4x_3 & + & 8x_4 & = & 2. \end{array}\right.$$

Problem 5.8
Reduce the following system to row echelon form and then find the solution.

$$\left\{\begin{array}{rcrcrcr} 2x_1 & - & 5x_2 & + & 3x_3 & = & -4 \\ x_1 & - & 2x_2 & - & 3x_3 & = & 3 \\ -3x_1 & + & 4x_2 & + & 2x_3 & = & -4. \end{array}\right.$$

Problem 5.9
Reduce the following system to reduced row echelon form and then find the solution.

$$\left\{\begin{array}{rcrcrcrcrcr} 2x_1 & + & 4x_2 & + & 2x_3 & + & 4x_4 & + & 2x_5 & = & 4 \\ 2x_1 & + & 4x_2 & + & 3x_3 & + & 3x_4 & + & 3x_5 & = & 4 \\ 3x_1 & + & 6x_2 & + & 6x_3 & + & 3x_4 & + & 6x_5 & = & 6 \\ & & & & x_3 & - & x_4 & - & x_5 & = & 4. \end{array}\right.$$

Problem 5.10
Using the Gauss-Jordan elimination method, solve the following linear system.

$$\left\{\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & + & 3x_5 & = & 1 \\ 2x_1 & + & 4x_2 & + & 6x_3 & + & 2x_4 & + & 6x_5 & = & 2 \\ 3x_1 & + & 6x_2 & + & 18x_3 & + & 9x_4 & + & 9x_5 & = & -6 \\ 4x_1 & + & 8x_2 & + & 12x_3 & + & 10x_4 & + & 12x_5 & = & 4 \\ 5x_1 & + & 10x_2 & + & 24x_3 & + & 11x_4 & + & 15x_5 & = & -4. \end{array}\right.$$

6. Homogeneous Systems of Linear Equations

A **homogeneous** linear system is any system of the form

$$\begin{array}{rl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = 0 \\ \dots\dots\dots\dots\dots\dots & \dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = 0. \end{array}$$

Every homogeneous system is consistent, since $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ is always a solution. This solution is called the **trivial solution**; any other solution is called **nontrivial.**
A homogeneous system has either a unique solution (the trivial solution) or infinitely many solutions. The following theorem provides a criterion where a homogeneous system is assured to have a nontrivial solution (and therefore infinitely many solutions).

Theorem 6.1
A homogeneous system in n unknowns and m equations has infinitely many solutions if either
(1) the rank of the coefficient matrix is less than n or
(2) the number of unknowns exceeds the number of equations, i.e. $m < n$. That is, the system is underdetermined.

Example 6.1
Solve the following homogeneous system using Gauss-Jordan elimination.

$$\left\{ \begin{array}{rcrcrcrcrcr} 2x_1 & + & 2x_2 & - & x_3 & & & + & x_5 & = & 0 \\ -x_1 & - & x_2 & + & 2x_3 & - & 3x_4 & + & x_5 & = & 0 \\ x_1 & + & x_2 & - & 2x_3 & & & - & x_5 & = & 0 \\ & & & & x_3 & + & x_4 & + & x_5 & = & 0. \end{array} \right.$$

Solution.
The reduction of the augmented matrix to reduced row-echelon form is outlined below.

$$\left[\begin{array}{rrrrrr} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right]$$

Step 1: $r_3 \leftarrow r_3 + r_2$

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Step 2: $r_3 \leftrightarrow r_4$ and $r_1 \leftrightarrow r_2$

$$\begin{bmatrix} -1 & -1 & 2 & -3 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix}$$

Step 3: $r_2 \leftarrow r_2 + 2r_1$ and $r_4 \leftarrow -\frac{1}{3}r_4$

$$\begin{bmatrix} -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 3 & -6 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 4: $r_1 \leftarrow -r_1$ and $r_2 \leftarrow \frac{1}{3}r_2$

$$\begin{bmatrix} 1 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5: $r_3 \leftarrow r_3 - r_2$

$$\begin{bmatrix} 1 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 6: $r_4 \leftarrow r_4 - \frac{1}{3}r_3$ and $r_3 \leftarrow \frac{1}{3}r_3$

$$\begin{bmatrix} 1 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 7: $r_1 \leftarrow r_1 - 3r_3$ and $r_2 \leftarrow r_2 + 2r_3$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 8: $r_1 \leftarrow r_1 + 2r_2$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system is

$$\left\{\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & & & & & + & x_5 & = & 0 \\ & & & & x_3 & & & + & x_5 & = & 0 \\ & & & & & & x_4 & & & = & 0 \end{array}\right.$$

The free variables are $x_2 = s, x_5 = t$ and the general solution is given by the formula: $x_1 = -s - t, x_2 = s, x_3 = -t, x_4 = 0, x_5 = t$ ■

Example 6.2
Solve the following homogeneous system using Gaussian elimination.

$$\left\{\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & + & 5x_3 & + & x_4 & = & 0 \\ 4x_1 & - & 7x_2 & - & 3x_3 & - & x_4 & = & 0 \\ 3x_1 & + & 2x_2 & + & 7x_3 & + & 8x_4 & = & 0 \end{array}\right.$$

Solution.
The augmented matrix for the system is

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 0 \\ 4 & -7 & -3 & -1 & 0 \\ 3 & 2 & 7 & 8 & 0 \end{bmatrix}$$

We reduce this matrix into a row-echelon form as follows.
Step 1: $r_2 \leftarrow r_2 - r_3$

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 0 \\ 1 & -9 & -10 & -9 & 0 \\ 3 & 2 & 7 & 8 & 0 \end{bmatrix}$$

Step 2: $r_2 \leftarrow r_2 - r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 0 \\ 0 & -12 & -15 & -10 & 0 \\ 0 & -7 & -8 & 5 & 0 \end{bmatrix}$$

Step 3: $r_2 \leftarrow -\frac{1}{12}r_2$

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 0 \\ 0 & 1 & \frac{5}{4} & \frac{5}{6} & 0 \\ 0 & -7 & -8 & 5 & 0 \end{bmatrix}$$

Step 4: $r_3 \leftarrow r_3 + 7r_2$

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 0 \\ 0 & 1 & \frac{5}{4} & \frac{5}{6} & 0 \\ 0 & 0 & \frac{3}{4} & \frac{65}{6} & 0 \end{bmatrix}$$

Step 5: $r_3 \leftarrow \frac{4}{3}r_3$

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 0 \\ 0 & 1 & \frac{5}{4} & \frac{5}{6} & 0 \\ 0 & 0 & 1 & \frac{130}{9} & 0 \end{bmatrix}$$

We see that $x_4 = t$ is the only free variable. Solving for the leading variables using back substitution we find $x_1 = \frac{176}{9}t, x_2 = \frac{155}{9}t$, and $x_3 = -\frac{130}{9}t$ ■

Remark 6.1
Part (2) of Theorem 6.1 applies only to homogeneous linear systems. A non-homogeneous system (right-hand side has non-zero entries) with more unknowns than equations need not be consistent as shown in the next example.

Example 6.3
Show that the following system is inconsistent.

$$\left\{ \begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 2x_2 & + & 2x_3 & = & 4. \end{array} \right.$$

Solution.
Multiplying the first equation by -2 and adding the resulting equation to the second we obtain $0 = 4$ which is impossible. So the system is inconsistent ■

Example 6.4
Show that if a homogeneous system of linear equations in n unknowns has a nontrivial solution then $rank(A) < n$, where A is the coefficient matrix.

Solution.
Since $rank(A) \leq n$, either $rank(A) = n$ or $rank(A) < n$. If $rank(A) < n$ then we are done. So suppose that $rank(A) = n$. Then there is a matrix B that is row equivalent to A and that has n nonzero rows. Moreover, B has the following form

$$\begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} & 0 \\ 0 & 1 & a_{23} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

The corresponding system is triangular and can be solved by back substitution to obtain the solution $x_1 = x_2 = \cdots = x_n = 0$ which is a contradiction. Thus we must have $rank(A) < n$ ∎

Practice Problems

Problem 6.1
Find the value(s) of a for which the following system has a nontrivial solution. Find the general solution.

$$\left\{\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ x_1 & + & 3x_2 & + & 6x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & ax_3 & = & 0 \end{array}\right.$$

Problem 6.2
Solve the following homogeneous system.

$$\left\{\begin{array}{rcrcrcrcl} x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 0 \\ 2x_1 & + & 2x_2 & & & - & x_4 & = & 0 \\ 3x_1 & + & x_2 & + & 2x_3 & + & x_4 & = & 0 \end{array}\right.$$

Problem 6.3
Solve the homogeneous linear system.

$$\left\{\begin{array}{rcrcrcl} x_1 & + & x_2 & - & 2x_3 & = & 0 \\ 3x_1 & + & 2x_2 & + & 4x_3 & = & 0 \\ 4x_1 & + & 3x_2 & + & 3x_3 & = & 0 \end{array}\right.$$

Problem 6.4
Solve the homogeneous linear system.

$$\left\{\begin{array}{rcrcrcl} x_1 & + & x_2 & - & 2x_3 & = & 0 \\ 3x_1 & + & 2x_2 & + & 4x_3 & = & 0 \\ 4x_1 & + & 3x_2 & + & 2x_3 & = & 0 \end{array}\right.$$

Problem 6.5
Solve the homogeneous linear system.

$$\left\{\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & - & 6x_3 & = & 0 \\ 4x_1 & + & 8x_2 & - & 12x_3 & = & 0. \end{array}\right.$$

Problem 6.6
Solve the homogeneous linear system.

$$\left\{\begin{array}{rcrcrcrcl} x_1 & + & x_2 & & & + & 3x_4 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & + & x_4 & = & 0 \\ 3x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 \end{array}\right.$$

Problem 6.7
Solve the homogeneous linear system.

$$\left\{\begin{array}{rcrcrcrcl} x_1 & + & x_2 & & & - & x_4 & = & 0 \\ -2x_1 & - & 3x_2 & + & 4x_3 & + & 5x_4 & = & 0 \\ 2x_1 & + & 4x_2 & & & - & 2x_4 & = & 0 \end{array}\right.$$

Matrices

Matrices are essential in the study of linear algebra. The concept of matrices has become a tool in all branches of mathematics, the sciences, and engineering. They arise in many contexts other than as augmented matrices for systems of linear equations. In this chapter we shall consider this concept as objects in their own right and develop their properties for use in our later discussions.

7. Matrices and Matrix Operations

In this section, we discuss several types of matrices. We also examine four operations on matrices- addition, scalar multiplication, trace, and the transpose operation- and give their basic properties. Also, we introduce symmetric, skew-symmetric matrices.

A **matrix A of size** $m \times n$ is a rectangular array of the form

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{array}\right]$$

where the a_{ij}'s are the **entries** of the matrix, m is the number of rows, n is the number of columns. The **zero matrix 0** is the matrix whose entries are all 0. The $n \times n$ **identity matrix** I_n is a square matrix whose main diagonal consists of $1's$ and the off diagonal entries are all 0. A matrix A can be represented with the following compact notation $A = [a_{ij}]$. The **ith row** of the matrix A is

$$[a_{i1}, a_{i2}, ..., a_{in}]$$

and the **jth column** is

$$\left[\begin{array}{c} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{array}\right]$$

In what follows we discuss the basic arithmetic of matrices.

Two matrices are said to be **equal** if they have the same size and their corresponding entries are all equal. If the matrix A is not equal to the matrix B we write $A \neq B$.

Example 7.1
Find x_1, x_2 and x_3 such that

$$\left[\begin{array}{ccc} x_1 + x_2 + 2x_3 & 0 & 1 \\ 2 & 3 & 2x_1 + 4x_2 - 3x_3 \\ 4 & 3x_1 + 6x_2 - 5x_3 & 5 \end{array}\right] = \left[\begin{array}{ccc} 9 & 0 & 1 \\ 2 & 3 & 1 \\ 4 & 0 & 5 \end{array}\right]$$

Solution.
Because corresponding entries must be equal, this gives the following linear system

$$\begin{cases} x_1 & + & x_2 & + & 2x_3 & = & 9 \\ 2x_1 & + & 4x_2 & - & 3x_3 & = & 1 \\ 3x_1 & + & 6x_2 & - & 5x_3 & = & 0 \end{cases}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

The reduction of this matrix to row-echelon form is

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_3 \leftarrow r_3 - 3r_1$

$$\left[\begin{array}{cccc} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

Step 2: $r_2 \leftrightarrow r_3$

$$\left[\begin{array}{cccc} 1 & 1 & 2 & 9 \\ 0 & 3 & -11 & -27 \\ 0 & 2 & -7 & -17 \end{array}\right]$$

Step 3: $r_2 \leftarrow r_2 - r_3$

$$\left[\begin{array}{cccc} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 2 & -7 & -17 \end{array}\right]$$

Step 4: $r_3 \leftarrow r_3 - 2r_2$

$$\left[\begin{array}{cccc} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The corresponding system is

$$\begin{cases} x_1 & + & x_2 & + & 2x_3 & = & 9 \\ & & x_2 & - & 4x_3 & = & -10 \\ & & & & x_3 & = & 3 \end{cases}$$

Using backward substitution we find: $x_1 = 1, x_2 = 2, x_3 = 3$ ■

Example 7.2
Solve the following matrix equation for a, b, c, and d

$$\begin{bmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$$

Solution.
Equating corresponding entries we get the system

$$\left\{\begin{array}{lllllllll} a & - & b & & & & & = & 8 \\ & & b & + & c & & & = & 1 \\ & & & & c & + & 3d & = & 7 \\ 2a & & & & & - & 4d & = & 6 \end{array}\right.$$

The augmented matrix is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 2 & 0 & 0 & -4 & 6 \end{bmatrix}$$

We next apply Gaussian elimination as follows.

Step 1: $r_4 \leftarrow r_4 - 2r_1$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 2 & 0 & -4 & -10 \end{bmatrix}$$

Step 2: $r_4 \leftarrow r_4 - 2r_2$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & -2 & -4 & -12 \end{bmatrix}$$

Step 3: $r_4 \leftarrow r_4 + 2r_3$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}$$

Using backward substitution to find: $a = -10, b = -18, c = 19, d = 1$ ■

Next, we introduce the operation of addition of two matrices. If A and B are two matrices of the same size, then the **sum** $A + B$ is the matrix obtained by adding together the corresponding entries in the two matrices. Matrices of different sizes cannot be added.

Example 7.3
Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

Compute, if possible, $A + B$, $A + C$ and $B + C$.

Solution.
We have

$$A + B = \begin{bmatrix} 4 & 2 \\ 6 & 10 \end{bmatrix}$$

$A + B$ and $B + C$ are undefined since A and B are of different sizes as well as A and C ■

From now on, a constant number will be called a **scalar.** If A is a matrix and c is a scalar, then the product cA is the matrix obtained by multiplying each entry of A by c. Hence, $-A = (-1)A$. We define, $A - B = A + (-B)$. The matrix cI_n is called a **scalar** matrix.

Example 7.4
Consider the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 & 7 \\ 1 & -3 & 5 \end{bmatrix}$$

Compute $A - 3B$.

Solution.
Using the above definitions we have

$$A - 3B = \begin{bmatrix} 2 & -3 & -17 \\ -2 & 11 & -14 \end{bmatrix} \blacksquare$$

Let M_{mn} be the collection of all $m \times n$ matrices. This set under the operations of addition and scalar multiplication satisfies algebraic properties which will remind us of the system of real numbers. The proofs of these properties depend on the properties of real numbers. Here we shall assume that the reader is familiar with the basic algebraic properties of $\mathbb{R}$. The following theorem list the properties of matrix addition and multiplication of a matrix by a scalar.

Theorem 7.1
Let A, B, and C be $m \times n$ and let c, d be scalars. Then

(i) $A + B = B + A$,
(ii) $(A + B) + C = A + (B + C) = A + B + C$,
(iii) $A + \mathbf{0} = \mathbf{0} + A = A$,
(iv) $A + (-A) = \mathbf{0}$,
(v) $c(A + B) = cA + cB$,
(vi) $(c + d)A = cA + dA$,
(vii) $(cd)A = c(dA)$,
(viii) $I_m A = AI_n = A$.

Example 7.5
Solve the following matrix equation.

$$\left[\begin{array}{cc} 3 & 2 \\ -1 & 1 \end{array}\right] + \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ -1 & 2 \end{array}\right]$$

Solution.
Adding and then equating corresponding entries we obtain $a = -2, b = -2, c = 0$, and $d = 1$ ■

If A is a square matrix then the sum of the entries on the main diagonal is called the **trace** of A and is denoted by $tr(A)$.

Example 7.6
Find the trace of the coefficient matrix of the system

$$\left\{\begin{array}{rcrcrcr} & - & x_2 & + & 3x_3 & = & 1 \\ x_1 & & & + & 2x_3 & = & 2 \\ -3x_1 & - & 2x_2 & & & = & 4 \end{array}\right.$$

Solution.
If A is the coefficient matrix of the system then

$$A = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$$

The trace of A is the number $tr(A) = 0 + 0 + 0 = 0$ ■

Two useful properties of the trace of a matrix are given in the following theorem.

Theorem 7.2
Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices and c be a scalar. Then
(i) $tr(A + B) = tr(A) + tr(B)$,
(ii) $tr(cA) = c\ tr(A)$.

Proof.
(i) $tr(A + B) = \sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = tr(A) + tr(B)$.
(ii) $tr(cA) = \sum_{i=1}^{n} ca_{ii} = c\sum_{i=1}^{n} a_{ii} = c\ tr(A)$ ■

If A is an $m \times n$ matrix then the **transpose** of A, denoted by A^T, is defined to be the $n \times m$ matrix obtained by interchanging the rows and columns of A, that is the first column of A^T is the first row of A, the second column of A^T is the second row of A, etc. Note that, if $A = (a_{ij})$ then $A^T = (a_{ji})$. Also, if A is a square matrix then the diagonal entries on both A and A^T are the same.

Example 7.7
Find the transpose of the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix},$$

Solution.
The transpose of A is the matrix

$$A^T = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 1 \end{bmatrix} ■$$

The following result lists some of the properties of the transpose of a matrix.

Theorem 7.3
Let $A = (a_{ij})$, and $B = (b_{ij})$ be two $m \times n$ matrices, $C = (c_{ij})$ be an $n \times n$ matrix, and c a scalar. Then
(i) $(A^T)^T = A$,
(ii) $(A + B)^T = A^T + B^T$,
(iii) $(cA)^T = cA^T$,
(iv) $tr(C^T) = tr(C)$.

Proof.
(i) $(A^T)^T = (a_{ji})^T = (a_{ij}) = A$.
(ii) $(A + B)^T = (a_{ij} + b_{ij})^T = (a_{ji} + b_{ji}) = (a_{ji}) + (b_{ji}) = A^T + B^T$.
(iii) $(cA)^T = (ca_{ij})^T = (ca_{ji}) = c(a_{ji}) = cA^T$.
(iv) $tr(C^T) = \sum_{i=1}^n c_{ii} = tr(C)$ ■

Example 7.8
A square matrix A is called **symmetric** if $A^T = A$. A square matrix A is called **skew-symmetric** if $A^T = -A$.
(a) Show that the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

is a symmetric matrix.
(b) Show that the matrix

$$A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

is a skew-symmetric matrix.
(c) Show that for any square matrix A the matrix $S = \frac{1}{2}(A + A^T)$ is symmetric and the matrix $K = \frac{1}{2}(A - A^T)$ is skew-symmetric.
(d) Show that if A is a square matrix, then $A = S + K$, where S is symmetric and K is skew-symmetric.
(e) Show that the representation in (d) is unique.

Solution.
(a) A is symmetric since

$$A^T = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix} = A$$

(b) A is skew-symmetric since

$$A^T = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix} = -A$$

(c) Because $S^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A + A^T)$ then S is symmetric. Similarly, $K^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -K$ so that K is skew-symmetric.
(d) $S + K = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = A$.
(e) Let S' be a symmetric matrix and K' be skew-symmetric such that $A = S' + K'$. Then $S + K = S' + K'$ and this implies that $S - S' = K - K'$. But the matrix $S - S'$ is symmetric and the matrix $K' - K$ is skew-symmetric. This equality is true only when $S - S'$ is the zero matrix. That is $S = S'$. Hence, $K = K'$ ■

Example 7.9
Let A be an $n \times n$ matrix.
(a) Show that if A is symmetric then A and A^T have the same main diagonal.
(b) Show that if A is skew-symmetric then the entries on the main diagonal are 0.
(c) If A and B are symmetric then so is $A + B$.

Solution.
(a) Let $A = (a_{ij})$ be symmetric. Let $A^T = (b_{ij})$. Then $b_{ij} = a_{ji}$ for all $1 \leq i, j \leq n$. In particular, when $i = j$ we have $b_{ii} = a_{ii}$. That is, A and A^T have the same main diagonal.
(b) Since A is skew-symmetric, we have $a_{ij} = -a_{ji}$. In particular, $a_{ii} = -a_{ii}$ and this implies that $a_{ii} = 0$.
(c) Suppose A and B are symmetric. Then $(A + B)^T = A^T + B^T = A + B$. That is, $A + B$ is symmetric ■

Example 7.10
Let A be an $m \times n$ matrix and α a real number. Show that if $\alpha A = \mathbf{0}$ then either $\alpha = 0$ or $A = \mathbf{0}$.

Solution.
Let $A = (a_{ij})$. Then $\alpha A = (\alpha a_{ij})$. Suppose $\alpha A = \mathbf{0}$. Then $\alpha a_{ij} = 0$ for all $0 \leq i \leq m$ and $0 \leq j \leq n$. If $\alpha \neq 0$ then $a_{ij} = 0$ for all indices i and j. In this case, $A = \mathbf{0}$ ■

Practice Problems

Problem 7.1
Compute the matrix

$$3\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^T - 2\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

Problem 7.2
Find w, x, y, and z.

$$\begin{bmatrix} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & z \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{bmatrix}$$

Problem 7.3
Determine two numbers s and t such that the following matrix is symmetric.

$$A = \begin{bmatrix} 2 & s & t \\ 2s & 0 & s+t \\ 3 & 3 & t \end{bmatrix}$$

Problem 7.4
Let A be the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Show that

$$A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 7.5
Let $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}$. If $rA + sB + tC = \mathbf{0}$ show that $s = r = t = 0$.

Problem 7.6
Compute

$$\begin{bmatrix} 1 & 9 & -2 \\ 3 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 8 & -4 & 3 \\ -7 & 1 & 6 \end{bmatrix}$$

Problem 7.7
Determine whether the matrix is symmetric or skew-symmetric.

$$A = \begin{bmatrix} 11 & 6 & 1 \\ 6 & 3 & -1 \\ 1 & -1 & -6 \end{bmatrix}$$

Problem 7.8
Determine whether the matrix is symmetric or skew-symmetric.

$$A = \begin{bmatrix} 0 & 3 & -1 & -5 \\ -3 & 0 & 7 & -2 \\ 1 & -7 & 0 & 0 \\ 5 & 2 & 0 & 0 \end{bmatrix}.$$

Problem 7.9
Consider the matrix

$$A = \begin{bmatrix} 0 & 3 & -1 & -5 \\ -3 & 0 & 7 & -2 \\ 1 & -7 & 0 & 0 \\ 5 & 2 & 0 & 0 \end{bmatrix}.$$

Find (a) $4tr(7A)$.

Problem 7.10
Consider the matrices

$$A = \begin{bmatrix} 11 & 6 & 1 \\ 6 & 3 & -1 \\ 1 & -1 & -6 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & 7 \\ 1 & -7 & 0 \end{bmatrix}.$$

Find $tr(A^T - 2B)$.

8. Matrix Multiplication

In the previous section we discussed some basic properties associated with matrix addition and scalar multiplication. Here we introduce another important operation involving matrices-the product.
Let $A = (a_{ij})$ be a matrix of size $m \times n$ and $B = (b_{ij})$ be a matrix of size $n \times p$. Then the **product** matrix is a matrix of size $m \times p$ and entries

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj},$$

that is, c_{ij} is obtained by multiplying componentwise the entries of the i^{th} row of A by the entries of the j^{th} column of B. It is very important to keep in mind that the number of columns of the first matrix must be equal to the number of rows of the second matrix; otherwise the product is undefined.
An interesting question associated with matrix multiplication is the following: If A and B are square matrices then is it always true that $AB = BA$?
The answer to this question is negative. In general, matrix multiplication is not commutative, as the following example shows.

Example 8.1
Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

Show that $AB \neq BA$. Hence, matrix multiplication is not commutative.

Solution.
Using the definition of matrix multiplication we find

$$AB = \begin{bmatrix} -4 & 7 \\ 0 & 5 \end{bmatrix}, BA = \begin{bmatrix} -1 & 2 \\ 9 & 2 \end{bmatrix}$$

Hence, $AB \neq BA$ ■

Example 8.2
Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, C = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$$

(a) Compare $A(BC)$ and $(AB)C$.
(b) Compare $A(B + C)$ and $AB + AC$.
(c) Compute I_2A and AI_2, where I_2 is the 2×2 identity matrix.

Solution.
(a)

$$A(BC) = (AB)C = \begin{bmatrix} 70 & 14 \\ 235 & 56 \end{bmatrix}$$

(b)

$$A(B+C) = AB + AC = \begin{bmatrix} 16 & 7 \\ 59 & 33 \end{bmatrix}$$

(c) $AI_2 = I_2A = A$ ■

Example 8.3
Let A be a 3×2 and B be a 2×4 matrices.Show that if
(a) B has a column of zeros then the same is true for AB.
(b) A has a row of zeros then the same is true for AB.

Solution.
Write

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12} + b_{21} & a_{11}b_{12} + a_{12} + b_{22} & a_{11}b_{13} + a_{12} + b_{23} & a_{11}b_{14} + a_{12} + b_{24} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} & a_{21}b_{14} + a_{22}b_{24} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} & a_{31}b_{14} + a_{32}b_{24} \end{bmatrix}$$

(a) Suppose that $b_{11} = b_{21} = 0$. Then

$$AB = \begin{bmatrix} 0 & a_{11}b_{12} + a_{12} + b_{22} & a_{11}b_{13} + a_{12} + b_{23} & a_{11}b_{14} + a_{12} + b_{24} \\ 0 & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} & a_{21}b_{14} + a_{22}b_{24} \\ 0 & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} & a_{31}b_{14} + a_{32}b_{24} \end{bmatrix}$$

(b) Suppose that $a_{21} = a_{22} = 0$. Then

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12} + b_{21} & a_{11}b_{12} + a_{12} + b_{22} & a_{11}b_{13} + a_{12} + b_{23} & a_{11}b_{14} + a_{12} + b_{24} \\ 0 & 0 & 0 & 0 \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} & a_{31}b_{14} + a_{32}b_{24} \end{bmatrix}$$ ■

Next, consider a system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n & = b_2 \\ .... & .... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n & = b_m \end{cases}$$

Then the matrix of the coefficients of the x_i's is called the **coefficient matrix:**

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

The matrix of the coefficients of the x_i's and the right hand side coefficients is called the **augmented** matrix:

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & b_2 \\ ... & ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} & b_m \end{bmatrix}$$

Now, if we let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

then the above system can be represented in **matrix notation** as

$$Ax = b.$$

Example 8.4
Consider the linear system

$$\left\{\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0\\ & & 2x_2 & - & 8x_3 & = & 8\\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9. \end{array}\right.$$

(a) Find the coefficient and augmented matrices of the linear system.
(b) Find the matrix notation.

Solution.
(a) The coefficient matrix of this system is

$$\left[\begin{array}{ccc} 1 & -2 & 1\\ 0 & 2 & -8\\ -4 & 5 & 9 \end{array}\right]$$

and the augmented matrix is

$$\left[\begin{array}{cccc} 1 & -2 & 1 & 0\\ 0 & 2 & -8 & 8\\ -4 & 5 & 9 & -9 \end{array}\right]$$

(b) We can write the given system in matrix form as

$$\left[\begin{array}{ccc} 1 & -2 & 1\\ 0 & 2 & -8\\ -4 & 5 & 9 \end{array}\right]\left[\begin{array}{c} x_1\\ x_2\\ x_3 \end{array}\right] = \left[\begin{array}{c} 0\\ 8\\ -9 \end{array}\right] \blacksquare$$

As the reader has noticed so far, most of the basic rules of arithmetic of real numbers also hold for matrices but a few do not. In Example 8.1 we have seen that matrix multiplication is not commutative. The following exercise shows that the cancellation law of numbers does not hold for matrix product.

Example 8.5
(a) Consider the matrices

$$A = \left[\begin{array}{cc} 1 & 0\\ 0 & 0 \end{array}\right], B = \left[\begin{array}{cc} 0 & 0\\ 1 & 0 \end{array}\right], C = \left[\begin{array}{cc} 0 & 0\\ 0 & 1 \end{array}\right]$$

Compare AB and AC. Is it true that $B = C$?
(b) Find two square matrices A and B such that $AB = \mathbf{0}$ but $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$.

Solution.
(a) Note that $B \neq C$ even though $AB = AC = \mathbf{0}$.
(b) The given matrices satisfy $AB = \mathbf{0}$ with $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$ ■

Matrix multiplication shares many properties of the product of real numbers which are listed in the following theorem

Theorem 8.1
Let A be a matrix of size $m \times n$. Then
(a) $A(BC) = (AB)C$, where B is of size $n \times p$, C of size $p \times q$.
(b) $A(B + C) = AB + AC$, where B and C are of size $n \times p$.
(c) $(B + C)A = BA + CA$, where B and C are of size $l \times m$.
(d) $c(AB) = (cA)B = A(cB)$, where c denotes a scalar.

The next theorem describes a property about the transpose of a matrix.

Theorem 8.2
Let $A = (a_{ij}), B = (b_{ij})$ be matrices of sizes $m \times n$ and $n \times m$ respectively. Then $(AB)^T = B^T A^T$.

Example 8.6
Let A be any matrix. Show that AA^T and $A^T A$ are symmetric matrices.

Solution.
First note that for any matrix A the matrices AA^T and $A^T A$ are well-defined. Since $(AA^T)^T = (A^T)^T A^T = AA^T$ then AA^T is symmetric. Similarly, $(A^T A)^T = A^T (A^T)^T = A^T A$ ■

Finally, we discuss the powers of a square matrix. Let A be a square matrix of size $n \times n$. Then the non-negative powers of A are defined as follows: $A^0 = I_n, A^1 = A$, and for $k \geq 2$, $A^k = (A^{k-1})A$.

Example 8.7
suppose that

$$A = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right]$$

Compute A^3.

Solution.
Multiplying the matrix A by itself three times we obtain

$$A^3 = \begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix} \blacksquare$$

Theorem 8.3
For any non-negative integers s, t we have
(a) $A^{s+t} = A^s A^t$
(b) $(A^s)^t = A^{st}$.

Example 8.8
Let A and B be two $n \times n$ matrices.
(a) Show that $tr(AB) = tr(BA)$.
(b) Show that $AB - BA = I_n$ is impossible.

Solution.
(a) Let $A = (a_{ij})$ and $B = (b_{ij})$. Then
$tr(AB) = \sum_{i=1}^n (\sum_{k=1}^n a_{ik}b_{ki}) = \sum_{i=1}^n (\sum_{k=1}^n b_{ik}a_{ki}) = tr(BA)$.
(b) If $AB - BA = I_n$ then $0 = tr(AB) - tr(BA) = tr(AB - BA) = tr(I_n) = n \geq 1$, a contradiction $\blacksquare$

Practice Problems

Problem 8.1
Write the linear system whose augmented matrix is given by

$$\left[\begin{array}{rrrr} 2 & -1 & 0 & -1 \\ -3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

Problem 8.2
Consider the linear system

$$\left\{\begin{array}{rrrrrrrrr} 2x_1 & + & 3x_2 & - & 4x_3 & + & x_4 & = & 5 \\ -2x_1 & & & + & x_3 & & & = & 7 \\ 3x_1 & + & 2x_2 & & & - & 4x_3 & = & 3 \end{array}\right.$$

(a) Find the coefficient and augmented matrices of the linear system.
(b) Find the matrix notation.

Problem 8.3
Let A be an arbitrary matrix. Under what conditions is the product AA^T defined?

Problem 8.4
An $n \times n$ matrix A is said to be **idempotent** if $A^2 = A$.
(a) Show that the matrix

$$A = \frac{1}{2}\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]$$

is idempotent.
(b) Show that if A is idempotent then the matrix $(I_n - A)$ is also idempotent.

Problem 8.5
The purpose of this exercise is to show that the rule $(ab)^n = a^n b^n$ does not hold with matrix multiplication. Consider the matrices

$$A = \left[\begin{array}{rr} 2 & -4 \\ 1 & 3 \end{array}\right], B = \left[\begin{array}{rr} 3 & 2 \\ -1 & 5 \end{array}\right]$$

Show that $(AB)^2 \neq A^2B^2$.

Problem 8.6
Show that $AB = BA$ if and only if $A^T B^T = B^T A^T$.

Problem 8.7
Let A and B be symmetric matrices. Show that AB is symmetric if and only if $AB = BA$.

Problem 8.8
A matrix B is said to be the **square root** of a matrix A if $BB = A$. Find two sqaure roots of the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Problem 8.9
Find k such that

$$\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0.$$

Problem 8.10
Express the matrix notation as a system of linear equations.

$$\begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 7 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}.$$

9. The Inverse of a Square Matrix

Most problems in practice reduces to a system with matrix notation $Ax = b$. Thus, in order to get x we must somehow be able to eliminate the coefficient matrix A. One is tempted to try to divide by A. Unfortunately such an operation has not been defined for matrices. In this section we introduce a special type of square matrices and formulate the matrix analogue of numerical division. Recall that the $n \times n$ identity square matrix is the matrix I_n whose main diagonal entries are 1 and off diagonal entries are 0.
A square matrix A of size n is called **invertible** or **non-singular** if there exists a square matrix B of the same size such that $AB = BA = I_n$. In this case B is called the **inverse** of A. A square matrix that is not invertible is called **singular.**

Example 9.1
Show that the matrix

$$B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

is the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution.
Using matrix multiplication one checks that $AB = BA = I_2$ ■

Example 9.2
Show that the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is singular.

Solution.
Let $B = (b_{ij})$ be a 2×2 matrix. If $BA = I_2$ then the $(2,2)$-th entry of BA is zero while the $(2,2)-$entry of I_2 is 1, which is impossible. Thus, A is singular ■

It is important to keep in mind that the concept of invertibility is defined only for square matrices. In other words, it is possible to have a matrix A of size $m \times n$ and a matrix B of size $n \times m$ such that $AB = I_m$. It would be wrong to conclude that A is invertible and B is its inverse.

Example 9.3
Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Show that $AB = I_2$.

Solution.
Simple matrix multiplication shows that $AB = I_2$. However, this does not imply that B is the inverse of A since BA is undefnied so that the condition $BA = I_2$ fails ■

Example 9.4
Show that the identity matrix is invertible but the zero matrix is not.

Solution.
Since $I_n I_n = I_n$, I_n is nonsingular and its inverse is I_n. Now, for any $n \times n$ matrix B we have $B\mathbf{0} = \mathbf{0} \neq I_n$ so that the zero matrix is not invertible ■

Now if A is a nonsingular matrix then how many different inverses does it possess? The answer to this question is provided by the following theorem.

Theorem 9.1
The inverse of a matrix is unique.

Proof.
Suppose A has two inverses B and C. We will show that $B = C$. Indeed, $B = BI_n = B(AC) = (BA)C = I_nC = C$ ■

Since an invertible matrix A has a unique inverse, we will denote it from now on by A^{-1}.
For an invertible matrix A one can now define the negative power of a square matrix as follows: For any positive integer $n \geq 1$, we define $A^{-n} = (A^{-1})^n$.
The next theorem lists some of the useful facts about inverse matrices.

Theorem 9.2
Let A and B be two square matrices of the same size $n \times n$.
(a) If A and B are invertible matrices then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
(b) If A is invertible then A^{-1} is invertible and $(A^{-1})^{-1} = A$.
(c) If A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Proof.
(a) If A and B are invertible then $AA^{-1} = A^{-1}A = I_n$ and $BB^{-1} = B^{-1}B = I_n$. In This case, $(AB)(B^{-1}A^{-1}) = A[B(B^{-1}A^{-1})] = A[(BB^{-1})A^{-1}] = A(I_nA^{-1}) = AA^{-1} = I_n$. Similarly, $(B^{-1}A^{-1})(AB) = I_n$. It follows that $B^{-1}A^{-1}$ is the inverse of AB.
(b) Since $A^{-1}A = AA^{-1} = I_n$, A is the inverse of A^{-1}, i.e. $(A^{-1})^{-1} = A$.
(c) Since $AA^{-1} = A^{-1}A = I_n$, by taking the transpose of both sides we get $(A^{-1})^TA^T = A^T(A^{-1})^T = I_n$. This shows that A^T is invertible with inverse $(A^{-1})^T$ ■

Example 9.5
(a) Under what conditions a diagonal matrix is invertible?
(b) Is the sum of two invertible matrices necessarily invertible?

Solution.
(a) Let $D = (d_{ii})$ be a diagonal $n \times n$ matrix. Let $B = (b_{ij})$ be an $n \times n$ matrix such that $DB = I_n$ and let $DB = (c_{ij})$. Then using matrix multiplication we find $c_{ij} = \sum_{k=1}^n d_{ik}b_{kj}$. If $i \neq j$ then $c_{ij} = d_{ii}b_{ij} = 0$ and $c_{ii} = d_{ii}b_{ii} = 1$. If $d_{ii} \neq 0$ for all $1 \leq i \leq n$ then $b_{ij} = 0$ for $i \neq j$ and $b_{ii} = \frac{1}{d_{ii}}$. Thus, if $d_{11}d_{22} \cdots d_{nn} \neq 0$ then D is invertible and its inverse is the diagonal matrix $D^{-1} = (\frac{1}{d_{ii}})$.
(b) The following two matrices are invertible but their sum , which is the zero matrix, is not.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \blacksquare$$

Example 9.6
Consider the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that if $ad - bc \neq 0$ then A^{-1} exists and is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Solution.
Let

$$B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

be a matrix such that $BA = I_2$. Then using matrix multiplication we find

$$\begin{bmatrix} ax+cy & bx+dy \\ az+cw & bz+dw \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding entries we obtain the following systems of linear equations in the unknowns x, y, z and w.

$$\begin{cases} ax + cy = 1 \\ bx + dy = 0 \end{cases}$$

and

$$\begin{cases} az + cw = 0 \\ bz + dw = 0 \end{cases}$$

In the first system, using elimination we find $(ad - bc)y = -b$ and $(ad - bc)x = d$. Similarly, using the second system we find $(ad - bc)z = -c$ and $(ad - bc)w = a$. If $ad - bc \neq 0$ then one can solve for x, y, z, and w and in this case $B = A^{-1}$ as given in the statement of the problem ■

Finally, we mention here that matrix inverses can be used to solve systems of linear equations as suggested by the following theorem.

Theorem 9.3
If A is an $n \times n$ invertible matrix and b is a column matrix then the equation $Ax = b$ has a unique solution $x = A^{-1}b$.

Proof.
Since $A(A^{-1}b) = (AA^{-1})b = I_n b = b$, we find that $A^{-1}b$ is a solution to the equation $Ax = b$. Now, if y is another solution then $y = I_n y = (A^{-1}A)y = A^{-1}(Ay) = A^{-1}b$ ■

Example 9.7
If A is invertible and $k \neq 0$ show that $(kA)^{-1} = \frac{1}{k}A^{-1}$.

Solution.
Suppose that A is invertible and $k \neq 0$. Then $(kA)A^{-1} = k(AA^{-1}) = kI_n$. This implies $(kA)(\frac{1}{k}A^{-1}) = I_n$. Thus, kA is invertible with inverse equals to $\frac{1}{k}A^{-1}$ ■

Practice Problems

Problem 9.1
(a) Find two 2×2 singular matrices whose sum in nonsingular.
(b) Find two 2×2 nonsingular matrices whose sum is singular.

Problem 9.2
Show that the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

is singular.

Problem 9.3
Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

. Find A^{-3}.

Problem 9.4
Let

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$$

Find A.

Problem 9.5
Let A and B be square matrices such that $AB = \mathbf{0}$. Show that if A is invertible then B is the zero matrix.

Problem 9.6
Find the inverse of the matrix

$$A = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$$

Problem 9.7
Find the matrix A given that

$$(I_2 + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}.$$

Problem 9.8
Find the matrix A given that

$$(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}.$$

Problem 9.9
Show that if a square matrix A satisfies the equation $A^2 - 3A + I_n = 0$ then $A^{-1} = 3I_n - A$.

Problem 9.10
Simplify: $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$.

10. Elementary Matrices

In this section we introduce a special type of invertible matrices, the so-called elementary matrices, and we discuss some of their properties. As we shall see, elementary matrices will be used in the next section to develop an algorithm for finding the inverse of a square matrix.
An $n \times n$ **elementary matrix** is a matrix obtained from the identity matrix by performing *one* single elementary row operation.

Example 10.1
Show that the following matrices are elementary matrices
(a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

(c)

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution.
We list the operations that produce the given elementary matrices.
(a) $r_1 \leftarrow 1r_1$.
(b) $r_2 \leftrightarrow r_3$.
(c) $r_1 \leftarrow r_1 + 3r_3$ ■

Example 10.2
Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

(a) Find the row equivalent matrix to A obtained by adding 3 times the first row of A to the third row. Call the equivalent matrix B.
(b) Find the elementary matrix E corresponding to the above elementary row operation.
(c) Compare EA and B.

Solution.
(a)

$$B = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

(b)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

(c) $EA = B$ ■

The conclusion of the above example holds for any matrix of size $m \times n$.

Theorem 10.1
If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product of EA is the matrix that results when this same row operation is performed on A.

It follows from the above theorem that a matrix A is row equivalent to a matrix B if and only if $B = E_k E_{k-1} \cdots E_1 A$, where $E_1, E_2, \cdots, E_k$ are elementary matrices.
The above theorem is primarily of theoretical interest and will be used for developping some results about matrices and systems of linear equations. From a computational point of view, it is preferred to perform row operations directly rather than multiply on the left by an elementary matrix. Also, this theorem says that an elementary row operation on A can be achieved by premultiplying A by the corresponding elementary matrix E.
Given any elementary row operation, there is another row operation (called its **inverse**) that reverse the effect of the first operation. The inverses are described in the following chart.

Type	*Operation*	*Inverse operation*
I	$r_i \leftarrow cr_i$	$r_i \leftarrow \frac{1}{c}r_i$
II	$r_j \leftarrow cr_i + r_j$	$r_j \leftarrow -cr_i + r_j$
III	$r_i \leftrightarrow r_j$	$r_i \leftrightarrow r_j$

The following theorem gives an important property of elementary matrices.

Theorem 10.2
Every elementary matrix is invertible, and the inverse is an elementary matrix.

Example 10.3
Write down the inverses of the following elementary matrices:

$$(a)E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, (b)E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}, (c)E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution.
(a) $E_1^{-1} = E_1$.
(b)

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}$$

(c)

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \blacksquare$$

Example 10.4
If E is an elementary matrix show that E^T is also an elementary matrix of the same type.

Solution.
Suppose that E is the elementary matrix obtained by interchanging rows i and j of I_n with $i < j$. This is equivalent to interchanging columns i and j of I_n. But then E^T is obtained by interchanging rows i and j of I_n and so is an elementary matrix. If E is obtained by multiplying the ith row of I_n by a

nonzero constant k then this is the same thing as multiplying the ith column of I_n by k. Thus, E^T is obtained by multiplying the ith row of I_n by k and so is an elementary matrix. Finally, if E is obtained by adding k times the ith row of I_n to the jth row then E^T is obtained by adding k times the jth row of I_n to the ith row. Note that if E is of Type I or Type III then $E^T = E$ ■

Practice Problems

Problem 10.1
Which of the following are elementary matrices?
(a)
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
(b)
$$\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$$
(c)
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$$
(d)
$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 10.2
Let A be a 4×3 matrix. Find the elementary matrix E, which as a premultiplier of A, that is, as EA, performs the following elementary row operations on A :
(a) Multiplies the second row of A by -2.
(b) Adds 3 times the third row of A to the fourth row of A.
(c) Interchanges the first and third rows of A.

Problem 10.3
For each of the following elementary matrices, describe the corresponding elementary row operation and write the inverse.
(a)
$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Problem 10.4
Consider the matrices

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$$

Find elementary matrices E_1, E_2, E_3, and E_4 such that
$(a) E_1 A = B, (b) E_2 B = A, (c) E_3 A = C, (d) E_4 C = A.$

Problem 10.5
What should we premultiply a 3×3 matrix if we want to interchange rows 1 and 3?

Problem 10.6
Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Find the corresponding inverse operations.

Problem 10.7
List all 3×3 elementary matrices corresponding to type I elementary row operations.

Problem 10.8
List all 3×3 elementary matrices corresponding to type II elementary row operations.

Problem 10.9
Write down the inverses of the following elementary matrices:

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 10.10
Consider the following elementary matrices:

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find

$$E_1 E_2 E_3 \begin{bmatrix} 1 & 0 & 2 \\ -2 & 3 & 4 \\ 0 & 5 & -3 \end{bmatrix}$$

11. Finding A^{-1} Using Elementary Matrices

Before we establish the main results of this section, we recall the reader of the following method of mathematical proofs. To say that statements $p_1, p_2, \cdots, p_n$ are all equivalent means that either they are all true or all false. To prove that they are equivalent, one assumes p_1 to be true and proves that p_2 is true, then assumes p_2 to be true and proves that p_3 is true, continuing in this fashion, assume that p_{n-1} is true and prove that p_n is true and finally, assume that p_n is true and prove that p_1 is true. This is known as the proof by circular argument.
Now, back to our discussion of inverses. The following result establishes relationships between square matrices and systems of linear equations. These relationships are very important and will be used many times in later sections.

Theorem 11.1
If A is an $n \times n$ matrix then the following statements are equivalent.
(a) A is invertible.
(b) $Ax = \mathbf{0}$ has only the trivial solution.
(c) A is row equivalent to I_n.
(d) $rank(A) = n$.

Proof.
$(a) \Rightarrow (b)$: Suppose that A is invertible and x_0 is a solution to $Ax = \mathbf{0}$. Then $Ax_0 = \mathbf{0}$. Multiply both sides of this equation by A^{-1} to obtain $A^{-1}Ax_0 = A^{-1}\mathbf{0}$, that is $x_0 = \mathbf{0}$. Hence, the trivial solution is the only solution.
$(b) \Rightarrow (c)$: Suppose that $Ax = \mathbf{0}$ has only the trivial solution. Then the reduced row-echelon form of the augmented matrix has no rows of zeros or free variables. Hence it must look like

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & 1 & 0 \end{bmatrix}$$

If we disregard the last column of the previous matrix we can conclude that A can be reduced to I_n by a sequence of elementary row operations, i.e. A is row equivalent to I_n.

$(c) \Rightarrow (d)$: Suppose that A is row equivalent to I_n. Then $rank(A) = rank(I_n) = n$.

$(d) \Rightarrow (a)$: Suppose that $rank(A) = n$. Then A is row equivalent to I_n. That is I_n is obtained by a finite sequence of elementary row operations performed on A. Then by Theorem 10.1, each of these operations can be accomplished by premultiplying on the left by an appropriate elementary matrix. Hence, obtaining

$$E_k E_{k-1} \dots E_2 E_1 A = I_n,$$

where k is the necessary number of elementary row operations needed to reduce A to I_n. Now, by Theorem 10.2, each E_i is invertible. Hence, $E_k E_{k-1} \dots E_2 E_1$ is invertible and $A^{-1} = E_k E_{k-1} \dots E_2 E_1$ ∎

Using the definition, to show that an $n \times n$ matrix A is invertible we find a matrix B of the same size such that $AB = I_n$ and $BA = I_n$. The next theorem shows that one of these equality is enough to assure invertibilty.

Theorem 11.2
If A and B are two square matrices of size $n \times n$ such that $AB = I_n$ then $BA = I_n$ and $B^{-1} = A$.

Proof
Suppose that $Bx = \mathbf{0}$. Multiply both sides by A to obtain $ABx = \mathbf{0}$. That is, $x = \mathbf{0}$. This shows that the homogenenous system $Bx = \mathbf{0}$ has only the trivial solution so by Theorem 11.1 we see that B is invertible, say with inverse C. Hence, $C = I_n C = (AB)C = A(BC) = AI_n = A$ so that $B^{-1} = A$. Thus, $BA = BB^{-1} = I_n$ ∎

As an application of Theorem 11.1, we describe an algorithm for finding A^{-1}. We perform elementary row operations on A until we get I_n; say that the product of the elementary matrices is $E_k E_{k-1} \dots E_2 E_1$. Then we have

$$\begin{aligned}(E_k E_{k-1} \dots E_2 E_1)[A|I_n] &= [(E_k E_{k-1} \dots E_2 E_1)A|(E_k E_{k-1} \dots E_2 E_1)I_n] \\ &= [I_n|A^{-1}]\end{aligned}$$

We ask the reader to carry the above algorithm in solving the following problems.

Example 11.1
Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution.
We first construct the matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

Applying the above algorithm to obtain

Step 1: $r_2 \leftarrow r_2 - 2r_1$ and $r_3 \leftarrow r_3 - r_1$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

Step 2: $r_3 \leftarrow r_3 + 2r_2$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

Step 3: $r_1 \leftarrow r_1 - 2r_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

Step 4: $r_2 \leftarrow r_2 - 3r_3$ and $r_1 \leftarrow r_1 + 9r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

Step 5: $r_3 \leftarrow -r_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

It follows that

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \blacksquare$$

Example 11.2
Show that the following homogeneous system has only the trivial solution.

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 0 \\ 2x_1 & + & 5x_2 & + & 3x_3 & = & 0 \\ x_1 & & & + & 8x_3 & = & 0. \end{array}$$

Solution.
The coefficient matrix of the given system is invertible by the previous example. Thus, by Theorem 11.1 the system has only the trivial solution $\blacksquare$

The following result exhibit a criterion for checking the singularity of a square matrix.

Theorem 11.3
If A is a square matrix with a row consisting entirely of zeros then A is singular.

Proof.
The reduced row-echelon form will have a row of zeros. So the rank of the coefficient matrix of the homogeneous system $Ax = \mathbf{0}$ is less than n. By Theorem 6.1, $Ax = \mathbf{0}$ has a nontrivial solution and as a result of Theorem 11.1, the matrix A must be singular $\blacksquare$

How can we tell when a square matrix A is singular? i.e., when does the algorithm of finding A^{-1} fail? The answer is provided by the following theorem

Theorem 11.4
An $n \times n$ matrix A is singular if and only if A is row equivalent to a matrix B that has a row of zeros.

Proof.
Suppose first that A is singular. Then by Theorem 11.1, A is not row equivalent to I_n. Thus, A is row equivalent to a matrix $B \neq I_n$ which is in reduced

echelon form. By Theorem 11.1, B must have a row of zeros.
Conversely, suppose that A is row equivalent to matrix B with a row consisting entirely of zeros. Then B is singular by Theorem 11.1. Now, $B = E_kE_{k-1}\dots E_2E_1A$. If A is nonsingular then B is nonsingular, a contradiction. Thus, A must be singular ■

The following theorem establishes a result of the solvability of linear systems using the concept of invertibility of matrices.

Theorem 11.5
An $n \times n$ square matrix A is invertible if and only if the linear system $Ax = b$ is consistent for every $n \times 1$ matrix b.

Proof.
Suppose first that A is invertible. Then for any $n \times 1$ matrix b the linear system $Ax = b$ has a unique solution, namely $x = A^{-1}b$.
Conversely, suppose that the system $Ax = b$ is solvable for any $n \times 1$ matrix b. In particular, $Ax_i = e_i, 1 \le i \le n$, has a solution, where e_i is the ith column of I_n. Construct the matrix

$$C = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

Then

$$AC = \begin{bmatrix} Ax_1 & Ax_2 & \cdots & Ax_n \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} = I_n.$$

Hence, by Theorem 11.2, A is non-singular ■

Example 11.3
Solve the following system by using the previous theorem

$$\left\{\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 5x_2 & + & 3x_3 & = & 3 \\ x_1 & & & + & 8x_3 & = & 17 \end{array}\right.$$

Solution.
Using Example 11.1 and Theorem 11.5 we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \blacksquare$$

Example 11.4
If P is an $n \times n$ matrix suxh that $P^TP = I_n$ then the matrix $H = I_n - 2PP^T$ is called the **Householder matrix**. Show that H is symmetric and $H^TH = I_n$.

Solution.
Taking the transpose of H we have $H^T = I_n^T - 2(P^T)^TP^T = H$. That is, H is symmetric. On the other hand, $H^TH = H^2 = (I_n - 2PP^T)^2 = I_n - 4PP^T + 4(PP^T)^2 = I_n - 4PP^T + 4P(P^TP)P^T = I_n - 4PP^T + 4PP^T = I_n$ ■

Example 11.5
Let A and B be two square matrices. Show that AB is nonsingular if and only if both A and B are nonsingular.

Solution.
Suppose that AB is nonsingular. Suppose that A is singular. Then $C = E_kE_{k-1}\cdots A$ with C having a row consisting entirely of zeros. But then $CB = E_kE_{k-1}\cdots(AB)$ and CB has a row consisting entirely of zeros (Example 8.3). This implies that AB is singular, a contradiction.
The converse is just Theorem 9.2 (a) ■

Practice Problems

Problem 11.1
Determine if the following matrix is invertible.

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Problem 11.2
For what values of a does the following homogeneous system have a nontrivial solution?

$$\left\{ \begin{array}{ccccc} (a-1)x_1 & + & 2x_2 & = & 0 \\ 2x_1 & + & (a-1)x_2 & = & 0 \end{array} \right.$$

Problem 11.3
Find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$$

Problem 11.4
Prove that if A is symmetric and nonsingular than A^{-1} is symmetric.

Problem 11.5
If

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

find D^{-1}.

Problem 11.6
Prove that a square matrix A is nonsingular if and only if A is a product of elementary matrices.

Problem 11.7
Prove that two $m \times n$ matrices A and B are row equivalent if and only if there exists a nonsingular matrix P such that $B = PA$.

Problem 11.8
Let A and B be two $n\times n$ matrices. Suppose A is row equivalent to B. Prove that A is nonsingular if and only if B is nonsingular.

Problem 11.9
Show that a 2×2 lower triangular matrix is invertible if and only if $a_{11}a_{22}\neq 0$ and in this case the inverse is also lower triangular.

Problem 11.10
Let A be an $n\times n$ matrix and suppose that the system $Ax=\mathbf{0}$ has only the trivial solution. Show that $A^kx=\mathbf{0}$ has only the trivial solution for any positive integer k.

Problem 11.11
Show that if A and B are two $n\times n$ invertible matrices then A is row equivalent to B.

Determinants

With each square matrix we can associate a real number called the determinant of the matrix. Determinants have important applications to the theory of systems of linear equations. More specifically, determinants give us a method (called Cramer's method) for solving linear systems. Also, determinant tells us whether or not a matrix is invertible.
Throughout this chapter we use only square matrices.

12. Determinants by Cofactor Expansion

The determinant of a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is the number

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

The determinant of a 3×3 matrix can be found using the determinants of $2\times$ matrices using a cofactor expansion which we discuss next.
If A is a square matrix of order n then the **minor of entry** a_{ij}, denoted by M_{ij}, is the determinant of the submatrix obtained from A by deleting the i^{th} row and the j^{th} column. The **cofactor of entry** a_{ij} is the number $C_{ij} = (-1)^{i+j}M_{ij}$.

Example 12.1
Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

Find the minor and the cofactor of the entry $a_{32} = 4$.

Solution.
The minor of the entry a_{32} is

$$M_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

and the cofactor is $C_{32} = (-1)^{3+2}M_{32} = -26$ ■

Example 12.2
Find the cofactors C_{11}, C_{12}, and C_{13} of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Solution.
We have

$$C_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{31}a_{23})$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22} \blacksquare$$

The determinant of a matrix A of order n can obtained by multiplying the entries of a row (or a column) by the corresponding cofactors and adding the resulting products. Any row or column chosen will result in the same answer. More precisely, we have the expansion along row i is

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The expansion along column j is given by

$$|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Any row or column chosen will result in the same answer.

Example 12.3
Find the determinant of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Solution.
Using the previous example, we can find the determinant using the cofactor along the first row to obtain

$$\begin{aligned} |A| =& a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ =& a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ =& a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}a_{21}a_{32} - a_{31}a_{22} \blacksquare \end{aligned}$$

Remark 12.1
In general, the best strategy for evaluating a determinant by cofactor expansion is to expand along a row or a column having the largest number of zeroes.

Example 12.4
Find the determinant of each of the following matrices.
(a)

$$A = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Solution.
(a) Expanding along the first column we find

$$|A| = a_{31}C_{31} = -a_{31}a_{22}a_{13}.$$

(b) Again, by expanding along the first column we obtain

$$|A| = a_{41}C_{41} = a_{41}a_{32}a_{23}a_{34} \blacksquare$$

(c) Expanding along the last column we find

$$|A| = a_{44}C_{44} = a_{11}a_{22}a_{33}a_{44} \blacksquare$$

Example 12.5
Evaluate the determinant of the following matrix.

$$\begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix}$$

Solution.
The given matrix is upper triangular so that the determinant is the product of entries on the main diagonal, i.e. equals to -1296 ■

Example 12.6
Use cofactor expansion along the first column to find $|A|$ where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

Solution.
Expanding along the first column we find

$$\begin{aligned} |A| &= 3C_{11} + C_{21} + 2C_{31} + 3C_{41} \\ &= 3M_{11} - M_{21} + 2M_{31} - 3M_{41} \\ &= 3(-54) + 78 + 2(60) - 3(18) = -18 \end{aligned}$$ ■

Practice Problems

Problem 12.1
Evaluate the determinant of each of the following matrices
(a)

$$A = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{bmatrix}$$

Problem 12.2
Find all values of t for which the determinant of the following matrix is zero.

$$A = \begin{bmatrix} t-4 & 0 & 0 \\ 0 & t & 0 \\ 0 & 3 & t-1 \end{bmatrix}$$

Problem 12.3
Solve for x

$$\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix}$$

Problem 12.4
Evaluate the determinant of the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 12.5
Let

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

Find M_{23} and C_{23}.

Problem 12.6
Find all values of λ for which $|A| = 0$, where

$$A = \begin{bmatrix} \lambda - 1 & 0 \\ 2 & \lambda + 1 \end{bmatrix}.$$

Problem 12.7
Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{bmatrix}$$

(a) along the first column.
(b) along the third row.

Problem 12.8
Evaluate the determinant of the matrix by a cofactor expansion along a row or column of your choice.

$$A = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$$

Problem 12.9
Evaluate the determinant of the following matrix by inspection.

$$A = \begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Problem 12.10
Evaluate the determinant of the following matrix.

$$A = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{bmatrix}$$

Problem 12.11
Find all values of λ such that $|A| = 0$.
(a)

$$A = \begin{bmatrix} \lambda - 1 & -2 \\ 1 & \lambda - 4 \end{bmatrix},$$

(b)

$$A = \begin{bmatrix} \lambda - 6 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 4 & \lambda - 4 \end{bmatrix}$$

13. Evaluating Determinants by Row Reduction

In this section we provide a simple procedure for finding the determinant of a matrix. The idea is to reduce the matrix into row-echelon form which in this case is a triangular matrix. Recall that a matrix is said to be **triangular** if it is upper triangular, lower triangular or diagonal. The following theorem provides a formula for finding the determinant of a triangular matrix.

Theorem 13.1
If A is an $n \times n$ triangular matrix then $|A| = a_{11}a_{22}\dots a_{nn}$.

Example 13.1
Compute $|A|$.
(a)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix},$$

(b)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix},$$

Solution.
(a) Since A is triangular, $|A| = (1)(4)(6) = 24$.
(b) $|A| = (1)(3)(6) = 18$ ■

Example 13.2
Compute the determinant of the identity matrix I_n.

Solution.
Since the identity matrix is triangular with entries equal to 1 on the main diagonal, $|I_n| = 1$ ■

The following theorem is of practical use. It provides a technique for evaluating determinants by greatly reducing the labor involved. We shall show that the determinant can be evaluated by reducing the matrix to row-echelon form.

Theorem 13.2
Let A be an $n \times n$ matrix.
(a) Let B be the matrix obtained from A by multiplying a row by a scalar c. Then $|B| = c|A|$.
(b) Let B be the matrix obtained from A by interchanging two rows of A. Then $|B| = -|A|$.
(c) Let B be the matrix obtained from A by adding c times a row to another row. Then $|B| = |A|$.
(d) If A is a square matrix then $|A^T| = |A|$.

Example 13.3
Use Theorem 13.2 to evaluate the determinant of the following matrix

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

Solution.
We use Gaussian elimination as follows.

Step 1: $r_1 \leftrightarrow r_2$

$$\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -|A|$$

Step 2: $r_1 \leftarrow r_1 - r_3$

$$\begin{vmatrix} 1 & -12 & 8 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -|A|$$

Step 3: $r_3 \leftarrow r_3 - 2r_1$

$$\begin{vmatrix} 1 & -12 & 8 \\ 0 & 1 & 5 \\ 0 & 30 & -15 \end{vmatrix} = -|A|$$

Step 4: $r_3 \leftarrow r_3 - 30r_2$

$$\begin{vmatrix} 1 & -12 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & -165 \end{vmatrix} = -|A|$$

Thus,

$$|A| = -\left|\begin{array}{ccc} 1 & -12 & 8 \\ 0 & 1 & 5 \\ 0 & 0 & -165 \end{array}\right| = 165 \blacksquare$$

Theorem 13.3
(a) If a square matrix has two identical rows or two identical columns then its determinant is zero.
(b) If a square matrix has a row or a column of zeroes then its determinant is zero.

Example 13.4
Find, by inspection, the determinant of the following matrix.

$$A = \left[\begin{array}{cccc} 3 & -1 & 4 & -2 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 6 \end{array}\right]$$

Solution.
Since the first and the fourth rows are proportional, the determinant is zero by the above theorem $\blacksquare$

Example 13.5
Show that if a square matrix has two proportional rows or two proportional columns then its determinant is zero.

Solution.
Suppose that A is a square matrix such that row j is k times row i with $k \neq 0$. By adding $-\frac{1}{k}r_j$ to r_i then the ith row will consist of 0. By Thereom 13.2 (c), $|A| = 0$ $\blacksquare$

Example 13.6
Show that if A is an $n \times n$ matrix and c is a scalar then $|cA| = c^n|A|$.

Solution.
The matrix cA is obtained from the matrix A by multiplying the rows of A by $c \neq 0$. By mutliplying the first row of cA by $\frac{1}{c}$ we obtain $|B| = \frac{1}{c}|cA|$ where B is obtained from the matrix A by multiplying all the rows of A, except the

first one, by c. Now, divide the second row of B by $\frac{1}{c}$ to obtain $|B'| = \frac{1}{c}|B|$, where B' is the matrix obtained from A by multiplying all the rows of A, except the first and the second, by c. Thus, $|B'| = \frac{1}{c^2}|cA|$. Repeating this process, we find $|A| = \frac{1}{c^n}|cA|$ or $|cA| = c^n|A|$ ■

Example 13.7
(a) Let E_1 be the elementary matrix corresponding to type I elementary row operation. Find $|E_1|$.
(b) Let E_2 be the elementary matrix corresponding to type II elementary row operation. Find $|E_2|$.
(c) Let E_3 be the elementary matrix corresponding to type III elementary row operation. Find $|E_3|$.

Solution.
(a) The matrix E_1 is obtained from the identity matrix by multiplying a row of I_n by a nonzero scalar c. In this case,$|E_1| = c|I_n| = c$.
(b) E_2 is obtained from I_n by adding a multiple of a row to another row. Thus, $|E_2| = |I_n| = 1$.
(c) The matrix E_3 is obtained from the matrix I_n by interchanging two rows. In this case, $|E_3| = -|I_n| = -1$ ■

Practice Problems

Problem 13.1
Use the row reduction technique to find the determinant of the following matrix.

$$A = \left[\begin{array}{cccc} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -6 & 4 & 3 \end{array}\right]$$

Problem 13.2
Given that

$$\left|\begin{array}{ccc} a & b & c \\ d & e & f \\ g & h & i \end{array}\right| = -6,$$

find
(a)

$$\left|\begin{array}{ccc} d & e & f \\ g & h & i \\ a & b & c \end{array}\right|,$$

(b)

$$\left|\begin{array}{ccc} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{array}\right|$$

(c)

$$\left|\begin{array}{ccc} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{array}\right|$$

(d)

$$\left|\begin{array}{ccc} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{array}\right|$$

Problem 13.3

Determine by inspection the determinant of the following matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix}$$

Problem 13.4
Let A be a 3×3 matrix such that $|2A| = 6$. Find $|A|$.

Problem 13.5
Find the determinant of the following elementary matrix by inspection.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 13.6
Find the determinant of the following elementary matrix by inspection.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 13.7
Find the determinant of the following elementary matrix by inspection.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 13.8
Use the row reduction technique to find the determinant of the following matrix.

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

Problem 13.9
Use row reduction to find the determinant of the following **Vandermonde** matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

Problem 13.10
Let a, b, c be three numbers such that $a+b+c=0$. Find the determinant of the following matrix.

$$A = \begin{bmatrix} b+c & a+c & a+b \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix}$$

14. Properties of the Determinant

In this section we shall exhibit some of the fundamental properties of the determinant. One of the immediate consequences of these properties will be an important determinant test for the invertibility of a square matrix.
The first result relates the invertibility of a square matrix to its determinant.

Theorem 14.1
If A is an $n \times n$ matrix then A is nonsingular if and only if $|A| \neq 0$.

Combining Theorem 11.1 with Theorem 14.1, we have

Theorem 14.2
The following statements are all equivalent:
(i) A is nonsingular.
(ii) $|A| \neq 0$.
(iii) A is row equivalent to I_n.
(iv) The homogeneous systen $Ax = \mathbf{0}$ has only the trivial solution.
(v) $rank(A) = n$.

Example 14.1
Prove that $|A| = 0$ if and only if $Ax = \mathbf{0}$ has a nontrivial solution.

Solution.
If $|A| = 0$ then according to Theorem 14.2 the homogeneous system $Ax = \mathbf{0}$ must have a nontrivial solution. Conversely, if the homogeneous system $Ax = \mathbf{0}$ has a nontrivial solution then A must be singular by Theorem 14.2. By Theorem 14.2 (a), $|A| = 0$ ■

Our next major result in this section concerns the determinant of a product of matrices.

Theorem 14.3
If A and B are $n \times n$ matrices then $|AB| = |A||B|$.

Example 14.2
Is it true that $|A + B| = |A| + |B|$?

Solution.
No. Consider the following matrices.

$$A = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right], B = \left[\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right]$$

Then $|A + B| = |\mathbf{0}| = 0$ and $|A| + |B| = -2$ ∎

Example 14.3
Show that if A is invertible then $|A^{-1}| = \frac{1}{|A|}$.

Solution.
If A is invertible then $A^{-1}A = I_n$. Taking the determinant of both sides we find $|A^{-1}||A| = 1$. That is, $|A^{-1}| = \frac{1}{|A|}$. Note that since A is invertible then $|A| \neq 0$ ∎

Example 14.4
Let A and B be two **similar** square matrices , i.e. there exists a nonsingular matrix P such that $A = P^{-1}BP$. Show that $|A| = |B|$.

Solution.
Using Theorem 14.3 and Example 14.3 we have, $|A| = |P^{-1}BP| = |P^{-1}||B||P| = \frac{1}{|P|}|B||P| = |B|$. Note that since P is nonsingular then $|P| \neq 0$ ∎

Practice Problems

Problem 14.1
Show that if n is any positive integer then $|A^n| = |A|^n$.

Problem 14.2
Show that if A is an $n \times n$ skew-symmetric and n is odd then $|A| = 0$.

Problem 14.3
Show that if A is **orthogonal**, i.e. $A^TA = AA^T = I_n$ then $|A| = \pm 1$. Note that $A^{-1} = A^T$.

Problem 14.4
If A is a nonsingular matrix such that $A^2 = A$, what is $|A|$?

Problem 14.5
Find out, without solving the system, whether the following system has a nontrivial solution

$$\left\{\begin{array}{ccccccc} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 0 \\ 3x_1 & + & x_2 & + & 2x_3 & = & 0 \end{array}\right.$$

Problem 14.6
For which values of c does the matrix

$$A = \left[\begin{array}{ccc} 1 & 0 & -c \\ -1 & 3 & 1 \\ 0 & 2c & -4 \end{array}\right]$$

have an inverse.

Problem 14.7
If $|A| = 2$ and $|B| = 5$, calculate $|A^3B^{-1}A^TB^2|$.

Problem 14.8
Show that $|AB| = |BA|$.

Problem 14.9
Show that $|A + B^T| = |A^T + B|$ for any $n \times n$ matrices A and B.

Problem 14.10
Let $A = (a_{ij})$ be a triangular matrix. Show that $|A| \neq 0$ if and only if $a_{ii} \neq 0$, for $1 \leq i \leq n$.

15. Finding A^{-1} Using Cofactor Expansions

In Section 14 we discussed the row reduction method for computing the determinant of a matrix. This method is well suited for computer evaluation of determinants because it is systematic and easily programmed. In this section we introduce a method for evaluating determinants that is useful for hand computations and is important theoretically. Namely, we will obtain a formula for the inverse of an invertible matrix as well as a formula for the solution of square systems of linear equations.

If A is an $n \times n$ square matrix and C_{ij} is the cofactor of the entry a_{ij} then the transpose of the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

is called the **adjoint** of A and is denoted by $adj(A)$.

Example 15.1
Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix},$$

Find $adj(A)$.

Solution.
We first find the matrix of cofactors of A.

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

The adjoint of A is the transpose of this cofactor matrix.

$$adj(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} \blacksquare$$

Our next goal is to find another method for finding the inverse of a nonsingular square matrix based on the adjoint. To this end, we need the following result.

Theorem 15.1
For $i \neq j$ we have

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = 0.$$

Proof.
Let B be the matrix obtained by replacing the jth row of A by the ith row of A. Then B has two identical rows and therefore $|B| = 0$(See Theorem 13.2 (c)). Expand $|B|$ along the jth row. The elements of the jth row of B are $a_{i1}, a_{i2}, \ldots, a_{in}$. The cofactors are $C_{j1}, C_{j2}, \ldots, C_{jn}$. Thus

$$0 = |B| = a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$$

This concludes a proof of the theorem ■

The following theorem states that the product $A \cdot adj(A)$ is a scalar matrix.

Theorem 15.2
If A is an $n \times n$ matrix then $A \cdot adj(A) = |A|I_n$.

Proof.
The (i, j) entry of the matrix

$$A.adj(A) = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{bmatrix}$$

is given by the sum

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = |A|$$

if $i = j$ and 0 if $i \neq j$. Hence,

$$A.adj(A) = \begin{bmatrix} |A| & 0 & \ldots & 0 \\ 0 & |A| & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & |A| \end{bmatrix} = |A|I_n.$$

This ends a proof of the theorem ■

The following theorem provides a way for finding the inverse of a matrix using the notion of the adjoint.

Theorem 15.3
If $|A| \neq 0$ then A is invertible and $A^{-1} = \frac{adj(A)}{|A|}$. Hence, $adj(A) = A^{-1}|A|$.

Proof.
By the previous theorem we have that $A(adj(A)) = |A|I_n$. If $|A| \neq 0$ then $A(\frac{adj(A)}{|A|}) = I_n$. By Theorem 11.2, A is invertible with inverse $A^{-1} = \frac{adj(A)}{|A|}$. ■

Example 15.2
Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

Use Theorem 15.3 to find A^{-1}.

Solution.
First we find the determinant of A given by $|A| = 64$. By Theorem 15.3

$$A^{-1} = \frac{1}{|A|}adj(A) = \begin{bmatrix} \frac{3}{16} & \frac{1}{16} & \frac{3}{16} \\ \frac{3}{32} & \frac{1}{32} & -\frac{5}{32} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \blacksquare$$

In the next theorem we discuss three properties of the adjoint matrix.

Theorem 15.4
Let A and B denote invertible $n \times n$ matrices. Then,
(a) $adj(A^{-1}) = (adj(A))^{-1}$.
(b) $adj(A^T) = (adj(A))^T$.
(c) $adj(AB) = adj(B)adj(A)$.

Proof.
(a) Since $A(adj(A)) = |A|I_n$, $adj(A)$ is invertible and $(adj(A))^{-1} = \frac{A}{|A|} = (A^{-1})^{-1}|A^{-1}| = adj(A^{-1})$.

(b) $adj(A^T) = (A^T)^{-1}|A^T| = (A^{-1})^T|A| = (adj(A))^T$.

(c) We have $adj(AB) = (AB)^{-1}|AB| = B^{-1}A^{-1}|A||B| = (B^{-1}|B|)(A^{-1}|A|) = adj(B)adj(A)$ ■

Example 15.3
Show that if A is singular then $A \cdot adj(A) = \mathbf{0}$, the zero matrix.

Solution.
If A is singular then $|A| = 0$. But then $A \cdot adj(A) = |A|I_n = \mathbf{0}$ ■

Practice Problems

Problem 15.1
Let

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$$

(a) Find adj(A).
(b) Compute $|A|$.

Problem 15.2
Let A be an $n \times n$ matrix. Show that $|adj(A)| = |A|^{n-1}$.

Problem 15.3
If

$$A^{-1} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$$

find $adj(A)$.

Problem 15.4
If $|A| = 2$, find $|A^{-1} + adj(A)|$.

Problem 15.5
Show that $adj(\alpha A) = \alpha^{n-1} adj(A)$.

Problem 15.6
Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{bmatrix}$$

(a) Find $|A|$.
(b) Find $adj(A)$.
(c) Find A^{-1}.

Problem 15.7
Prove that if A is symmetric then $adj(A)$ is also symmetric.

Problem 15.8
Prove that if A is a nonsingular triangular matrix then $adj(A)$ is a lower triangular matrix.

Problem 15.9
Prove that if A is a nonsingular triangular matrix then A^{-1} is also triangular.

Problem 15.10
Let A be an $n \times n$ matrix.
(a) Show that if A has integer entries and $|A| = 1$ then A^{-1} has integer entries as well.
(b) Let $Ax = b$. Show that if the entries of A and b are integers and $|A| = 1$ then the entries of x are also integers.

16. Application of Determinants to Systems: Cramer's Rule

Cramer's rule is another method for solving a linear system of n equations in n unknowns. This method is reasonable for inverting, for example, a 3×3 matrix by hand; however, the inversion method discussed before is more efficient for larger matrices.

Theorem 16.1
Let $Ax = b$ be a matrix equation with $A = (a_{ij}), x = (x_i), b = (b_i)$. Then we have the following matrix equation

$$\left[\begin{array}{c} |A|x_1 \\ |A|x_2 \\ \vdots \\ |A|x_n \end{array}\right] = \left[\begin{array}{c} |A_1| \\ |A_2| \\ \vdots \\ |A_n| \end{array}\right]$$

where A_i is the matrix obtained from A by replacing its i^{th} column by b. It follows that

(1) If $|A| \neq 0$ then the above system has a unique solution given by

$$x_i = \frac{|A_i|}{|A|},$$

where $1 \leq i \leq n$.
(2) If $|A| = 0$ and $|A_i| \neq 0$ for some i then the system has no solution.
(3) If $|A| = |A_1| = \cdots = |A_n| = 0$ then the system has an infinite number of solutions.

Proof.
We have the following chain of equalities

$$\begin{array}{rcl} |A|x & = & |A|(I_n x) \\ & = & (|A|I_n)x \\ & = & adj(A)Ax \\ & = & adj(A)b \end{array}$$

The i^{th} entry of the vector $|A|x$ is given by

$$|A|x_i = b_1C_{1i} + b_2C_{2i} + \cdots + b_nC_{ni}.$$

On the other hand by expanding $|A_i|$ along the i^{th} column we find that

$$|A_i| = C_{1i}b_1 + C_{2i}b_2 + \cdots + C_{ni}b_n.$$

Hence

$$|A|x_i = |A_i|.$$

Now, (1), (2), and (3) follow easily. This ends a proof of the theorem ■

Example 16.1
Use Cramer's rule to solve

$$\begin{cases} -2x_1 & + & 3x_2 & - & x_3 & = & 1 \\ x_1 & + & 2x_2 & - & x_3 & = & 4 \\ -2x_1 & - & x_2 & + & x_3 & = & -3. \end{cases}$$

Solution.
By Cramer's rule we have

$$A = \left[\begin{array}{ccc} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{array}\right], |A| = -2.$$

$$A_1 = \left[\begin{array}{ccc} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{array}\right], |A_1| = -4.$$

$$A_2 = \left[\begin{array}{ccc} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{array}\right], |A_2| = -6.$$

$$A_3 = \left[\begin{array}{ccc} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{array}\right], |A_3| = -8.$$

Thus, $x_1 = \frac{|A_1|}{|A|} = 2, x_2 = \frac{|A_2|}{|A|} = 3, x_3 = \frac{|A_3|}{|A|} = 4$ ■

Example 16.2
Use Cramer's rule to solve

$$\left\{\begin{array}{rcrcrcr} 5x_1 & - & 3x_2 & - & 10x_3 & = & -9 \\ 2x_1 & + & 2x_2 & - & 3x_3 & = & 4 \\ -3x_1 & - & x_2 & + & 5x_3 & = & 1. \end{array}\right.$$

Solution.
By Cramer's rule we have

$$A = \begin{bmatrix} 5 & -3 & -10 \\ 2 & 2 & -3 \\ -3 & -1 & 5 \end{bmatrix}, |A| = -2.$$

$$A_1 = \begin{bmatrix} -9 & -3 & -10 \\ 4 & 2 & -3 \\ 1 & -1 & 5 \end{bmatrix}, |A_1| = 66.$$

$$A_2 = \begin{bmatrix} 5 & -9 & -10 \\ 2 & 4 & -3 \\ -3 & 1 & 5 \end{bmatrix}, |A_2| = -16.$$

$$A_3 = \begin{bmatrix} 5 & -3 & -9 \\ 2 & 2 & 4 \\ -3 & -1 & 1 \end{bmatrix}, |A_3| = 36.$$

Thus, $x_1 = \frac{|A_1|}{|A|} = -33, x_2 = \frac{|A_2|}{|A|} = 8, x_3 = \frac{|A_3|}{|A|} = -18$ ■

Practice Problems

Problem 16.1
Use Cramer's Rule to solve

$$\left\{\begin{array}{rcrcrcr} x_1 & & & + & 2x_3 & = & 6 \\ -3x_1 & + & 4x_2 & + & 6x_3 & = & 30 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 8 \end{array}\right.$$

Problem 16.2
Use Cramer's Rule to solve

$$\left\{\begin{array}{rcrcrcr} 5x_1 & + & x_2 & - & x_3 & = & 4 \\ 9x_1 & + & x_2 & - & x_3 & = & 1 \\ x_1 & - & x_2 & + & 5x_3 & = & 2 \end{array}\right.$$

Problem 16.3
Use Cramer's Rule to solve

$$\left\{\begin{array}{rcrcrcr} 4x_1 & - & x_2 & + & x_3 & = & -5 \\ 2x_1 & + & 2x_2 & + & 3x_3 & = & 10 \\ 5x_1 & - & 2x_2 & + & 6x_3 & = & 1. \end{array}\right.$$

Problem 16.4
Use Cramer's Rule to solve

$$\left\{\begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & 5x_3 & = & -2 \\ -4x_1 & + & x_2 & + & 7x_3 & = & 10 \\ 2x_1 & + & 4x_2 & - & x_3 & = & 3. \end{array}\right.$$

Problem 16.5
Use Cramer's Rule to solve

$$\left\{\begin{array}{rcrcrcr} -x_1 & + & 2x_2 & + & 3x_3 & = & -7 \\ -4x_1 & - & 5x_2 & + & 6x_3 & = & -13 \\ 7x_1 & - & 8x_2 & - & 9x_3 & = & 39. \end{array}\right.$$

Problem 16.6
Use Cramer's Rule to solve

$$\left\{\begin{array}{rcrcrcr} 3x_1 & - & 4x_2 & + & 2x_3 & = & 18 \\ 4x_1 & + & x_2 & - & 5x_3 & = & -13 \\ 2x_1 & - & 3x_2 & + & x_3 & = & 11. \end{array}\right.$$

Problem 16.7
Use Cramer's Rule to solve

$$\begin{cases} 5x_1 & - & 4x_2 & + & x_3 & = & 17 \\ 6x_1 & + & 2x_2 & - & 3x_3 & = & 1 \\ x_1 & - & 4x_2 & + & 3x_3 & = & 15. \end{cases}$$

Problem 16.8
Use Cramer's Rule to solve

$$\begin{cases} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 3x_1 & + & 2x_2 & - & x_3 & = & 16 \\ x_1 & - & 5x_2 & + & 3x_3 & = & -7. \end{cases}$$

Problem 16.9
Use Cramer's Rule to solve

$$\begin{cases} x_1 & - & 2x_2 & + & 2x_3 & = & 5 \\ 3x_1 & + & 2x_2 & - & 3x_3 & = & 13 \\ 2x_1 & - & 5x_2 & + & x_3 & = & 2. \end{cases}$$

Problem 16.10
Use Cramer's Rule to solve

$$\begin{cases} 5x_1 & - & x_2 & + & 3x_3 & = & 10 \\ 6x_1 & + & 4x_2 & - & x_3 & = & 19 \\ x_1 & - & 7x_2 & + & 4x_3 & = & -15. \end{cases}$$

The Theory of Vector Spaces

In Chapter 2, we saw that the operations of addition and scalar multiplication on the set M_{mn} of $m \times n$ matrices possess many of the same algebraic properties as addition and scalar multiplication on the set $\mathbb{R}$ of real numbers. In fact, there are many other sets with operations that share these same properties. Instead of studying these sets individually, we study them as a class.

In this chapter, we define vector spaces to be sets with algebraic operations having the properties similar to those of addition and scalar multiplication on $\mathbb{R}$ and M_{mn}. We then establish many important results that apply to all vector spaces, not just $\mathbb{R}$ and M_{mn}.

17. Vector Spaces and Subspaces

In this section, we define vector spaces to be sets with algebraic operations having the properties similar to those of addition and scalar multiplication on $\mathbb{R}^n$ and M_{mn}.

Let n be a positive integer. Let $\mathbb{R}^n$ be the collection of elements of the form $(x_1, x_2, \ldots, x_n)$, where the x_is are real numbers. Define the following operations on $\mathbb{R}^n$:

(a) Addition: $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$
(b) Multiplication of a vector by a scalar:

$$\alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n).$$

The basic properties of addition and scalar multiplication of vectors in $\mathbb{R}^n$ are listed in the following theorem.

Theorem 17.1
The following properties hold, for u, v, w in $\mathbb{R}^n$ and α, β scalars:
(a) $u + v = v + u$
(b) $u + (v + w) = (u + v) + w$
(c) $u + 0 = 0 + u = u$ where $0 = (0, 0, \ldots, 0)$
(d) $u + (-u) = 0$
(e) $\alpha(u + v) = \alpha u + \alpha v$
(f) $(\alpha + \beta)u = \alpha u + \beta u$
(g) $\alpha(\beta u) = (\alpha\beta)u$
(h) $1u = u$.

The set $\mathbb{R}^n$ with the above operations and properties is called the **Euclidean space.**

A **vector space** is a set V together with the following operations:
(i) Addition: If $u, v \in V$ then $u + v \in V$. We say that V is **closed under addition**.
(ii) Multiplication of an element by a scalar: If $\alpha \in \mathbb{R}$ and $u \in V$ then $\alpha u \in V$. That is, V is **closed under scalar multiplication.**
(iii) These operations satisfy the properties (a) - (h) of Theorem 17.1.

Example 17.1
Let M_{mn} be the collection of all $m \times n$ matrices. Show that M_{mn} is a vector space using matrix addition and scalar multiplication.

Solution.
(i) $A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) = (b_{ij} + a_{ij}) = (b_{ij}) + (a_{ij}) = B + A$, since addition of scalars is commutative.
(ii) Use the fact that addition of scalars is associative.
(iii) $A + \mathbf{0} = (a_{ij}) + (0) = (a_{ij} + 0) = (a_{ij}) = A$.
We leave the proofs of the remaining properties to the reader ■

Example 17.2
Let $V = \{(x, y) : x \geq 0, y \geq 0\}$. Show that the set V fails to be a vector space under the standard operations on $\mathbb{R}^2$.

Solution.
For any $(x, y) \in V$ with $x, y > 0$ we have $-(x, y) \notin V$. Thus, V is not a vector space ■

The following theorem exhibits some properties which follow directly from the axioms of the definition of a vector space and therefore hold for every vector space.

Theorem 17.2
Let V be a vector space, u a vector in V and α is a scalar. Then the following properties hold:
(a) $0u = 0$.
(b) $\alpha 0 = 0$
(c) $(-1)u = -u$
(d) If $\alpha u = 0$ then $\alpha = 0$ or $u = 0$.

Proof.
(a) For any scalar $\alpha \in \mathbb{R}$ we have $0u = (\alpha + (-\alpha))u = \alpha u + (-\alpha)u = \alpha u + (-(\alpha u)) = 0$.
(b) Let $u \in V$. Then $\alpha 0 = \alpha(u + (-u)) = \alpha u + \alpha(-u) = \alpha u + (-(\alpha u)) = 0$.
(c) $u + (-u) = u + (-1)u = 0$. So that $-u = (-1)u$.
(d) Suppose $\alpha u = 0$. If $\alpha \neq 0$ then α^{-1} exists and $u = 1u = (\alpha^{-1}\alpha)u = \alpha^{-1}(\alpha u) = \alpha^{-1}0 = 0$ ■

Now, it is possible that a vector space in contained in a larger vector space. A subset W of a vector space V is called a **subspace** of V if the following two properties are satisfied:

(i) If u, v are in W then $u + v$ is also in W.
(ii) If α is a scalar and u is in W then αu is also in W.

Every vector space V has at least two subspaces:V itself and the subspace consisting of the zero vector of V. These are called the **trivial** subspaces of V.

Example 17.3
Show that a subspace of a vector space is itself a vector space.

Solution.
All the axioms of a vector space hold for the elements of a subspace ■

The following example provides a criterion for deciding whether a subset S of a vector space V is a subspace of V.

Example 17.4
Show that W is a subspace of V if and only if $\alpha u + v \in W$ for all $u, v \in W$ and $\alpha \in \mathbb{R}$.

Solution.
Suppose that W is a subspace of V. If $u, v \in W$ and $\alpha \in \mathbb{R}$ then $\alpha u \in W$ and therefore $\alpha u + v \in W$. Conversely, suppose that for all $u, v \in W$ and $\alpha \in \mathbb{R}$ we have $\alpha u + v \in \mathbb{R}$. In particular, if $\alpha = 1$ then $u + v \in W$. If $v = 0$ then $\alpha u + v = \alpha u \in W$. Hence, W is a subspace ■

Example 17.5
Let M_{22} be the collection of 2×2 matrices. Show that the set W of all 2×2 matrices having zeroes on the main diagonal is a subspace of M_{22}.

Solution.
The set W is the set

$$W = \left\{ \left[\begin{array}{cc} 0 & a \\ b & 0 \end{array} \right] : a, b \in \mathbb{R} \right\}$$

Clearly, the 2×2 zero matrix belongs to W. Also,

$$\alpha \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & a' \\ b' & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha a + a' \\ \alpha b + b' & 0 \end{bmatrix} \in W$$

Thus, W is a subspace of M_{22} ■

Practice Problems

Problem 17.1
Let $D([a,b])$ be the collection of all differentiable functions on $[a,b]$. Show that $D([a,b])$ is a subspace of the vector space of all functions defined on $[a,b]$.

Problem 17.2
Let A be an $m \times n$ matrix. Show that the set $S = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

Problem 17.3
Let $\mathbf{P}$ be the collection of polynomials in the indeterminate x. Let $p(x) = a_0 + a_x + a_2x^2 + \cdots$ and $q(x) = b_0 + b_1x + b_2x^2 + c\ldots$ be two polynomials in $\mathbf{P}$. Define the operations:
(a) Addition: $p(x) + q(x) = a_0 + b_0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$
(b) Multiplication by a scalar: $\alpha p(x) = \alpha a_0 + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots$.
Show that $\mathbf{P}$ is a vector space.

Problem 17.4
Let $F(\mathbb{R})$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$. Define the operations

$$(f + g)(x) = f(x) + g(x)$$

and

$$(\alpha f)(x) = \alpha f(x).$$

Show that $F(\mathbb{R})$ is a vector space under these operations.

Problem 17.5
Define on $\mathbb{R}^2$ the following operations:
(i) $(x, y) + (x', y') = (x + x', y + y,)$;
(ii) $\alpha(x, y) = (\alpha y, \alpha x)$.
Show that $\mathbb{R}^2$ with the above operations is not a vector space.

Problem 17.6
Let $U = \{p(x) \in \mathbf{P} : p(3) = 0\}$. Show that U is a subspace of **P.**

Problem 17.7
Let P_n denote the collection of all polynomials of degree n. Show that P_n is a subspace of $\mathbf{P}$.

Problem 17.8
Show that the set $S = \{(x, y) : x \leq 0\}$ is not a vector space of $\mathbb{R}^2$ under the usual operations of $\mathbb{R}^2$.

Problem 17.9
Show that the collection $C([a, b])$ of all continuous functions on $[a, b]$ with the operations:

$$\begin{array}{rcl}(f + g)(x) & = & f(x) + g(x) \\ (\alpha f)(x) & = & \alpha f(x)\end{array}$$

is a vector space.

Problem 17.10
Let $S = \{(a, b, a + b) : a, b \in \mathbb{R}\}$. Show that S is a subspace of $\mathbb{R}^3$ under the usual operations.

Problem 17.11
Let V be a vector space. Show that if $u, v, w \in V$ are such that $u + v = u + w$ then $v = w$.

Problem 17.12
Let H and K be subspaces of a vector space V.
(a) The **intersection** of H and K, denoted by $H \cap K$, is the subset of V that consists of elements that belong to both H and K. Show that $H \cap V$ is a subspace of V.
(b) The **union** of H and K, denoted by $H \cup K$, is the susbet of V that consists of all elements that belong to either H or K. Give, an example of two subspaces of V such that $H \cup K$ is not a subspace.
(c) Show that if $H \subset K$ or $K \subset H$ then $H \cup K$ is a subspace of V.

18. Basis and Dimension

The concepts of linear combination, spanning set, and basis for a vector space play a major role in the investigation of the structure of any vector space. In this section we introduce and discuss these concepts.
The concept of linear combination will allow us to generate vector spaces from a given set of vectors in a vector space .
Let V be a vector space and $v_1, v_2, \cdots, v_n$ be vectors in V. A vector $w \in V$ is called a **linear combination** of the vectors $v_1, v_2, \ldots, v_n$ if it can be written in the form

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Example 18.1
Show that the vector $\vec{w} = (9, 2, 7)$ is a linear combination of the vectors $\vec{u} = (1, 2, -1)$ and $\vec{v} = (6, 4, 2)$ whereas the vector $\vec{w'} = (4, -1, 8)$ is not.

Solution.
We must find numbers s and t such that

$$(9, 2, 7) = s(1, 2, -1) + t(6, 4, 2)$$

This leads to the system

$$\begin{cases} s & + & 6t & = & 9 \\ 2s & + & 4t & = & 2 \\ -s & + & 2t & = & 7 \end{cases}$$

Solving the first two equations one finds $s = -3$ and $t = 2$ both values satisfy the third equation.
Turning to $(4, -1, 8)$, the question is whether s and t can be found such that $(4, -1, 8) = s(1, 2, -1) + t(6, 4, 2)$. Equating components gives

$$\begin{cases} s & + & 6t & = & 4 \\ 2s & + & 4t & = & -1 \\ -s & + & 2t & = & 8 \end{cases}$$

Solving the first two equations one finds $s = -\frac{11}{4}$ and $t = \frac{9}{8}$ and these values do not satisfy the third equation. That is the system is inconsistent ■

The process of forming linear combinations leads to a method of constructing subspaces, as follows.

Theorem 18.1
Let $W = \{v_1, v_2, \dots, v_n\}$ be a subset of a vector space V. Let $span(W)$ be the collection of all linear combinations of elements of W. Then $span(W)$ is a subspace of V.

Example 18.2
Show that $P_n = span\{1, x, x^2, \cdots, x^n\}$.

Solution.
If $p(x) \in P_n$ then there are scalars $a_0, a_1 \cdots, a_n$ such that $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in span\{1, x, \cdots, x^n\}$ ■

Example 18.3
Show that $\mathbb{R}^n = span\{e_1, e_2, \cdots, e_n\}$ where e_i is the vector with 1 in the ith component and 0 otherwise.

Solution.
We must show that if $u \in \mathbb{R}^n$ then u is a linear combination of the $e_i's$. Indeed, if $u = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ then

$$u = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

Hence u lies in $span\{e_1, e_2, \cdots, e_n\}$ ■

If every element of V can be written as a linear combination of elements of W then we have $V = span(W)$ and in this case we say that W is a **span** of V or W **generates** V.

Example 18.4
(a)Determine whether $\vec{v_1} = (1, 1, 2), \vec{v_2} = (1, 0, 1)$ and $\vec{v_3} = (2, 1, 3)$ span $\mathbb{R}^3$.
(b) Show that the vectors $\vec{i} = (1, 0, 0), \vec{j} = (0, 1, 0)$, and $\vec{k} = (0, 0, 1)$ span $\mathbb{R}^3$.

Solution.
(a) We must show that an arbitrary vector $\vec{v} = (a, b, c)$ in $\mathbb{R}^3$ is a linear combination of the vectors $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$. That is $\vec{v} = s\vec{v_1} + t\vec{v_2} + w\vec{v_3}$. Expressing this equation in terms of components gives

$$\left\{\begin{array}{ccccccc} s & + & t & + & 2w & = & a \\ s & + & & + & w & = & b \\ 2s & + & t & + & 3w & = & c \end{array}\right.$$

The problem is reduced of showing that the above system is consistent. This system will be consistent if and only if the coefficient matrix A

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

is invertible. Since $|A| = 0$, the system is inconsistent and therefore $\mathbb{R}^3 \neq span\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.
(b) See Example 18.3 ■

Next, we introduce a concept which guarantees that any vector in the span of a set S has only one representation as a linear combination of vectors in S. Spanning sets with this property play a fundamental role in the study of vector spaces as we shall see later in this section.
If $v_1, v_2, \ldots, v_n$ are vectors in a vector space with the property that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

holds only for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ then the vectors are said to be **linearly independent.** If there are scalars not all 0 such that the above equation holds then the vectors are called **linearly dependent.**

Example 18.5
Show that the set $S = \{1, x, x^2, \cdots, x^n\}$ is a linearly independent set in P_n.

Solution.
Suppose that $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ for all $x \in \mathbb{R}$. By the Fundamental Theorem of Algebra, a polynomial of degree n has at most n roots. But by the above equation, every real number is a root of the equation. This forces the numbers $a_0, a_1, \cdots, a_n$ to be 0 ■

Example 18.6
Let u be a nonzero vector. Show that $\{u\}$ is linearly independent.

Solution.
Suppose that $\alpha u = 0$. If $\alpha \neq 0$ then we can multiply both sides by α^{-1} and obtain $u = 0$. But this contradicts the fact that u is a nonzero vector ■

Example 18.7
(a) Show that the vectors $\vec{v_1} = (1, 0, 1, 2), \vec{v_2} = (0, 1, 1, 2)$, and $\vec{v_3} = (1, 1, 1, 3)$ are linearly independent.
(b) Show that the vectors $\vec{v_1} = (1, 2, -1), \vec{v_2} = (1, 2, -1)$, and $\vec{v_3} = (1, -2, 1)$ are linearly dependent.

Solution.
(a) Suppose that s, t, and w are real numbers such that $s\vec{v_1} = t\vec{v_2} + w\vec{v_3} = \mathbf{0}$. Then equating components gives

$$\begin{cases} s \quad\quad + \; w = 0 \\ \quad t + w = 0 \\ s + t + w = 0 \\ 2s + 2t + 3w = 0 \end{cases}$$

The second and third equation leads to $s = 0$. The first equation gives $w = 0$ and the second equation gives $t = 0$. Thus, the given vectors are linearly independent.
(b) These vectors are linearly dependent since $\vec{v_1} + \vec{v_2} - 2\vec{v_3} = \mathbf{0}$ ■

Example 18.8
Show that the unit vectors $e_1, e_2, \cdots, e_n$ in $\mathbb{R}^n$ are linearly independent.

Solution.
Suppose that $x_1e_1 + x_2e_2 + \cdots + x_ne_n = (0, 0, \cdots, 0)$. Then $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$ and this leads to $x_1 = x_2 = \cdots = x_n = 0$. Hence the vectors $e_1, e_2, \cdots, e_n$ are linearly independent ■

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a subset of a vector space V. We say that S is a **basis** for V if

(i) S is linearly independent set.
(ii) $V = span(S)$.

Example 18.9
Let e_i be the vector of $\mathbb{R}^n$ whose ith component is 1 and zero otherwise. Show that the set $S = \{e_1, e_2, \ldots, e_n\}$ is a basis for $\mathbb{R}^n$. This is called the **standard basis** of $\mathbb{R}^n$.

Solution.
By Example 18.3, we have $\mathbb{R}^n = span\{e_1, e_2, \cdots, e_n\}$. By Example 18.8, the vectors $e_1, e_2, \cdots, e_n$ are linearly independent. Thus $\{e_1, e_2, \cdots, e_n\}$ is a basis of $\mathbb{R}^3$ ■

Example 18.10
Show that $\{1, x, x^2, \cdots, x^n\}$ is a basis of P_n.

Solution.
By Example 18.2, $P_n = span\{1, x, x^2, \cdots, x^n\}$ and by Example 18.5, the set $S = \{1, x, x^2, \cdots, x^n\}$ is linearly independent. Thus, S is a basis of P_n ■

If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for V then we say that V is a **finite dimensional space of dimension** n. We write $dim(V) = n$. A vector space which is not finite dimensional is said to be **infinite dimensional** vector space. We define the zero vector space to have dimension zero. The vector spaces $M_{mn}, \mathbb{R}^n$, and P_n are finite-dimensional spaces whereas the space P of all polynomials and the vector space of all real-valued functions defined on $\mathbb{R}$ are inifinite dimensional vector spaces.
Unless otherwise specified, the term vector space shall always mean a finite-dimensional vector space.

Example 18.11
Determine a basis and the dimension for the solution space of the homogeneous system

$$\left\{\begin{array}{rcrcrcrcrcr}
2x_1 & + & 2x_2 & - & x_3 & + & & + & x_5 & = & 0\\
-x_1 & - & x_2 & + & 2x_3 & - & 3x_4 & + & x_5 & = & 0\\
x_1 & + & x_2 & - & 2x_3 & & & - & x_5 & = & 0\\
 & & & & x_3 & + & x_4 & + & x_5 & = & 0
\end{array}\right.$$

Solution.
By Example 15.3, we found that $x_1 = -s - t, x_2 = s, x_3 = -t, x_4 = 0, x_5 = t$. So if S is the vector space of the solutions to the given system then $S = \{(-s-t, s, -t, 0, t) : s, t \in \mathbb{R}\} = \{s(-1, 1, 0, 0, 0) + t(-1, 0, -1, 0, 1) : s, t \in \mathbb{R}\} = span\{(-1, 1, 0, 0, 0), (-1, 0, -1, 0, 1)\}$. Moreover, if $s(-1, 1, 0, 0, 0) + t(-1, 0, -1, 0, 1) = (0, 0, 0, 0, 0)$ then $s = t = 0$. Thus the set $\{(-1, 1, 0, 0, 0), (-1, 0, -1, 0, 1)\}$ is a basis for the solution space of the homogeneous system ■

The following theorem will indicate the importance of the concept of a basis in investigating the structure of vector spaces. In fact, a basis for a vector space V determines the representation of each vector in V in terms of the vectors in that basis.

Theorem 18.2
If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for V then any element of V can be written in one and only one way as a linear combination of the vectors in S.

Remark 18.1
A vector space can have different bases; however, all of them have the same number of elements.

Practice Problems

Problem 18.1
Let $W = span\{v_1, v_2, \cdots, v_n\}$, where $v_1, v_2, \cdots, v_n$ are vectors in V. Show that any subspace U of V containing the vectors $v_1, v_2, \cdots, v_n$ must contain W, i.e. $W \subset U$. That is, W is the smallest subspace of V containing $v_1, v_2, \cdots, v_n$.

Problem 18.2
Show that the polynomials $p_1(x) = 1 - x, p_2(x) = 5 + 3x - 2x^2$, and $p_3(x) = 1 + 3x - x^2$ are linearly dependent vectors in P_2.

Problem 18.3
Express the vector $\vec{u} = (-9, -7, -15)$ as a linear combination of the vectors $\vec{v_1} = (2, 1, 4), \vec{v_2} = (1, -1, 3), \vec{v_3} = (3, 2, 5)$.

Problem 18.4
(a) Show that the vectors $\vec{v_1} = (2, 2, 2), \vec{v_2} = (0, 0, 3)$, and $\vec{v_3} = (0, 1, 1)$ span $\mathbb{R}^3$.
(b) Show that the vectors $\vec{v_1} = (2, -1, 3), \vec{v_2} = (4, 1, 2)$, and $\vec{v_3} = (8, -1, 8)$ do not span $\mathbb{R}^3$.

Problem 18.5
Show that

$$M_{22} = span\{\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]\}$$

Problem 18.6
Show that the vectors $\vec{v_1} = (2, -1, 0, 3), \vec{v_2} = (1, 2, 5, -1)$, and $\vec{v_3} = (7, -1, 5, 8)$ are linearly dependent.

Problem 18.7
Show that the vectors $\vec{v_1} = (4, -1, 2)$ and $\vec{v_2} = (-4, 10, 2)$ are linearly independent.

Problem 18.8
Show that the $\{u, v\}$ is linearly dependent if and only if one is a scalar multiple of the other.

Problem 18.9
Let V be the vector of all real-valued functions with domain $\mathbb{R}$. If f, g, h are twice differentiable functions then we define $w(x)$ by the determinant

$$w(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

We call $w(x)$ the **Wronskian** of f, g, and h. Prove that f, g, and h are linearly independent if and only if $w(x) \neq 0$.

Problem 18.10
Use the Wronskian to show that the functions e^x, xe^x, x^2e^x are linearly independent.

Problem 18.11
Find a basis for the vector space M_{22} of 2×2 matrices.

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors arise in many physical applications such as the study of vibrations, electrical systems, genetics, chemical reactions, quantum mechanics, economics, etc. In this chapter we introduce these two concepts and we show how to find them.

19. The Eigenvalues of a Square Matrix

Consider the following linear system

$$\begin{aligned}\frac{dx_1}{dt} =& x_1 - 2x_2\\ \frac{dx_2}{dt} =& 3x_1 - 4x_2.\end{aligned}$$

In matrix form, this sysem can be written as

$$\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]' = \left[\begin{array}{cc} 1 & -2 \\ 3 & -4 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$$

A solution to this system has the form $\mathbf{x} = e^{\lambda t}\mathbf{y}$ where

$$\mathbf{x} = \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] \text{ and } \mathbf{y} = \left[\begin{array}{c} y_1 \\ y_2 \end{array}\right]$$

That is, $\mathbf{x}$ is known once we know λ and $\mathbf{y}$. Substituting, we have

$$\lambda e^{\lambda t}\mathbf{y} = e^{\lambda t}A\mathbf{y}$$

where

$$A = \left[\begin{array}{cc} 1 & -2 \\ 3 & -4 \end{array}\right]$$

or

$$A\mathbf{y} = \lambda\mathbf{y}.$$

Thus, we need to find λ and $\mathbf{y}$ from this matrix equation.

If A is an $n \times n$ matrix and x is a nonzero vector in $\mathbb{R}^n$ such that $Ax = \lambda x$ for some real number λ then we call x an **eigenvector** corresponding to the **eigenvalue** λ.

Example 19.1
Show that $x = \left[\begin{array}{c} 1 \\ 2 \end{array}\right]$ is an eigenvector of the matrix

$$A = \left[\begin{array}{cc} 3 & 0 \\ 8 & -1 \end{array}\right]$$

corresponding to the eigenvalue $\lambda = 3$.

Solution.
The value $\lambda = 3$ is an eigenvalue of A with eigenvector x since

$$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3x \blacksquare$$

Eigenvalues can be either real numbers or complex numbers. To find the eigenvalues of a square matrix A we rewrite the equation $Ax = \lambda x$ as

$$Ax = \lambda I_n x$$

or equivalently

$$(\lambda I_n - A)x = 0.$$

For λ to be an eigenvalue, there must be a nonzero solution to the above homogeneous system. But, the above system has a nontrivial solution if and only if the coefficient matrix $(\lambda I_n - A)$ is singular, that is , if and only if

$$|\lambda I_n - A| = 0.$$

This equation is called the **characteristic equation** of A.

Example 19.2
Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Solution.
The characteristic equation of A is the equation

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = 0$$

That is, the equation: $\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \blacksquare$

It can be shown that

$$\begin{aligned} p(\lambda) =& |\lambda I_n - A| \\ =& \lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \text{ terms of lower degree} \end{aligned} \quad (19.1)$$

That is, $p(\lambda)$ is is a polynomial function in λ of degree n and leading coefficient 1. This is called the **characteristic polynomial** of A.

Example 19.3
Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$$

Solution.
The characteristic polynomial of A is

$$p(\lambda) = \begin{vmatrix} \lambda - 5 & -8 & -16 \\ -4 & \lambda - 1 & -8 \\ 4 & 4 & \lambda + 11 \end{vmatrix}$$

Expanding this determinant we obtain $p(\lambda) = (\lambda + 3)(\lambda^2 + 2\lambda - 3) = (\lambda + 3)^2(\lambda - 1)$ ■

Example 19.4
Show that the constant term in the characteristic polynomial of a matrix A is $(-1)^n|A|$.

Solution.
The constant term of the polynomial $p(\lambda)$ corresponds to $p(0)$. It follows that $p(0) = \text{constant term} = |-A| = (-1)^n|A|$ ■

Example 19.5
Find the eigenvalues of the matrices

(a)

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

(b)

$$B = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$$

Solution.
(a) The characteristic equation of A is given by

$$\left|\begin{array}{cc} \lambda-3 & -2 \\ 1 & \lambda \end{array}\right| = 0$$

Expanding the determinant and simplifying, we obtain

$$\lambda^2 - 3\lambda + 2 = 0$$

or

$$(\lambda-1)(\lambda-2) = 0.$$

Thus, the eigenvalues of A are $\lambda = 2$ and $\lambda = 1$.

(b) The characteristic equation of the matrix B is

$$\left|\begin{array}{cc} \lambda+2 & 1 \\ -5 & \lambda-2 \end{array}\right| = 0$$

Expanding the determinant and simplifying, we obtain

$$\lambda^2 - 9 = 0$$

and the eigenvalues are $\lambda = \pm 3$ ■

Example 19.6
Find the eigenvalues of the matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{array}\right]$$

Solution.
According to Example 19.2 the characteristic equation of A is $\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$. Using the rational root test we find that $\lambda = 4$ is a solution to this equation. Using synthetic division of polynomials we find

$$(\lambda-4)(\lambda^2 - 4\lambda + 1) = 0.$$

The eigenvalues of the matrix A are the solutions to this equation, namely, $\lambda = 4, \lambda = 2+\sqrt{3}$, and $\lambda = 2-\sqrt{3}$ ■

Example 19.7
Show that $\lambda = 0$ is an eigenvalue of a matrix A if and only if A is singular.

Solution.
If $\lambda = 0$ is an eigenvalue of A then it must satisfy $|0I_n - A| = |-A| = 0$. That is $|A| = 0$ and this implies that A is singular. Conversely, if A is singular then $0 = |A| = |0I_n - A|$ and therefore 0 is an eigenvalue of A ■

Example 19.8
(a) Show that the eigenvalues of a triangular matrix are the entries on the main diagonal.
(b) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & -8 & -\frac{1}{4} \end{bmatrix}$$

Solution.
(a) Suppose that A is upper triangular $n \times n$ matrix. Then the matrix $\lambda I_n - A$ is also upper triangular with entries on the main diagonal are $\lambda - a_{11}, \lambda - a_{22}, \cdots, \lambda a_{nn}$. Since the determinant of a triangular matrix is just the product of the entries of the main diaginal, the characteristic equation of A is

$$(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}) = 0.$$

Hence, the eigenvalues of A are $a_{11}, a_{22}, \cdots, a_{nn}$.
(b) Using (a), the eigenvalues of A are $\lambda = \frac{1}{2}, \lambda = \frac{2}{3}$, and $\lambda = -\frac{1}{4}$ ■

Example 19.9
Show that A and A^T have the same characteristic polynomial and hence the same eigenvalues.

Solution.
We use the fact that a matrix and its transpose have the same determinant. Hence,

$$|\lambda I_n - A^T| = |(\lambda I_n - A)^T| = |\lambda I_n - A|.$$

Thus, A and A^T have the same characteristic equation and therefore the same eigenvalues ■

The **algebraic multiplicity** of an eigenvalue λ of a matrix A is the multiplicity of λ as a root of the characteristic polynomial.

Example 19.10
Find the algebraic multiplicity of the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution.
The characteristic equation of the matrix A is

$$\begin{bmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ -2 & -3 & \lambda - 1 \end{bmatrix}$$

Expanding the determinant and simplifying we obtain

$$(\lambda - 2)^2(\lambda - 1) = 0.$$

The eigenvalues of A are $\lambda = 2$ (of algebraic multiplicity 2) and $\lambda = 1$ (of algebraic multiplicity 1) ■

There are many matrices with real entries but with no real eigenvalues. An example is given next.

Example 19.11
Show that the following matrix has no real eigenvalues.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Solution.
The characteristic equation of the matrix A is

$$\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}$$

Expanding the determinant we obtain

$$\lambda^2 + 1 = 0.$$

The solutions to this equation are the imaginary complex numbers $\lambda = i$ and $\lambda = -i$ ■

We next introduce a concept for square matrices that will be fundamental in the next section. We say that two $n \times n$ matrices A and B are **similar** if there exists a nonsingular matrix P such that $B = P^{-1}AP$. We write $A \sim B$. The matrix P is not unique. For example, if $A = B = I_n$ then any invertible matrix P will satisfy the definition.

Example 19.12
Let A and B be similar matrices. Show the following:
(a) $|A| = |B|$.
(b) $tr(A) = tr(B)$.
(c) $|\lambda I_n - A| = |\lambda I_n - B|$.

Solution.
Since $A \sim B$, there exists an invertible matrix P such that $B = P^{-1}AP$.
(a) $|B| = |P^{-1}AP| = |P^{-1}||A||P| = |A|$ since $|P^{-1}| = |P|^{-1}$.
(b) $tr(B) = tr(P^{-1}(AP)) = tr((AP)P^{-1}) = tr(A)$ (See Example 9.9(a)).
(c) Indeed, $|\lambda I_n - B| = |\lambda I_n - P^{-1}AP| = |P^{-1}(\lambda I_n - A)P| = |\lambda I_n - A|$. It follows that two similar matrices have the same eigenvalues ■

Example 19.13
Show that the following matrices are not similar.

$$A = \left[\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}\right], B = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]$$

Solution.
The eigenvalues of A are $\lambda = 3$ and $\lambda = -1$. The eigenvalues of B are $\lambda = 0$ and $\lambda = 2$. According to Example 19.12 (c), these two matrices cannot be similar ■

Example 19.14
Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ including repetitions. Show the following.
(a) $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.
(b) $|A| = \lambda_1 \lambda_2 \cdots \lambda_n$.

Solution.
Factoring the characteristic polynomial of A we find

$$\begin{aligned}p(\lambda) =&(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)\\ =&\lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} + \cdots + (-1)^n\lambda_1\lambda_2\cdots\lambda_n\end{aligned}$$

(a) By Equation 19.1, $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.
(b) $|-A| = p(0) = (-1)^n\lambda_1\lambda_2\cdots\lambda_n$. But $|-A| = (-1)^n|A|$. Hence, $|A| = \lambda_1\lambda_2\cdots\lambda_n$ ■

Example 19.15
(a) Find the characteristic polynomial of

$$A = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right]$$

(b) Find the matrix $A^2 - 5A - -2I_2$.
(c) Compare the result of (b) with (a).

Solution.
(a) $p(\lambda) = \left|\begin{array}{cc} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{array}\right| = \lambda^2 - 5\lambda - 2.$
(b) Simple algebra shows $A^2 - 5A - -2I_2 = \mathbf{0}$.
(c) A satisfies $p(A) = \mathbf{0}$. That is, A satisfies its own characteristic equation ■
More generally, we have

Theorem 19.1 *(Cayley-Hamilton)*
Every square matrix is the zero of its characteristic polynomial.

Example 19.16
Use the Cayley-Hamilton theorem to find the inverse of the matrix

$$A = \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right]$$

Solution.
Since $|A| = 4 - 6 = -2 \neq 0$, A^{-1} exists. By Cayley-Hamilton Theorem we

have

$$\begin{aligned} A^2 - 5A - 2I_2 =& \mathbf{0} \\ 2I_2 =& A^2 - 5A \\ 2A^{-1} =& A - 5I_2 \\ 2A^{-1} =& \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array}\right] - \left[\begin{array}{cc} 5 & 0 \\ 0 & 5 \end{array}\right] \\ =& \left[\begin{array}{cc} -4 & 2 \\ 3 & -1 \end{array}\right] \\ A^{-1} =& \left[\begin{array}{cc} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{array}\right] \blacksquare \end{aligned}$$

Example 19.17
Show that if D is a diagonal matrix then D^k, where k is a positive integer, is a diagonal matrix whose entries are the entries of D raised to the power k.

Solution.
We will show by induction on k that if

$$D = \left[\begin{array}{cccc} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{array}\right]$$

then

$$D^k = \left[\begin{array}{cccc} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{array}\right]$$

Indeed, the result is true for $k = 1$. Suppose true up to $k - 1$ then

$$\begin{aligned} D^k =& D^{k-1}D = \begin{bmatrix} d_{11}^{k-1} & 0 & \cdots & 0 \\ 0 & d_{22}^{k-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn}^{k-1} \end{bmatrix} \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \\ =& \begin{bmatrix} d_{11}^{k} & 0 & \cdots & 0 \\ 0 & d_{22}^{k} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn}^{k} \end{bmatrix} \blacksquare \end{aligned}$$

Practice Problems

Problem 19.1
Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$$

Problem 19.2
Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Problem 19.3
Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}$$

Problem 19.4
Find the eigenvalues of the matrix

$$A = \begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$$

Problem 19.5
Show that if λ is a nonzero eigenvalue of an invertible matrix A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Problem 19.6
Show that if λ is an eigenvalue of a matrix A then λ^m is an eigenvalue of A^m for any positive integer m.

Problem 19.7
Show that if A is similar to a diagonal matrix D then A^k is similar to D^k.

Problem 19.8
Show that the identity matrix I_n has exactly one eigenvalue.

Problem 19.9
Let A be an $n \times n$ **nilpotent** matrix, i.e. $A^k = \mathbf{0}$ for some positive ineteger k.

(a) Show that $\lambda = 0$ is the only eigenvalue of A.
(b) Show that $p(\lambda) = \lambda^n$.

Problem 19.10
Suppose that A and B are $n \times n$ similar matrices and $B = P^{-1}AP$. Show that if λ is an eigenvalue of A with corresponding eigenvector x then λ is an eigenvalue of B with corresponding eigenvector $P^{-1}x$.

Problem 19.11
Let A be an $n \times n$ matrix with n odd. Show that A has at least one real eigenvalue.

Problem 19.12
Consider the following $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}$$

Show that the characterisitc polynomial of A is given by $p(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$. Hence, every monic polynomial (i.e. the coefficient of the highest power of λ is 1) is the characteristic polynomial of some matrix. A is called the **companion matrix** of $p(\lambda)$.

20. Finding Eigenvectors and Eigenspaces

In this section, we turn to the problem of finding the eigenvectors of a square matrix. Recall that an eigenvector is a nontrivial solution to the matrix equation $(\lambda I_n - A)\mathbf{x} = 0$.
For a square matrix of size $n \times n$, the set of all eigenvectors together with the zero vector is a vector space as shown in the next result.

Theorem 20.1
Let V_λ denote the set of eigenvectors of a matrix corresponding to an eigenvalue λ. The set $V^\lambda = V_\lambda \cup \{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$. This subspace is called the **eigenspace** of A corresponding to λ.

Proof.
Let $V_\lambda = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \lambda\mathbf{x}\}$. We will show that $V^\lambda = V_\lambda \cup \{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.
(i) Let $\mathbf{u}, \mathbf{v} \in V^\lambda$. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ then the sum is either $\mathbf{u}, \mathbf{v}$, or $\mathbf{0}$ which belongs to V^λ. So assume that both $\mathbf{u}, \mathbf{v} \in V_\lambda$. We have $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \lambda\mathbf{u} + \lambda\mathbf{v} = \lambda(\mathbf{u}+\mathbf{v})$. That is $\mathbf{u}+\mathbf{v} \in V^\lambda$.
(ii) Let $\mathbf{u} \in V^\lambda$ and $\alpha \in \mathbb{R}$. Then $A(\alpha\mathbf{u}) = \alpha A\mathbf{u} = \lambda(\alpha\mathbf{u})$ so $\alpha\mathbf{u} \in V^\lambda$. Hence, V^λ is a subspace of $\mathbb{R}^n$ $\blacksquare$

By the above theorem, determining the eigenspaces of a square matrix is reduced to two problems: First find the eigenvalues of the matrix, and then find the corresponding eigenvectors which are solutions to linear homogeneous systems.
Consider the matrix

$$A = \left[\begin{array}{ccc} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{array}\right]$$

The characteristic equation of the matrix A is

$$\left[\begin{array}{ccc} \lambda-3 & 2 & 0 \\ 2 & \lambda-3 & 0 \\ 0 & 0 & \lambda-5 \end{array}\right]$$

Expanding the determinant and simplifying we obtain

$$(\lambda-5)^2(\lambda-1) = 0.$$

The eigenvalues of A are $\lambda = 5$ and $\lambda = 1$.
A vector $\mathbf{x} = (x_1, x_2, x_3)^T$ is an eigenvector corresponding to an eigenvalue λ if and only if $\mathbf{x}$ is a nontrivial solution to the homogeneous system

$$\left\{\begin{array}{lllll}(\lambda-3)x_1 & + & 2x_2 & & = 0\\ 2x_1 & + & (\lambda-3)x_2 & & = 0\\ & & & (\lambda-5)x_3 & = 0\end{array}\right. \tag{20.1}$$

If $\lambda = 1$, then the above system becomes

$$\left\{\begin{array}{lllll}-2x_1 & + & 2x_2 & & = 0\\ 2x_1 & - & 2x_2 & & = 0\\ & & & -\ 4x_3 & = 0\end{array}\right.$$

Solving this system yields

$$x_1 = s, x_2 = s, x_3 = 0$$

The eigenspace corresponding to $\lambda = 1$ is

$$V^1 = \left\{\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R}\right\}$$

If $\lambda = 5$, then (20.1) becomes

$$\left\{\begin{array}{lllll}2x_1 & + & 2x_2 & & = 0\\ 2x_1 & + & 2x_2 & & = 0\\ & & & 0x_3 & = 0\end{array}\right.$$

Solving this system yields

$$x_1 = -t, x_2 = t, x_3 = s$$

The eigenspace corresponding to $\lambda = 5$ is

$$\begin{aligned} V^5 &= \left\{\begin{bmatrix} -t \\ t \\ s \end{bmatrix} : s \in \mathbb{R}\right\} \\ &= \left\{t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R}\right\}\end{aligned}$$

Now, a vector $\mathbf{v}$ in $\mathbb{R}^n$ is said to be a **linear combination** of the vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_m}$ if there are real numbers $\alpha_1, \alpha_2, \cdots, \alpha_m$ such that

$$\mathbf{v} = \alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_m\mathbf{v_m}.$$

The set of all linear combinations of the vectos $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_m}$ is a subspace of $\mathbb{R}^n$ denoted by

$$\text{span}\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_m}\}$$

Thus, in the example above, we have

$$V^1 = \text{span}\left\{\left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right]\right\}$$

and

$$V^5 = \text{span}\left\{\left[\begin{array}{c} -1 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]\right\}.$$

Now recall that if $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ are vectors in a vector space with the property that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

holds only for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ then the vectors are said to be **linearly independent.** If there are scalars not all 0 such that the above equation holds then the vectors are called **linearly dependent.**

Example 20.1
(a) Show that $\mathbf{x} = [1, 1, 0]^T$ is linearly independent.
(b) Show that the vectors $\mathbf{x} = [-1, 1, 0]^T$ and $\mathbf{y} = [0, 0, 1]^T$ are linearly indenpent.

Solution.
(a) Suppose that $\alpha\mathbf{x} = \mathbf{0}$. Then $[\alpha, \alpha, 0]^T = [0, 0, 0]^T$ and this implies that $\alpha = 0$.
(b) Suppose that $\alpha\mathbf{x} + \beta\mathbf{y} = \mathbf{0}$. This implies

$$\begin{aligned} -\alpha =& 0 \\ \alpha =& 0 \\ \beta =& 0 \ \blacksquare \end{aligned}$$

Now, since

$$V^1 = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a linearly independent set, we say that

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a **basis** of V^1 and we call the number of elements in the basis the **dimension** of V^1. We write $\dim(V^1) = 1$.

Example 20.2
Find the dimension of V^5.

Solution.
We have that

$$V^5 = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a linearly indepndent set so that

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis of V^5. Hence $\dim(V^5) = 2$ ■

Example 20.3
Find bases for the eigenspaces of the matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution.
The characteristic equation of the matrix A is

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix}$$

Expanding the determinant and simplifying we obtain

$$(\lambda - 2)^2(\lambda - 1) = 0.$$

The eigenvalues of A are $\lambda = 2$ and $\lambda = 1$.
A vector $\mathbf{x} = [x_1, x_2, x_3]^T$ is an eigenvector corresponding to an eigenvalue λ if and only if $\mathbf{x}$ is a solution to the homogeneous system

$$\left\{ \begin{array}{rcrcrcl} \lambda x_1 & & & + & 2x_3 & = & 0 \\ -x_1 & + & (\lambda - 2)x_2 & - & x_3 & = & 0 \\ -x_1 & & & + & (\lambda - 3)x_3 & = & 0 \end{array} \right. \qquad (20.2)$$

If $\lambda = 1$, then (20.2) becomes

$$\left\{ \begin{array}{rcrcrcl} x_1 & & & + & 2x_3 & = & 0 \\ -x_1 & - & x_2 & - & x_3 & = & 0 \\ -x_1 & & & - & 2x_3 & = & 0 \end{array} \right. \qquad (20.3)$$

Solving this system yields

$$x_1 = -2s, x_2 = s, x_3 = s$$

The eigenspace corresponding to $\lambda = 1$ is

$$V^1 = \left\{ \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and $[-2,1,1]^T$ is a basis for V^1. Hence, $\dim(V^1) = 1$.

If $\lambda = 2$, then (20.2) becomes

$$\left\{\begin{array}{rcrcl} 2x_1 & + & 2x_3 & = & 0 \\ -x_1 & - & x_3 & = & 0 \\ -x_1 & - & x_3 & = & 0 \end{array}\right. \qquad (20.4)$$

Solving this system yields

$$x_1 = -s, x_2 = t, x_3 = s$$

The eigenspace corresponding to $\lambda = 2$ is

$$\begin{aligned} V^2 = & \left\{ \begin{bmatrix} -s \\ t \\ s \end{bmatrix} : s \in \mathbb{R} \right\} \\ = & \left\{ s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\} \\ =& \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \end{aligned}$$

One can easily check that the vectors $[-1,0,1]^T$ and $[0,1,0]^T$ are linearly independent and therefore these vectors form a basis for V^2 ■

The **algebraic multiplicity** of an eigenvalue λ of a matrix A is the multiplicity of λ as a root of the characteristic polynomial, and the dimension of the eigenspace corresponding to λ is called the **geometric multiplicity** of λ.

Example 20.4
Find the algebraic and the geometric multiplicity of the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Solution.
The characteristic equation of the matrix A is

$$\left[\begin{array}{ccc} \lambda-2 & -1 & 0 \\ 0 & \lambda-2 & 0 \\ -2 & -3 & \lambda-1 \end{array}\right]$$

Expanding the determinant and simplifying we obtain

$$(\lambda-2)^2(\lambda-1)=0.$$

The eigenvalues of A are $\lambda=2$ (of algebraic multiplicity 2) and $\lambda=1$ (of algebraic multiplicity 1).
A vector $\mathbf{x}=[x_1,x_2,x_3]^T$ is an eigenvector corresponding to an eigenvalue λ if and only if $\mathbf{x}$ is a solution to the homogeneous system

$$\left\{\begin{array}{rcrcrcl} (\lambda-2)x_1 & - & x_2 & & & = & 0 \\ & & (\lambda-2)x_2 & & & = & 0 \\ -2x_1 & - & 3x_2 & + & (\lambda-1)x_3 & = & 0 \end{array}\right. \qquad (20.5)$$

If $\lambda=1$, then (20.5) becomes

$$\left\{\begin{array}{rcrcl} -x_1 & - & x_2 & = & 0 \\ & - & x_2 & = & 0 \\ -2x_1 & - & 3x_2 & = & 0 \end{array}\right.$$

Solving this system yields

$$x_1=0, x_2=0, x_3=s$$

The eigenspace corresponding to $\lambda=1$ is

$$V^1=\left\{\left[\begin{array}{c} 0 \\ 0 \\ s \end{array}\right] : s\in\mathbb{R}\right\}=\text{span}\left\{\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]\right\}$$

and $[0,0,1]^T$ is a basis for V^1. The geometric multiplicity of $\lambda=1$ is $\dim(V^1)=1$.

If $\lambda=2$, then (20.5) becomes

$$\left\{\begin{array}{rcrcrcl} & - & x_2 & & & = & 0 \\ -2x_1 & - & 3x_2 & + & x_3 & = & 0 \end{array}\right.$$

Solving this system yields

$$x_1 = \frac{1}{2}s, x_2 = 0, x_3 = s$$

The eigenspace corresponding to $\lambda = 2$ is

$$\begin{aligned} V^2 &= \left\{ \begin{bmatrix} \frac{1}{2}s \\ 0 \\ s \end{bmatrix} : s \in \mathbb{R} \right\} \\ &= span \left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

and the vector $[\frac{1}{2}, 0, 1]^T$ is a basis for V^2 so that the geometric multiplicity of $\lambda = 2$ is 1 ■

Example 20.5
Solve the homogeneous linear system

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 - 2x_2 \\ \frac{dx_2}{dt} &= 3x_1 - 4x_2. \end{aligned}$$

using eigenvalues and eigenvectors.

Solution.
In matrix form, this sysem can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A solution to this system has the form $\mathbf{x} = e^{\lambda t}\mathbf{y}$ where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

That is, $\mathbf{x}$ is known once we know λ and $\mathbf{y}$. Substituting, we have

$$\lambda e^{\lambda t}\mathbf{y} = e^{\lambda t} A\mathbf{y}$$

where

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

or

$$A\mathbf{y} = \lambda\mathbf{y}.$$

To find λ, we solve the characteristic equation

$$|\lambda I_2 - A| = \lambda^2 + 3\lambda + 2 = 0.$$

The eigenvalues are $\lambda = -1$ and $\lambda = -2$. Next, we find the eigenspaces of A. A vector $\mathbf{x} = [x_1, x_2, x_3]^T$ is an eigenvector corresponding to an eigenvalue λ if and only if $\mathbf{x}$ is a solution to the homogeneous system

$$\left\{ \begin{array}{rcrcl} (\lambda - 1)x_1 & + & 2x_2 & = & 0 \\ -3x_1 & + & (\lambda + 4)x_2 & = & 0 \end{array} \right. \tag{20.6}$$

If $\lambda = -1$, then (20.6) becomes

$$\left\{ \begin{array}{rcrcl} -2x_1 & + & 2x_2 & = & 0 \\ -3x_1 & + & 3x_2 & = & 0 \end{array} \right.$$

Solving this system yields

$$x_1 = s, x_2 = s.$$

The eigenspace corresponding to $\lambda = -1$ is

$$V^{-1} = \left\{ \begin{bmatrix} s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

If $\lambda = -2$, then (20.6) becomes

$$\left\{ \begin{array}{rcrcl} -3x_1 & + & 2x_2 & = & 0 \\ -3x_1 & + & 2x_2 & = & 0 \end{array} \right.$$

Solving this system yields

$$x_1 = \frac{2}{3}s, x_2 = s.$$

The eigenspace corresponding to $\lambda = -2$ is

$$\begin{aligned} V^{-2} =& \left\{ \begin{bmatrix} \frac{3}{2}s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} \\ =& \text{span} \left\{ \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} \right\} \end{aligned}$$

The general solution to the system is

$$\mathbf{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + \frac{3}{2} c_2 e^{-2t} \\ c_1 e^{-t} + c_2 e^{-2t} \end{bmatrix} \blacksquare$$

Practice Problems

Problem 20.1
Show that $\lambda = -3$ is an eigenvalue of the matrix

$$A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$$

and then find the corresponding eigenspace V^{-3}.

Problem 20.2
Find the eigenspaces of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Problem 20.3
Find the eigenspaces of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}$$

Problem 20.4
Find the bases of the eigenspaces of the matrix

$$A = \begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$$

Problem 20.5
Find the eigenvectors and the eigenspaces of the matrix

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

Problem 20.6
Find the eigenvectors and the eigenspaces of the matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Problem 20.7
Find the eigenvectors and the eigenspaces of the matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Problem 20.8
Find the eigenvectors and the eigenspaces of the matrix

$$A = \begin{bmatrix} -1 & 1 & 1 & -2 \\ -1 & 1 & 3 & 2 \\ 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

Problem 20.9
Find the geometric and algebraic multiplicities of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 20.10
When an $n \times n$ matrix has a eigenvalue whose geometric multiplicity is less than the algebraic multiplicity, then it is called a **defective** matrix. Is A defective?

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

21. Diagonalization of a Matrix

In this section we shall discuss a method for finding a basis of $\mathbb{R}^n$ consisting of eigenvectors of a given $n \times n$ matrix A. It turns out that this is equivalent to finding an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. The latter statement suggests the following terminology.

A square matrix A is called **diagonalizable** if A is similar to a diagonal matrix. That is, there exists an invertible matrix P such that $P^{-1}AP = D$ is a diagonal matrix.

The next theorem gives a characterization of diagonalizable matrices and tells how to construct a suitable characterization. In fact, it supports our statement mentioned at the beginning of this section that the problem of finding a basis of $\mathbb{R}^n$ consisting of eigenvectors of A is equivalent to diagonalizing A.

Theorem 21.1

If A is an $n \times n$ square matrix, then the following statements are all equivalent.

(a) A is diagonalizable.

(b) A has n linearly independent eigenvectors.

How do we find P and D? The following is a procedure for diagonalizing a diagonalizable matrix.

Step 1. Find n linearly independent eigenvectors of A, say $p_1, p_2, \cdots, p_n$.

Step 2. Form the matrix P having $p_1, p_2, \cdots, p_n$ as its column vectors.

Step 3. The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2, \dots, \lambda_n$ as its diagonal entries, where λ_i is the eigenvalue corresponding to p_i, $1 \le i \le n$.

Example 21.1

Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution.

From Section 20, the eigenspaces corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 5$ are

$$V^1 = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and

$$\begin{aligned} V^5 &= \left\{ \begin{bmatrix} -t \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\} \\ &= \left\{ t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\} \\ &= span \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

Let $\vec{v_1} = [1, 1, 0]^T, \vec{v_2} = [-1, 1, 0]^T$, and $\vec{v_3} = [0, 0, 1]^T$. It is easy to verify that these vectors are linearly independent. The matrices

$$P = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

satisfy $AP = PD$ or $D = P^{-1}AP$ ■

Example 21.2
Show that the matrix

$$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

is not diagonalizable.

Solution.
The characteristic equation of the matrix A is

$$\begin{bmatrix} \lambda + 3 & -2 \\ 2 & \lambda - 1 \end{bmatrix}$$

Expanding the determinant and simplifying we obtain

$$(\lambda + 1)^2 = 0.$$

The only eigenvalue of A is $\lambda = -1$.
A vector $x = (x_1, x_2)^T$ is an eigenvector corresponding to an eigenvalue λ if and only if x is a solution to the homogeneous system

$$\left\{\begin{array}{rcrcl} (\lambda+3)x_1 & - & 2x_2 & = & 0 \\ 2x_1 & + & (\lambda-1)x_2 & = & 0 \end{array}\right. \tag{21.1}$$

If $\lambda = -1$, then (21.1) becomes

$$\left\{\begin{array}{rcrcl} 2x_1 & - & 2x_2 & = & 0 \\ 2x_1 & - & 2x_2 & = & 0 \end{array}\right. \tag{21.2}$$

Solving this system yields $x_1 = s, x_2 = s$. Hence the eigenspace corresponding to $\lambda = -1$ is

$$V^{-1} = \left\{ \left[\begin{array}{c} s \\ s \end{array} \right] : s \in \mathbb{R} \right\} = span \left\{ \left[\begin{array}{c} 1 \\ 1 \end{array} \right] \right\}$$

Since $dim(V^{-1}) = 1$, A does not have two linearly independent eigenvectors and is therefore not diagonalizable ■

In many applications one is concerned only with knowing whether a matrix is diagonalizable without the need of finding the matrix P. The answer is provided with the following theorem.

Theorem 21.2
If A is an $n \times n$ matrix with n distinct eigenvalues then A is diagonalizable.

Example 21.3
Show that the following matrix is diagonalizable.

$$A = \left[\begin{array}{rrr} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 1 & -1 & 0 \end{array} \right]$$

Solution.
The characteristic equation of the matrix A is

$$\left[\begin{array}{ccc} \lambda-1 & 0 & 0 \\ -1 & \lambda-2 & 3 \\ -1 & 1 & \lambda \end{array} \right]$$

Expanding the determinant and simplifying we obtain

$$(\lambda - 1)(\lambda - 3)(\lambda + 1) = 0$$

The eigenvalues are $1, 3$ and -1, so A is diagonalizable by Theorem 21.1 ■
The converse of Theorem 21.2 is false. See Example 21.1.

Example 21.4
Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution.
The eigenspaces corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 2$ are

$$V^1 = span\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$$

and

$$V^2 = span\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

Let $\vec{v_1} = (-2, 1, 1)^T, \vec{v_2} = (-1, 0, 1)$, and $\vec{v_3} = (0, 1, 0)^T$. It is easy to verify that these vectors are linearly independent. The matrices

$$P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

satisfy $AP = PD$ or $D = P^{-1}AP$ ■

Practice Problems

Problem 21.1
Recall that a matrix A is similar to a matrix B if and only if there is an invertible matrix P such that $P^{-1}AP = B$. In symbol, we write $A \sim B$. Show that if $A \sim B$ then
(a) $A^T \sim B^T$.
(b) $A^{-1} \sim B^{-1}$.

Problem 21.2
If A is invertible show that $AB \sim BA$ for all B.

Problem 21.3
Show that the matrix A is diagonalizable.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Problem 21.4
Show that the matrix A is not diagonalizable.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix}$$

Problem 21.5
Show that the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

is diagonalizable with only one eigenvalue.

Problem 21.6
Show that A is diagonalizable if and only if A^T is diagonalizable.

Problem 21.7
Show that if A and B are similar then A is diagonalizable if and only if B is diagonalizable.

Problem 21.8
Give an example of two diagonalizable matrices A and B such that $A+B$ is not diagonalizable.

Problem 21.9
Find P and D such that $P^{-1}AP = D$ where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 21.10
Find P and D such that $P^{-1}AP = D$ where

$$A = \begin{bmatrix} -1 & 1 & 1 & -2 \\ -1 & 1 & 3 & 2 \\ 1 & 1 & -1 & -2 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

Linear Transformations

In this chapter we shall discuss a special class of functions whose domains and ranges are vector spaces. Such functions are referred to as linear transformations, a concept to be defined in Section 23.

22. An Example of Motivation

Linear transformations play an important role in many areas of mathematics, the physical and social sciences, engineering, and economics. Let's look at an application in Cryptography theory.

Suppose we want to send the following message to our friend,

MEET TOMORROW

For the security, we first code the alphabet as follows:

A	B	C	···	X	Y	Z
1	2	3	···	24	25	26

Thus, the code message is

M	E	E	T	T	O	M	O	R	R	O	W
13	5	5	20	20	15	13	15	18	18	15	23

The sequence

$$13 \quad 5 \quad 5 \quad 20 \quad 20 \quad 15 \quad 13 \quad 15 \quad 18 \quad 18 \quad 15 \quad 23$$

is the original code message. To encrypt the original code message, we can apply a linear transformation to original code message. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Then, we break the original message into 4 vectors first,

$$\begin{bmatrix} 13 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 20 \\ 20 \\ 15 \end{bmatrix}, \begin{bmatrix} 13 \\ 15 \\ 18 \end{bmatrix}, \begin{bmatrix} 18 \\ 15 \\ 23 \end{bmatrix}$$

and use the linear transformation to obtain the encrypted code message

$$T\left(\begin{bmatrix} 13 \\ 5 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 13 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 38 \\ 28 \\ 15 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 20 \\ 20 \\ 15 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 105 \\ 70 \\ 50 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 13 \\ 15 \\ 18 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 13 \\ 15 \\ 18 \end{bmatrix} = \begin{bmatrix} 97 \\ 64 \\ 51 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 18 \\ 15 \\ 23 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 18 \\ 15 \\ 23 \end{bmatrix} = \begin{bmatrix} 117 \\ 79 \\ 61 \end{bmatrix}$$

Then, we can send the encrypted message code

$$38 \quad 28 \quad 15 \quad 105 \quad 70 \quad 50 \quad 97 \quad 64 \quad 51 \quad 117 \quad 79 \quad 61$$

Suppose our friend wants to encode the encrypted message code. Our friend can find the inverse matrix of A first

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

and then

$$A^{-1} \begin{bmatrix} 38 \\ 28 \\ 15 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 38 \\ 28 \\ 15 \end{bmatrix} = \begin{bmatrix} 13 \\ 5 \\ 5 \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} 105 \\ 70 \\ 50 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 105 \\ 70 \\ 50 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 15 \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} 97 \\ 64 \\ 51 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 97 \\ 64 \\ 51 \end{bmatrix} = \begin{bmatrix} 13 \\ 15 \\ 18 \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} 117 \\ 79 \\ 61 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 117 \\ 79 \\ 61 \end{bmatrix} = \begin{bmatrix} 18 \\ 15 \\ 23 \end{bmatrix}$$

Thus, our friend can find the original message code

$$13 \quad 5 \quad 5 \quad 20 \quad 20 \quad 15 \quad 13 \quad 15 \quad 18 \quad 18 \quad 15 \quad 23$$

via the inverse matrix of A.

Example 22.1
What is the decrypted message for

77 54 38 71 49 29 68 51 33 76 48 40 86 53 52

Solution.
We first break the message into 5 vectors,

$$\begin{bmatrix} 77 \\ 54 \\ 38 \end{bmatrix}, \begin{bmatrix} 71 \\ 49 \\ 29 \end{bmatrix}, \begin{bmatrix} 68 \\ 51 \\ 33 \end{bmatrix}, \begin{bmatrix} 76 \\ 48 \\ 40 \end{bmatrix}, \begin{bmatrix} 86 \\ 53 \\ 52 \end{bmatrix}.$$

and then the original message code can be obtained by

$$A^{-1}\begin{bmatrix} 77 \\ 54 \\ 38 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 77 \\ 54 \\ 38 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \\ 15 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 71 \\ 49 \\ 29 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 71 \\ 49 \\ 29 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \\ 7 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 68 \\ 61 \\ 33 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 68 \\ 61 \\ 33 \end{bmatrix} = \begin{bmatrix} 18 \\ 1 \\ 16 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 76 \\ 48 \\ 40 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 76 \\ 48 \\ 40 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 12 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 86 \\ 53 \\ 52 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 86 \\ 53 \\ 52 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \\ 19 \end{bmatrix}.$$

Thus, the original message from our friend is

16	8	15	20	15	7	18	1	16	8	16	12	1	14	19
P	H	O	T	O	G	R	A	P	H	P	L	A	N	S

23. Linear Transformation: Definition and Elementary Properties

A **linear transformation** T from a vector space V to a vector space W is a function $T : V \to W$ that satisfies the following two conditions
(i) $T(u+v) = T(u) + T(v)$, for all u, v in V.
(ii) $T(\alpha u) = \alpha T(u)$ for all u in V and scalar α.

If $W = \mathbb{R}$ then we call T a **linear functional** on V.
It is important to keep in mind that the addition in $u+v$ refers to the addition operation in V whereas that in $T(u)+T(v)$ refers to the addition operation in W. Similar remark for the scalar multiplication.

Example 23.1
Show that $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} x \\ x+y \\ x-y \end{array}\right]$$

is a linear transformation.

Solution.
We verify the two conditions of the definition. Given $[x_1, y_1]^T$ and $[x_2, y_2]^T$ in $\mathbb{R}^2$, compute

$$\begin{aligned}
T\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right] + \left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right) =& T\left(\left[\begin{array}{c} x_1+y_1 \\ x_2+y_2 \end{array}\right]\right) \\
=& \left[\begin{array}{c} x_1+x_2 \\ x_1+x_2+y_1+y_2 \\ x_1+x_2-y_1-y_2 \end{array}\right] \\
=& \left[\begin{array}{c} x_1 \\ x_1+y_1 \\ x_1-y_1 \end{array}\right] + \left[\begin{array}{c} x_2 \\ x_2+y_2 \\ x_2-y_2 \end{array}\right] \\
=& T\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]\right) + T\left(\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right)
\end{aligned}$$

This proves the first condition. For the second condition, we let $\alpha \in \mathbb{R}$ and compute

$$\begin{aligned}
T\left(\alpha\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]\right) =& T\left(\left[\begin{array}{c} \alpha x_1 \\ \alpha y_1 \end{array}\right]\right)\\
=& \left[\begin{array}{c} \alpha x_1 \\ \alpha x_1 + \alpha y_1 \\ \alpha x_1 - \alpha y_1 \end{array}\right] = \alpha\left[\begin{array}{c} x_1 \\ x_1 + y_1 \\ x_1 - y_1 \end{array}\right]\\
=& \alpha T\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]\right)
\end{aligned}$$

Hence T is a linear transformation ■

Example 23.2
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} x \\ y \\ 1 \end{array}\right].$$

Show that T is not linear.

Solution.
We show that the first condition of the definition is violated. Indeed, for any two vectors $[x_1, y_1]^T$ and $[x_2, y_2]^T$ we have

$$\begin{aligned}
T\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right] + \left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right) =& T\left(\left[\begin{array}{c} x_1 + x_2 \\ y_1 + y_2 \end{array}\right]\right)\\
=& \left[\begin{array}{c} x_1 + x_2 \\ y_1 + y_2 \\ 1 \end{array}\right]\\
\neq& \left[\begin{array}{c} x_1 \\ y_1 \\ 1 \end{array}\right] + \left[\begin{array}{c} x_2 \\ y_2 \\ 1 \end{array}\right] = T\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]\right) + T\left(\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right)
\end{aligned}$$

Hence the given transformation is not linear ■

Example 23.3
Show that an $m \times n$ matrix defines a linear transforamtion from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Solution.
Given $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^m$ and $\alpha \in \mathbb{R}$, matrix arithmetic yields $T(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}+\mathbf{y}) = A\mathbf{x}+A\mathbf{y} = T\mathbf{x}+T\mathbf{y}$ and $T(\alpha\mathbf{x}) = A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha T\mathbf{x}$. Thus, T is linear ■

Example 23.4
(a) Show that the identity transformation defined by $I(v) = v$ for all $v \in V$ is a linear transformation.
(b) Show that the zero transformation is linear.

Solution.
(a) For all $u, v \in V$ and $\alpha \in \mathbb{R}$ we have $I(u+v) = u+v = Iu+Iv$ and $I(\alpha u) = \alpha u = \alpha I u$. So I is linear.
(b) For all $u, v \in V$ and $\alpha \in \mathbb{R}$ we have $\mathbf{0}(u+v) = \mathbf{0} = \mathbf{0}u + \mathbf{0}v$ and $\mathbf{0}(\alpha u) = \mathbf{0} = \alpha 0 u$. So $\mathbf{0}$ is linear ■

The next theorem collects four useful properties of all linear transformations.

Theorem 23.1
If $T : V \to W$ is a linear transformation then

(a) $T(0) = 0$
(b) $T(-u) = -T(u)$
(c) $T(u-w) = T(u) - T(w)$
(d) $T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_n T(u_n)$.

The following theorem provides a criterion for showing that a transformation is linear.

Theorem 23.2
A function $T : V \to W$ is linear if and only if $T(\alpha u + v) = \alpha T(u) + T(v)$ for all $u, v \in V$ and $\alpha \in \mathbb{R}$.

Example 23.5
Let M_{mn} denote the vector space of all $m \times n$ matrices.
(a) Show that $T : M_{mn} \to M_{nm}$ defined by $T(A) = A^T$ is a linear transformation.
(b) Show that $T : M_{nn} \to \mathbb{R}$ defined by $T(A) = tr(A)$ is a linear functional.

Solution.
(a) For any $A, B \in M_{mn}$ and $\alpha \in \mathbb{R}$ we find $T(\alpha A + B) = (\alpha A + B)^T = \alpha A^T + B^T = \alpha T(A) + T(B)$. Hence, T is a linear transformation.
(b) For any $A, B \in M_{nn}$ and $\alpha \in \mathbb{R}$ we have $T(\alpha A + b) = tr(\alpha A + B) = \alpha tr(A) + tr(B) = \alpha T(A) + T(B)$ so T is a linear functional ■

Example 23.6
Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V and let $T : V \to W$ be a linear transformation. Show that if $T(v_1) = T(v_2) = \cdots = T(v_n) = 0$ then $T(v) = 0$ for any vector v in V.

Solution.
Let $v \in V$. Then there exist scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$. Since T is linear then $T(v) = \alpha T v_1 + \alpha T v_2 + \cdots + \alpha_n T v_n = 0$ ■

Example 23.7
Let $S : V \to W$ and $T : V \to W$ be two linear transformations. Show the following:
(a) $S + T$ and $S - T$ are linear transformations.
(b) αT is a linear transformation where α denotes a scalar.

Solution.
(a) Let $u, v \in V$ and $\alpha \in \mathbb{R}$ then

$$\begin{aligned}(S \pm T)(\alpha u + v) =& S(\alpha u + v) \pm T(\alpha u + v)\\ =& \alpha S(u) + S(v) \pm (\alpha T(u) + T(v))\\ =& \alpha(S(u) \pm T(u)) + (S(v) \pm T(v))\\ =& \alpha(S \pm T)(u) + (S \pm T)(v)\end{aligned}$$

(b) Let $u, v \in V$ and $\beta \in \mathbb{R}$ then

$$\begin{aligned}(\alpha T)(\beta u + v) =& (\alpha T)(\beta u) + (\alpha T)(v)\\ =& \alpha\beta T(u) + \alpha T(v)\\ =& \beta(\alpha T(u)) + \alpha T(v)\\ =& \beta(\alpha T)(u) + (\alpha T)(v)\end{aligned}$$

Hence, αT is a linear transformation ■

The following theorem shows that two linear transformations defined on V are equal whenever they have the same effect on a basis of the vector space V.

Theorem 23.3
Let $V = span\{v_1, v_2, \cdots, v_n\}$. If T and S are two linear transformations from V into a vector space W such that $T(v_i) = S(v_i)$ for each i then $T = S$.

The following very useful theorem tells us that once we say what a linear transformation does to a basis for V, then we have completely specified T.

Theorem 23.4
Let V be an $n-$dimensional vector space with basis $\{v_1, v_2, \cdots, v_n\}$. If $T : V \longrightarrow W$ is a linear transformation then for any $v \in V$, Tv is completely determined by $\{Tv_1, Tv_2, \cdots, Tv_n\}$.

Theorem 23.5
Let V and W be two vector spaces and $\{e_1, e_2, \cdots, e_n\}$ be a basis of V. Given any vectors $w_1, w_2, \cdots, w_n$ in W, there exists a unique linear transformation $T : V \longrightarrow W$ such that $T(e_i) = w_i$ for each i.

Example 23.8
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that there exists an $m \times n$ matrix A such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$. The matrix A is called the **standard matrix** of T.

Solution.
Consider the standard basis of $\mathbb{R}^n$, $\{e_1, e_2, \cdots, e_n\}$ where e_i is the vector with 1 at the i^{th} component and 0 otherwise. Let $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in \mathbb{R}^n$. Then $\mathbf{x} = x_1e_1 + x_2e_2 + \cdots + x_ne_n$. Thus,

$$T(\mathbf{x}) = x_1T(e_1) + x_2T(e_2) + \cdots + x_nT(e_n) = A\mathbf{x}$$

where $A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$ ■

Example 23.9
Find the standard matrix of $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 2y + z \\ x - z \end{bmatrix}$$

Solution.
Indeed, by simple inspection one finds that

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \blacksquare$$

Practice Problems

Problem 23.1
Consider the matrix

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

Show that the transformation

$$T_E\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$$

is linear. This transformation is a **reflection** in the line $y = x$.

Problem 23.2
Consider the matrix

$$F = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix},$$

Show that

$$T_F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \alpha x \\ y \end{bmatrix}$$

is linear. Such a transformation is called an **expansion** if $\alpha > 1$ and a **compression** if $\alpha < 1$.

Problem 23.3
Consider the matrix

$$G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Show that

$$T_G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \alpha x + y \\ y \end{bmatrix}$$

is linear. This transformation is called a **shear**

Problem 23.4
Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x - 2y \\ 3x \end{bmatrix}$$

is a linear transformation.

Problem 23.5
(a) Show that $D : P_n \longrightarrow P_{n-1}$ given by $D(p) = p'$ is a linear transformation.
(b) Show that $I : P_n \longrightarrow P_{n+1}$ given by $I(p) = \int_0^x p(t)dt$ is a linear transformation.

Problem 23.6
If $T : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a linear transformation with $T\left(\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}\right) = 5$ and $T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = 2$. Find $T\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$.

Problem 23.7
Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the transformation

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Show that T is linear. This transformation is called a **projection.**

Problem 23.8
Show that the following transformation is not linear: $T : M_{nn} \longrightarrow \mathbb{R}$.

Problem 23.9
If $T_1 : U \longrightarrow V$ and $T_2 : V \longrightarrow W$ are linear transformations, then $T_2 \circ T_1 : U \longrightarrow W$ is also a linear transformation.

Problem 23.10
Let T be a linear transformation on a vector space V such that $T(v-3v_1) = w$ and $T(2v - v_1) = w_1$. Find $T(v)$ and $T(v_1)$ in terms of w and w_1.

24. Kernel and Range of a Linear Transformation

In this section we discuss two important subspaces associated with a linear transformation T, namely the kernel of T and the range of T. Also, we discuss some further properties of T as a function such as, the concepts of one-one, onto and the inverse of T.
Let $T : V \to W$ be a linear transformation. The **kernel** of T (denoted by $ker(T)$) and the **range** of T (denoted by $R(T)$) are defined by

$$\begin{aligned} ker(T) =& \{v \in V : T(v) = 0\} \\ R(T) =& \{w \in W : T(v) = w, v \in V\} \end{aligned}$$

The following theorem asserts that $ker(T)$ and $R(T)$ are subspaces.

Theorem 24.1
Let $T : V \to W$ be a linear transformation. Then
(a) $ker(T)$ is a subspace of V.
(b) $R(T)$ is a subspace of W.

Proof.
(a) Let $v_1, v_2 \in ker(T)$ and $\alpha \in \mathbb{R}$. Then $T(\alpha v_1 + v_2) = \alpha Tv_1 + Tv_2 = 0$. That is, $\alpha v_1 + v_2 \in ker(T)$. This proves that $ker(T)$ is a subspace of V.
(b) Let $w_1, w_2 \in R(T)$. Then there exist $v_1, v_2 \in V$ such that $Tv_1 = w_1$ and $Tv_2 = w_2$. Let $\alpha \in \mathbb{R}$. Then $T(\alpha v_1 + v_2) = \alpha Tv_1 + Tv_2 = \alpha w_1 + w_2$. Hence, $\alpha w_1 + w_2 \in R(T)$. This shows that $R(T)$ is a subspace of W ■

Example 24.1
If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T\left(\left[\begin{array}{c} x \\ y \\ z \end{array}\right]\right) = \left[\begin{array}{c} x - y \\ z \\ y - x \end{array}\right],$$

find $ker(T)$ and $R(T)$.

Solution.
If $\left[\begin{array}{c} x \\ y \\ z \end{array}\right] \in ker(T)$ then

$$\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right] = T\left(\left[\begin{array}{c} x \\ y \\ z \end{array}\right]\right) = \left[\begin{array}{c} x-y \\ z \\ y-x \end{array}\right].$$

This leads to the system

$$\left\{\begin{array}{rcrcrcl} x & - & y & & & = & 0 \\ -x & + & y & & & = & 0 \\ & & & & z & = & 0 \end{array}\right.$$

The general solution is given by $\left[\begin{array}{c} s \\ s \\ 0 \end{array}\right]$ and therefore

$$ker(T) = \text{span}\left\{\left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right]\right\}.$$

Now, let $\left[\begin{array}{c} u \\ v \\ w \end{array}\right] \in R(T)$ be given. Then there is a vector $\left[\begin{array}{c} x \\ y \\ z \end{array}\right] \in \mathbb{R}^3$ such that $T\left(\left[\begin{array}{c} x \\ y \\ z \end{array}\right]\right) = \left[\begin{array}{c} u \\ v \\ w \end{array}\right]$. This yields the following system

$$\left\{\begin{array}{rcrcrcl} x & - & y & & & = & u \\ -x & + & y & & & = & w \\ & & & & z & = & v \end{array}\right.$$

and the solution is given by $\left[\begin{array}{c} u \\ v \\ -u \end{array}\right]$. Hence,

$$R(T) = \text{span}\left\{\left[\begin{array}{c} 1 \\ 0 \\ -1 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]\right\} \blacksquare$$

Example 24.2
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be given by $Tx = Ax$. Find $ker(T)$ and $R(T)$.

Solution.
We have
$$ker(T) = \{x \in \mathbb{R}^n : Ax = \mathbf{0}\}$$
and
$$R(T) = \{Ax : x \in \mathbb{R}^n\} \blacksquare$$

Example 24.3
Let V be any vector space and α be a scalar. Let $T : V \to V$ be the transformation defined by $T(v) = \alpha v$.

(a) Show that T is linear.
(b) What is the kernel of T?
(c) What is the range of T?

Solution.
(a) Let $u, v \in V$ and $\beta \in \mathbb{R}$. Then $T(\beta u + v) = \alpha(\beta u + v) = \alpha\beta u + \alpha v = \beta T(u) + T(v)$. Hence, T is linear
(b) If $v \in ker(T)$ then $0 = T(v) = \alpha v$. If $\alpha = 0$ then $ker(T) = V$. If $\alpha \neq 0$ then $ker(T) = \{0\}$.
(c) If $\alpha = 0$ then $R(T) = \{0\}$. If $\alpha \neq 0$ then $R(T) = V$ since $T(\frac{1}{\alpha}v) = v$ for all $v \in V$ $\blacksquare$

Since the kernel and the range of a linear transformation are subspaces of given vector spaces, we may speak of their dimensions. The dimension of the kernel is called the **nullity** of T (denoted $nullity(T)$) and the dimension of the range of T is called the **rank** of T (denoted $rank(T)$).

Example 24.4
Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by
$$T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} x \\ x + y \\ y \end{array}\right].$$

(a) Show that T is linear.
(b) Find $nullity(T)$ and $rank(T)$.

Solution.
(a) Let $\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]$ and $\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]$ be two vectors in $\mathbb{R}^2$. Then for any $\alpha \in \mathbb{R}$ we have

$$\begin{aligned} T\left(\alpha\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]+\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right) =&T\left(\left[\begin{array}{c} \alpha x_1 + x_2 \\ \alpha y_1 + y_2 \end{array}\right]\right) \\ =&\left[\begin{array}{c} \alpha x_1 + x_2 \\ \alpha x_1 + x_2 + \alpha y_1 + y_2 \\ \alpha y_1 + y_2 \end{array}\right] \\ =&\left[\begin{array}{c} \alpha x_1 \\ \alpha(x_1 + y_1) \\ \alpha y_1 \end{array}\right] + \left[\begin{array}{c} x_2 \\ x_2 + y_2 \\ y_2 \end{array}\right] \\ =&\alpha T\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]\right) + T\left(\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right) \end{aligned}$$

(b) Let $\left[\begin{array}{c} x \\ y \end{array}\right] \in ker(T)$. Then $\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right] = T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} x \\ x+y \\ y \end{array}\right]$ and this leads to $ker(T) = \left\{\left[\begin{array}{c} 0 \\ 0 \end{array}\right]\right\}$. Hence, $nullity(T) = 0$.

Now, let $\left[\begin{array}{c} u \\ v \\ w \end{array}\right] \in R(T)$. Then there exists $\left[\begin{array}{c} x \\ y \end{array}\right] \in \mathbb{R}^2$ such that $\left[\begin{array}{c} x \\ x+y \\ y \end{array}\right] = T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} u \\ v \\ w \end{array}\right]$. Hence, $R(T) = \left\{\left[\begin{array}{c} x \\ x+y \\ y \end{array}\right] : x, y \in \mathbb{R}\right\} = \text{span}\left\{\left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right]\right\}$.
Thus, $rank(T) = 2$ ■

Since linear transformations are functions, it makes sense to talk about one-to-one and onto functions. We say that a linear transformation $T : V \to W$ is **one-to-one** if $Tv = Tw$ implies $v = w$. We say that T is **onto** if $R(T) = W$. If T is both one-to-one and onto we say that T is an **isomorphism** and the vector spaces V and W are said to be **isomorphic** and we write $V \cong W$. The identity transformation is an isomorphism of any vector space onto itself. That is, if V is a vector space then $V \cong V$.

The following theorem is used as a criterion for proving that a linear transformation is one-to-one.

Theorem 24.2
Let $T : V \to W$ be a linear transformation. Then T is one-to-one if and only if $ker(T) = \{0\}$.

Proof.
Suppose first that T is one-to-one. Let $v \in ker(T)$. Then $Tv = 0 = T0$. Since T is one-to-one, $v = 0$. Hence, $ker(T) = \{0\}$.
Conversely, suppose that $ker(T) = \{0\}$. Let $u, v \in V$ be such that $Tu = Tv$, i.e $T(u-v) = 0$. This says that $u-v \in ker(T)$, which implies that $u-v = 0$. Thus, T is one-to-one ■

Another criterion of showing that a linear transformation is one-to-one is provided by the following theorem.

Theorem 24.3
Let $T : V \to W$ be a linear transformation. Then the following are equivalent:
(a) T is one-to-one.
(b) If S is linearly independent set of vectors then $T(S)$ is also linearly independent.

Proof.
(a) ⇒ (b): Let $S = \{v_1, v_2, \cdots, v_3\}$ consists of linearly independent vectors. Then $T(S) = \{T(v_1), T(v_2), \cdots, T(v_n)\}$. Suppose that $\alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) = 0$. Then we have

$$\begin{aligned} T(0) = 0 = &\alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) \\ = &T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) \end{aligned}$$

Since T is one-to-one, we must have $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$. Since the vectors $v_1, v_2, \cdots, v_n$ are linear, we have $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. This shows that $T(S)$ consists of linearly independent vectors.
(b) ⇒ (a): Suppose that $T(S)$ is linearly independent for any linearly independent set S. Let v be a nonzero vector of V. Since $\{v\}$ is linearly independent, $\{Tv\}$ is linearly independent. That is, $Tv \neq 0$. Hence, $kerT = \{0\}$ and by Theorem 21.2, T is one-to-one ■

Example 24.5
Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$

(a) Show that T is linear.
(b) Show that T is onto but not one-to-one.

Solution.
(a) Let $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be two vectors in $\mathbb{R}^3$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T\left(\alpha \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) =& T\left(\begin{bmatrix} \alpha x_1 + x_2 \\ \alpha y_1 + y_2 \\ \alpha z_1 + z_2 \end{bmatrix}\right) \\ =& \begin{bmatrix} \alpha x_1 + x_2 + \alpha y_1 + y_2 \\ \alpha x_1 + x_2 - \alpha y_1 - y_2 \end{bmatrix} \\ =& \begin{bmatrix} \alpha(x_1 + y_1) \\ \alpha(x_1 - y_1) \end{bmatrix} + \begin{bmatrix} x_2 + y_2 \\ x_2 - y_2 \end{bmatrix} \\ =& \alpha T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) \end{aligned}$$

(b) Since $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in ker(T)$, by Theorem 24.2 T is not one-to-one. Now, let $\begin{bmatrix} u \\ v \\ w \end{bmatrix} \in \mathbb{R}^3$ be such that $T\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$. In this case, $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$. Hence, $R(T) = \mathbb{R}^3$ so that T is onto ■

Example 24.6

Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \\ x \end{bmatrix}$.

(a) Show that T is linear.
(b) Show that T is one-to-one but not onto.

Solution.
(a) Let $\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]$ and $\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]$ be two vectors in $\mathbb{R}^2$. Then for any $\alpha \in \mathbb{R}$ we have

$$\begin{aligned}
T\left(\alpha\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]+\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right) =&T\left(\left[\begin{array}{c} \alpha x_1+x_2 \\ \alpha y_1+y_2 \end{array}\right]\right)\\
=&\left[\begin{array}{c} \alpha x_1+x_2+\alpha y_1+y_2 \\ \alpha x_1+x_2-\alpha y_1-y_2 \\ \alpha x_1+x_2 \end{array}\right]\\
=&\left[\begin{array}{c} \alpha(x_1+y_1) \\ \alpha(x_1-y_1) \\ \alpha x_1 \end{array}\right]+\left[\begin{array}{c} x_2+y_2 \\ x_2-y_2 \\ x_2 \end{array}\right]\\
=&\alpha T\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]\right)+T\left(\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right)
\end{aligned}$$

Hence, T is linear.
(b) If $\left[\begin{array}{c} x \\ y \end{array}\right] \in ker(T)$ then $\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right] = T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} x+y \\ x-y \\ x \end{array}\right]$ and this leads to $\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$. Hence, $ker(T) = \left\{\left[\begin{array}{c} 0 \\ 0 \end{array}\right]\right\}$ so that T is one-to-one.
To show that T is not onto, take the vector $\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] \in \mathbb{R}^3$. Suppose that $\left[\begin{array}{c} x \\ y \end{array}\right] \in \mathbb{R}^2$ is such that $T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$. This leads to $x = 1$ and $x = 0$ which is impossible. Thus, T is not onto $\blacksquare$

Example 24.7
Let $T : V \to W$ be a one-one linear transformation. Show that if $\{v_1, v_2, \ldots, v_n\}$ is a basis for V then $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for $R(T)$.

Solution.
The fact that $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is linearly independent follows from Theorem 24.3. It remains to show that $R(T) = span\{T(v_1), T(v_2), \cdots, T(v_n)\}$. Indeed, let $w \in R(T)$. Then there exists $v \in V$ such that $T(v) = w$. Since $\{v_1, v_2, \cdots, v_n\}$ is a basis of V, v can be written uniquely in the form

$v = \alpha v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$. Hence, $w = T(v) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n)$. That is, $w \in span\{T(v_1), T(v_2), \cdots, T(v_n)\}$. We conclude that $\{T(v_1), T(v_2), \cdots, T(v_n)\}$ is a basis of $R(T)$ ■

The following important result is called the **dimension theorem**.

Theorem 24.4
If $T : V \to W$ is a linear transformation with $dim(V) = n$, then

$$nullity(T) + rank(T) = n.$$

Theorem 24.5
If W is a subspace of a finite dimensional vector space V and $dim(W) = dim(V)$ then $W = V$.

We have seen that a linear transformation $T : V \to W$ can be one-to-one and onto, one-to-one but not onto, and onto but not one-to-one. The foregoing theorem shows that each of these properties implies the other if the vector spaces V and W have the same dimension.

Theorem 24.6
Let $T : V \to W$ be a linear transformation such that $dim(V) = dim(W) = n$. Then
(a) if T is one - one, then T is onto;
(b) if T is onto, then T is one-one.

Proof.
(a) If T is one-one then $ker(T) = \{0\}$. Thus, $dim(ker(T)) = 0$. By Theorem 24.4 we have $dim(R(T)) = n$. Hence, $R(T) = W$. That is, T is onto.
(b) If T is onto then $dim(R(T)) = n$. By Theorem 24.4, $dim(ker(T)) = 0$. Hence, $ker(T) = \{0\}$, i.e. T is one-one ■

A linear transformation $T : V \to W$ is said to be **invertible** if and only if there exists a unique function $T^{-1} : W \to V$ such that $T \circ T^{-1} = id_W$ and $T^{-1} \circ T = id_V$.

Theorem 24.7
Let $T : V \to W$ be an invertible linear transformation. Then
(a) T^{-1} is linear.
(b) $(T^{-1})^{-1} = T$.

Proof.
(a) Suppose $T^{-1}(w_1) = v_1, T^{-1}(w_2) = v_2$ and $\alpha \in \mathbb{R}$. Then $\alpha w_1 + w_2 = \alpha T(v_1) + T(v_2) = T(\alpha v_1 + v_2$. That is, $T^{-1}(\alpha w_1 + w_2) = \alpha v_1 + v_2 = \alpha T^{-1}(w_1) + T^{-1}(w_2)$.
(b) Follows from the definition of invertible functions ■

What types of linear transformations are invertible?

Theorem 24.8
A linear transformation $T : V \to W$ is invertible if and only if $ker(T) = \{0\}$ and $R(T) = W$.

Example 24.8
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x) = Ax$ where A is the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

(a) Prove that T is invertible.
(b) What is $T^{-1}(x)$?

Solution.
(a) We must show that T is one-to-one and onto. Let $x = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \in ker(T)$. Then $Tx = Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since $|A| = -1 \neq 0$, A is invertible and therefore $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Hence, $ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. Now since A is invertible the system $Ax = b$ is always solvable. This shows that $R(T) = \mathbb{R}^3$. Hence, by the above theorem, T is invertible.
(b) $T^{-1}x = A^{-1}x$ ■

Practice Problems

Problem 24.1
Let $T : M_{mn} \to M_{mn}$ be given by $T(X) = AX$ for all $X \in M_{mn}$, where A is an $m \times m$ invertible matrix. Show that T is both one-one and onto.

Problem 24.2
Let $T : V \longrightarrow W$ be a linear transformation. Show that if the vectors

$$T(v_1), T(v_2), \cdots, T(v_n)$$

are linearly independent then the vectors $v_1, v_2, \cdots, v_n$ are also linearly independent.

Problem 24.3
Show that the projection transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$ is not one-one.

Problem 24.4
Let M_{nn} be the vector space of all $n \times n$ matrices. Let $T : M_{nn} \to M_{nn}$ be given by $T(A) = A - A^T$.
(a) Show that T is linear.
(b) Find $ker(T)$ and $R(T)$.

Problem 24.5
Let $T : V \to W$. Prove that T is one-one if and only if $dim(R(T)) = dim(V)$.

Problem 24.6
Show that the linear transformation $T : M_{nn} \to M_{nn}$ given by $T(A) = A^T$ is an isomorphism.

Problem 24.7
Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ x + 2y \\ y \end{bmatrix}.$$

Show that T is one-to-one.

Problem 24.8
Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find a basis for $Ker(T)$.

Problem 24.9
Consider the linear transformation $T : M_{22} \to M_{22}$ defined by $T(X) = AX - XA$. Find the rank and nullity of T.

Problem 24.10
Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x \end{bmatrix}.$$

Find $ker(T)$ and $R(T)$.

25. Matrix Representation of a Linear Transformation

In this section we shall see the relation between linear transformation, basis and matrices.
Let $S = \{v_1, v_2, \cdots, v_n\}$ be an ordered basis of a vector space V. Then for any vector $v \in V$ there are unique scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

The **coordinate vector of** v **relative to** S is defined by

$$[v]_S = \left[\begin{array}{c} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_n \end{array}\right]$$

Example 25.1
Let $v_1 = (1,0,0), v_2 = (1,1,0)$, and $v_3 = (1,1,1)$. The set $S = \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$. Find the coordinate vector $v = (x, y, z)$ relative to S.

Solution.
We want to find scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$(x, y, z) = \alpha_1(1,0,0) + \alpha_2(1,1,0) + \alpha_3(1,1,1).$$

This leads to the system

$$\left\{\begin{array}{ccccccc} \alpha_1 & + & \alpha_2 & + & \alpha_3 & = & x \\ & & \alpha_2 & + & \alpha_3 & = & y \\ & & & & \alpha_3 & = & z \end{array}\right.$$

Solving this system we find $\alpha_1 = x - y, \alpha_2 = y - z, \alpha_3 = z$. Thus, the coordinate vector v with respect to S is

$$[v]_S = \left[\begin{array}{c} x-y \\ y-z \\ z \end{array}\right] \blacksquare$$

Theorem 25.1
Let V and W be two vector spaces such that $dim(V) = n$ and $dim(W) = m$. Let $T : V \to W$ be a linear transformation. Let $S = \{v_1, v_2, \cdots, v_n\}$ and $S' = \{u_1, u_2, \cdots, u_m\}$ be ordered bases for V and W respectively. Then there is a unique $m \times n$ matrix A such that $T(x) = Ax$. That is,

$$[T(v)]_{S'} = [T]_S^{S'}[v]_S.$$

The $j-$th column of $[T]_S^{S'}$ is the coordinate vector of $T(v_j)$ with respect to the basis S'.

The matrix $[T]_S^{S'}$ is called the **matrix representation of T relative to the ordered bases S and S'.**

Example 25.2
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the formula

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ 2x - y \end{bmatrix}.$$

Let $S = \{\mathbf{e_1}, \mathbf{e_2}\}$ be the standard basis of $\mathbb{R}^2$. Find the matrix representation of T relative to S.

Solution.
We have the following computation

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Thus, the matrix representation of T with respect to S is

$$[T]_S^{S'} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \blacksquare$$

Example 25.3
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by the formula

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ x + z \\ x - z \end{bmatrix}.$$

Let and $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ and $S' = \{(1,0,0), (1,1,0), (1,1,1)\}$. Find the matrix representation of T relative to S and S'.

Solution.
We have the following computation

$$\begin{aligned}
[T(1,0,0)]_{S'} =& T\left(\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]\right) \\
=& \left[\begin{array}{c} 3 \\ 1 \\ 1 \end{array}\right] \\
=& 2(1,0,0) + 0(1,1,0) + 1(1,1,1) \\
[T(0,1,0)]_{S'} =& T\left(\left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right]\right) = \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right] \\
=& 1(1,0,0) + 0(1,1,0) + 0(1,1,1) \\
[T(0,0,1)]_{S'} =& T\left(\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]\right) = \left[\begin{array}{c} 0 \\ 1 \\ -1 \end{array}\right] \\
=& -1(1,0,0) + 2(1,1,0) - 1(1,1,1).
\end{aligned}$$

Thus, the matrix representation of T relative to S and S' is

$$[T]_S^{S'} = \left[\begin{array}{ccc} 2 & 1 & -1 \\ 0 & 0 & 2 \\ 1 & 0 & -1 \end{array}\right] \blacksquare$$

Example 25.4
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the formula

$$T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} x+2y \\ 2x-y \end{array}\right].$$

Let $S = \{(1,0),(0,1)\}$ and $S' = \{(-1,2),(2,0)\}$. Find the matrix representation of T relative to S and S'.

Solution.
We have

$$\begin{aligned}
T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) &= \begin{bmatrix}1\\2\end{bmatrix}\\
&= \begin{bmatrix}-1\\2\end{bmatrix} + \begin{bmatrix}2\\0\end{bmatrix}\\
T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) &= \begin{bmatrix}2\\-1\end{bmatrix}\\
&= -\frac{1}{2}\begin{bmatrix}-1\\2\end{bmatrix} + \frac{3}{4}\begin{bmatrix}2\\0\end{bmatrix}
\end{aligned}$$

Thus,

$$[T]_S^{S'} = \begin{bmatrix}1 & -\frac{1}{2}\\ 1 & \frac{3}{4}\end{bmatrix} \blacksquare$$

Matrices of Composition of Linear Transformations
Let V, W, and Z be finite-dimensional vector spaces with ordered bases S, S', and S'' respectively. Let $T : V \to W$ and $U : W \to Z$ be linear transformations. Then

$$[UT]_S^{S''} = [U]_{S'}^{S''}[T]_S^{S'}.$$

Example 25.5
Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}^4$ be defined by $T(x, y, z) = (x - y + 2z, 2x + y - 4z)$ and $U(x, y) = (2x + 3y, 5x, 4y, 3x - y)$.
(a) Find the matrix representation of UT relative to the standard bases.
(b) Find a formula for $(UT)(x, y, z)$.

Solution.
Let

$$\begin{aligned}
S =&\{(1,0,0),(0,1,0),(0,0,1)\},\\
S' =&\{(1,0),(0,1)\},\\
S'' =&\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}.
\end{aligned}$$

We have

$$\begin{aligned}
T(1,0,0) =&(1,2)\\
T(0,1,0) =&(-1,1)\\
T(0,0,1) =&(2,-4)
\end{aligned}$$

Thus,

$$[T]_S^{S'} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -4 \end{bmatrix}.$$

Likewise,

$$\begin{aligned} U(1,0) =&(2,5,0,3) \\ U(0,1) =&(3,0,4,-1) \end{aligned}$$

Thus,

$$[U]_{S'}^{S''} = \begin{bmatrix} 2 & 3 \\ 5 & 0 \\ 0 & 4 \\ 3 & -1 \end{bmatrix}.$$

Finally,

$$[US]_S^{S''} = \begin{bmatrix} 2 & 3 \\ 5 & 0 \\ 0 & -4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -4 \end{bmatrix}$$

(b) We have

$$(UT)(x,y,z) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8x+y-8z \\ 5x-5y+10z \\ 8x+4y-16z \\ x-4y+10z \end{bmatrix} \blacksquare$$

Matrices of Inverse Linear Transformations

Let S be an ordered basis of a vector space V and S' and ordered basis of a vector space W. Let $T : V \to W$ be an invertible linear transformation. Then T^{-1} is a linear transformation from W to V and $[T^{-1}]_{S'}^S = ([T]_S^{S'})^{-1}$.

Example 25.6

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y+3z \\ x+y+2z \\ y+2z \end{bmatrix}.$$

(a) Prove that T is invertible.
(b) Find matrix representation of T relative to the standard basis of $\mathbb{R}^3$.

Solution.
(a) Note that $Tx = Ax$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

We must show that T is one-to-one and onto. Let $x = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \in ker(T)$. Then $Tx = Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since $|A| = -1 \neq 0$, A is invertible and therefore $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Hence, $ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$. Now since A is invertible the system $Ax = b$ is always solvable. This shows that $R(T) = \mathbb{R}^3$.
(b) We have

$$[T]_S = A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

So that

$$[T^{-1}]_S = A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix} \blacksquare$$

Practice Problems

Problem 25.1
Let $T : P_2 \rightarrow P_1$ be the linear transformation $Tp = p'$. Consider the standard ordered bases $S = \{1, x, x^2\}$ and $S' = \{1, x\}$. Find the matrix representation of T with respect to the basis S and S'.

Problem 25.2
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} x \\ -y \end{array}\right]$$

Find the matrix representation of T with respect to the standard basis S of $\mathbb{R}^2$.

Problem 25.3
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$T\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} x \\ -y \end{array}\right)$$

Let

$$S' = \{\left(\begin{array}{c} 1 \\ 1 \end{array}\right), \left(\begin{array}{c} -1 \\ 1 \end{array}\right)\}$$

and S the standard basis of $\mathbb{R}^2$. Find the matrix representation of T with repspect to the bases S and S'.

Problem 25.4
Let V be the vector space of continuous functions on $\mathbb{R}$ with the ordered basis $S = \{\sin t, \cos t\}$. Find the matrix representation of the linear transformation $T : V \rightarrow V$ defined by $T(f) = f'$ with respect to S.

Problem 25.5
Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation whose matrix representation with the respect to the standard basis of $\mathbb{R}^3$ is given by

$$A = \left[\begin{array}{ccc} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

Find

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

.

Problem 25.6
Consider the linear transformation $T : P_4(x) \to P_4(x)$ defined by $T(p) = p'' + 3p'$, where $P_4(x)$ is the vector space of polynomials of degree 4. Find the matrix representation of T relative to the basis $S = \{1, x, x^2, x^3, x^4\}$.

Problem 25.7
Consider the linear transformations $S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ 3x - y \end{bmatrix} \text{ and } T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 4y \\ -5x + 7y \end{bmatrix}$$

(a) Find a formula for the composition TS.
(b) Find the matrix representation of TS relative to the standard basis S of $\mathbb{R}^2$.

Problem 25.8
Consider the linear transformation $T : P_2(x) \to P_1(x)$ defined by $T(p) = p'$. Let $S = \{1, x, x^2\}$ be an ordered basis of $P_2(x)$ and $S' = \{1, x\}$ be an ordered basis of $P_1(x)$. Find $[T]_S^{S'}$.

Problem 25.9
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix representation with the respect to the standard basis of $\mathbb{R}^3$ is given by

$$[T]_S = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Find

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

.

Problem 25.10

Let V be a vector space with ordered basis $S = \{v_1, v_2, \cdots, v_n\}$. Consider the linear transformation $T : V \to V$ defined by

$$T(v_i) = \lambda_i v_i, \ i = 1, 2, \cdots, n.$$

Find $[T]_S$.

Answer Key

Section 1

1.1
(a) Linear (b) Non-linear (c) Non-linear.

1.2
Substituting these values for $x_1, x_2, x_3,$ and x_4 in each equation.

$$\begin{array}{rclcl} 2x_1 + 5x_2 + 9x_3 + 3x_4 & = & 2(2s + 12t + 13) + 5s + 9(-s - 3t - 3) + 3t & = & -1 \\ x_1 + 2x_2 + 4x_3 & = & (2s + 12t + 13) + 2s + 4(-s - 3t - 3) & = & 1. \end{array}$$

Since both equations are satisfied, it is a solution for all s and t.

1.3
(a) The two lines intersect at the point $(3, 4)$ so the system is consistent.
(b) The two equations represent the same line. Hence, $x_2 = s$ is a parameter. Solving for x_1 we find $x_1 = \frac{5+3t}{2}$.The system is consistent.
(c) The two lines are parallel. So the given system is inconsistent.

1.4
(a) Non-linear because of the term $\ln x_1$.
(b) Linear.

1.5
$x_1 = 1 + 5w - 3t - 2s,\ x_2 = w,\ x_3 = t,\ x_4 = s.$

1.6

(a)

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ x_1 + 2x_2 + 2x_3 = 4 \\ 3x_1 + 6x_2 - 5x_3 = 0. \end{cases}$$

Note that the first two equations imply $2 = 9$ which is impossible.
(b)

$$\begin{cases} x_1 + x_2 + 2x_3 = 9 \\ 2x_1 + 4x_2 - 3x_3 = 1 \\ 3x_1 + 6x_2 - 5x_3 = 0. \end{cases}$$

Solving for x_3 in the third equation, we find $x_3 = \frac{3}{5}x_1 + \frac{6}{5}x_2$. Substituting this into the first two equations we find the system

$$\begin{cases} 11x_1 + 17x_2 = 45 \\ x_1 + 2x_2 = 5. \end{cases}$$

Solving this system by elimination, we find $x_1 = 1$ and $x_2 = 2$. Finally, $x_3 = 3$.
(c)

$$\begin{cases} x_1 + x_2 + 2x_3 = 1 \\ 2x_1 + 2x_2 + 4x_3 = 2 \\ -3x_1 - 3x_2 - 6x_3 = 3. \end{cases}$$

The three equations reduce to the single equation $x_1 + x_2 + 2x_3 = 1$. Letting $x_3 = t,\ x_2 = s$, we find $x_1 = 1 - s - 2t$.

1.7
(a) The system has no solutions if $k \neq 6$ and $h = 9$.
(b) The system has a unique solution if $h = 9$ and any k. In this case, $x_2 = \frac{k-6}{h-9}$ and $x_1 = 2 - \frac{3(k-6)}{h-9}$.
(c) The system has infinitely many solutions if $h = 9$ and $k = 6$ since in this case the two equations reduces to the single equation $x_1 + 3x_2 = 2$. All solutions to this equation are given by the parametric equations $x_1 = 2 - 3t,\ x_2 = t$.

1.8
(a) True (b) False (c) True (d) False.

1.9

$x - 2y = 5.$

1.10
$c = a + b.$

Section 2

2.1
(a) The unique solution is $x_1 = 3,\ x_2 = 4$.
(b) The system is consistent. The general solution is given by the parametric equations: $x_1 = \frac{5+3t}{2},\ x_2 = t$.
(c) System is inconsistent.

2.2
$A = -\frac{1}{9}, B = -\frac{5}{9}$, and $C = \frac{11}{9}$.

2.3
$a = 3, b = -2$, and $c = 4$.

2.4
$x_1 = -\frac{11}{2}\ , x_2 = -6,\ x_3 = -\frac{5}{2}$.

2.5
$x_1 = \frac{1}{9}, x_2 = \frac{10}{9}, x_3 = -\frac{7}{3}$.

2.6
Thus $x_3 = s$ and $x_4 = t$ are parameters. Solving one finds $x_1 = 1 - s + t$ and $x_2 = 2 + s + t,\ x_3 = s,\ x_4 = t$.

2.7
$a = 1, b = 2, c = -1$.

2.8
Solving both systems using backward-substitution technique, we find that both systems have the same solution $x_1 = 1,\ x_2 = 4,\ x_3 = 3$.

2.9
$x_1 = 2,\ x_2 = 1,\ x_3$.

2.10
$x_1 = 2,\ x_2 = -1,\ x_3 = 1$.

Section 3

3.1
$x_1 = 2, x_2 = -1, x_3 = 1.$

3.2
$-5g + 4h + k = 0.$

3.3
$x_1 = -\frac{11}{2}.\ x_2 = -6,\ x_3 = -\frac{5}{2}.$

3.4
$x_1 = \frac{1}{9}, x_2 = \frac{10}{9}, x_3 = -\frac{7}{3}.$

3.5
$x_1 = -s, x_2 = s, x_3 = s,$ and $x_4 = 0.$

3.6
$x_1 = 9s$ and $x_2 = -5s,\ x_3 = s.$

3.7
$x_1 = 3, x_2 = 1, x_3 = 2.$

3.8
Because of the last row the system is inconsistent.

3.9
$x_1 = 8 + 7s,\ x_2 = 2 - 3s,\ x_3 = -5 - s,\ x_4 = s.$

3.10

$$\left[\begin{array}{cccc} -1 & -1 & 0 & 0 \\ 0 & -4 & -2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

3.11
$x_1 = 4 - 3t, x_2 = 5 + 2t, x_3 = t, x_4 = -2.$

Section 4

4.1

$$\begin{bmatrix} 1 & -2 & 3 & 1 & -3 \\ 0 & 3 & -3 & -3 & 6 \end{bmatrix}$$

4.2

$$\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -7 & 9 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

4.3

$$\begin{bmatrix} 1 & -1 & -3 & 8 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & -\frac{5}{2} \end{bmatrix}$$

4.4

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{7}{3} \end{bmatrix}$$

4.5
(a) No, because the matrix fails condition 1 of the definition. Rows of zeros must be at the bottom of the matrix.
(b) No, because the matrix fails condition 2 of the definition. Leading entry in row 2 must be 1 and not 2.
(c) Yes. The given matrix satisfies conditions 1 - 4.

4.6

$$\begin{bmatrix} 1 & -3 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.7

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4.8

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.9

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & .5 & -.25 \\ 0 & 0 & 1 & 1.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

4.10

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

4.11
(a) 3 (b) 2.

Section 5

5.1
$x_1 = 3, x_2 = 1, x_3 = 2.$

5.2
$x_1 = 8 + 7s, x_2 = 2 - 3s,$ and $x_3 = -5 - s.$

5.3
The system is inconsistent.

5.4
$x_1 = 4 - 3t, x_2 = 5 + 2t, x_3 = t, x_4 = -2.$

5.5
$x_1 = \frac{1}{9}, x_2 = \frac{10}{9}, x_3 = -\frac{7}{3}.$

5.6
$x_1 = -\frac{11}{2},\ x_2 = -6,\ x_3 = -\frac{5}{2}.$ **5.7**
$x_1 = 1,\ x_2 = -2,\ x_3 = 1,\ x_4 = 3.$

5.8
$x_1 = 2,\ x_2 = 1,\ x_3 = -1.$

5.9
$x_1 = 2 - 2t - 3s,\ x_2 = t,\ x_3 = 2 + s,\ x_4 = s,\ x_5 = -2.$

5.10
$x_1 = 4 - 2s - 3t,\ x_2 = s,\ x_3 = -1,\ x_4 = 0,\ x_5 = t.$

Section 6

6.1
$x_1 = 9s,\ x_2 = -5s,\ x_3 = s.$

6.2
$x_1 = -s,\ x_2 = s,\ x_3 = s,\ x_4 = 0.$

6.3
$x_1 = x_2 = x_3 = 0.$

6.4
Infinitely many solutions: $x_1 = -8t,\ x_2 = 10t,\ x_3 = t.$

6.5
$x_1 = -s + 3t,\ x_2 = s,\ x_3 = t.$

6.6
$x_1 = -\frac{7}{3}t,\ x_2 = -\frac{2}{3}t,\ x_3 = -\frac{13}{3}t,\ x_4 = t.$

6.7
$x_1 = 8s + 7t,\ x_2 = -4s - 3t,\ x_3 = s,\ x_4 = t.$

Section 7

7.1

$$\begin{bmatrix} 4 & -1 \\ -1 & -6 \end{bmatrix}$$

7.2

$w = -1, x = -3, y = 0$, and $z = 5$.

7.3

$s = 0$ and $t = 3$.

7.4

We have

$$\begin{array}{rl} a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & = \\ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} & = \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A & \end{array}$$

7.5

A simple arithmetic yields the matrix

$$rA + sB + tC = \begin{bmatrix} r + 3t & r + s & -r + 2s + t \end{bmatrix}$$

The condition $rA + sB + tC = \mathbf{0}$ yields the system

$$\left\{ \begin{array}{rcrcrcl} r & & & + & 3t & = & 0 \\ r & + & s & & & = & 0 \\ -r & + & 2s & + & t & = & 0 \end{array} \right.$$

The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \end{bmatrix}$$

Step 1: $r_2 \leftarrow r_2 - r_1$ and $r_3 \leftarrow r_3 + r_1$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & 4 & 0 \end{bmatrix}$$

Step 2: $r_3 \leftarrow r_3 - 2r_2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 10 & 0 \end{bmatrix}$$

Solving the corresponding system we find $r = s = t = 0$

7.6

$$\begin{bmatrix} 9 & 5 & 1 \\ -4 & 7 & 6 \end{bmatrix}$$

7.7
The transpose of A is equal to A.

7.8
$A^T = 0A$ so the matrix is skew-symmetric.

7.9
$4tr(7A) = 0$.

7.10
$tr(A^T - 2B) = 8$.

Section 8

8.1

$$\left\{\begin{array}{rcrcrcr} 2x_1 & - & x_2 & & & = & -1 \\ -3x_1 & + & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 3 \end{array}\right.$$

8.2
(a) If A is the coefficient matrix and B is the augmented matrix then

$$A = \left[\begin{array}{rrrr} 2 & 3 & -4 & 1 \\ -2 & 0 & 1 & 0 \\ 3 & 2 & 0 & -4 \end{array}\right], B = \left[\begin{array}{rrrrr} 2 & 3 & -4 & 1 & 5 \\ -2 & 0 & 1 & 0 & 7 \\ 3 & 2 & 0 & -4 & 3 \end{array}\right]$$

(b) The given system can be written in matrix form as follows

$$\left[\begin{array}{rrrr} 2 & 3 & -4 & 1 \\ -2 & 0 & 1 & 0 \\ 3 & 2 & 0 & -4 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] = \left[\begin{array}{r} 0 \\ 8 \\ -9 \end{array}\right]$$

8.3
AA^T is always defined.

8.4
(a) Easy calculation shows that $A^2 = A$.
(b) Suppose that $A^2 = A$ then $(I_n - A)^2 = I_n - 2A + A^2 = I_n - 2A + A = I_n - A$.

8.5
We have

$$(AB)^2 = \left[\begin{array}{cc} 100 & -432 \\ 0 & 289 \end{array}\right]$$

and

$$A^2B^2 = \left[\begin{array}{cc} 160 & -460 \\ -5 & 195 \end{array}\right]$$

8.6
$AB = BA$ if and only if $(AB)^T = (BA)^T$ if and only if $B^TA^T = A^TB^T$.

8.7

AB is symmetric if and only if $(AB)^T = AB$ if and only if $B^T A^T = AB$ if and only if $AB = BA$.

8.8

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

8.9
$k = -1$.

8.10

$$\left\{ \begin{array}{rcrcrcr} 3x_1 & - & x_2 & + & 2x_3 & = & 2 \\ 4x_1 & + & 3x_2 & + & 7x_3 & = & -1 \\ -2x_1 & + & x_2 & + & 5x_3 & = & 4. \end{array} \right.$$

Section 9

9.1
(a)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

9.2
If B is a 3×3 matrix such that $BA = I_3$ then

$$b_{31}(0) + b_{32}(0) + b_{33}(0) = 0$$

But this is equal to the $(3,3)$ entry of I_3 which is 1. A contradiction.

9.3
$\begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$

9.4
$\begin{bmatrix} \frac{5}{13} & \frac{1}{13} \\ -\frac{3}{13} & \frac{2}{13} \end{bmatrix}.$

9.5
If A is invertible then $B = I_nB = (A^{-1}A)B = A^{-1}(AB) = A^{-1}\mathbf{0} = \mathbf{0}$.

9.6
$A^{-1} = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}.$

9.7
$A = \begin{bmatrix} -\frac{9}{13} & \frac{1}{13} \\ \frac{2}{13} & -\frac{6}{13} \end{bmatrix}.$

9.8
$(5A^T)^{-1} = -\frac{1}{25}\begin{bmatrix} 10 & -25 \\ 5 & -15 \end{bmatrix}.$

9.9
We have

$$A(A - 3I_n) = I_n \text{ and } (A - 3I_n)A = I_n.$$

Hence, A is invertible with $A^{-1} = A - 3I_n$.

9.10
B^{-1}.

Section 10

10.1
(a) No. This matrix is obtained by performing two operations: $r_2 \leftrightarrow r_3$ and $r_1 \leftarrow r_1 + r_3$.
(b) Yes: $r_2 \leftarrow r_2 - 5r_1$.
(c) Yes: $r_2 \leftarrow r_2 + 9r_3$.
(d) No: $r_1 \leftarrow 2r_1$ and $r_1 \leftarrow r_1 + 2r_4$.

10.2
(a)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

10.3
(a) $r_1 \leftrightarrow r_3$, $E^{-1} = E$.
(b) $r_2 \leftarrow r_2 - 2r_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $r_3 \leftarrow 5r_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}.$$

10.4
(a)

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(b) $E_2 = E_1$.
(c)

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

(d)

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

10.5

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

10.6
$r_2 \leftarrow \frac{1}{2}r_2,\ r_1 \leftarrow -r_2 + r_1,\ r_2 \leftrightarrow r_3.$

10.7
$$\begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & a \end{bmatrix}.$$

10.8
$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix},\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix}.$$

10.9
(a) $E_1^{-1} = E_1$.

(b)

$$E_2^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c)

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & -0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10.10

$$\begin{bmatrix} 0 & 5 & -3 \\ -4 & 3 & 0 \\ 3 & 0 & 2 \end{bmatrix}.$$

Section 11

11.1
The matrix is singular.

11.2
$a = -1$ or $a = 3$.

11.3

$$A^{-1} = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}.$$

11.4
Let A be an invertible and symmetric $n \times n$ matrix. Then $(A^{-1})^T = (A^T)^{-1} = A^{-1}$. That is, A^{-1} is symmetric.

11.5
According to Example 9.5(a), we have

$$D^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

11.6
Suppose first that A is nonsingular. Then by Theorem 11.1, A is row equivalent to I_n. That is, there exist elementary matrices $E_1, E_2, \cdots, E_k$ such that $I_n = E_k E_{k-1} \cdots E_1 A$. Then $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$. But each E_i^{-1} is an elementary matrix by Theorem 10.2.
Conversely, suppose that $A = E_1 E_2 \cdots E_k$. Then $(E_1 E_2 \cdots E_k)^{-1} A = I_n$. That is, A is nonsingular.

11.7
Suppose that $A \sim B$. Then there exist elementary matrices $E_1, E_2, \cdots, E_k$ such that $B = E_k E_{k-1} \cdots E_1 A$. Let $P = E_k E_{k-1} \cdots E_1$. Then by Theorem 10.2 and Theorem 9.2 (a), P is nonsingular.
Conversely, suppose that $B = PA$, for some nonsingular matrix P. By Theorem 11.1, P is row equivalent to I_n. That is, $I_n = E_k E_{k-1} \cdots E_1 P$. Thus,

$B = E_1^{-1}E_2^{-1}\cdots E_k^{-1}A$ and this implies that A is row equivalent to B.

11.8
Suppose that A is row equivalent to B. Then by the previous exercise, $B = PA$, with P nonsingular. If A is nonsingular then by Theorem 9.2 (a), B is nonsingular. Conversely, if B is nonsingular then $A = P^{-1}B$ is nonsingular.

11.9
$$A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 \\ -\frac{a_{21}}{a_{11}a_{22}} & \frac{1}{a_{22}} \end{bmatrix}.$$

11.10
Since $Ax = \mathbf{0}$ has only the trivial solution, A is invertible. By induction on k and Theorem 9.2(a), A^k is invertible and consequently the system $A^kx = \mathbf{0}$ has only the trivial solution by Theorem 11.1.

11.11
Since A is invertible, by Theorem 11.1, A is row equivalent to I_n. That is, there exist elementary matrices $E_1, E_2, \cdots, E_k$ such that $I_n = E_kE_{k-1}\cdots E_1A$. Similarly, there exist elementary matrices $F_1, F_2, \cdots, F_l$ such that $I_n = F_lF_{l-1}\cdots F_1B$. Hence, $A = E_1^{-1}E_2^{-1}\cdots E_k^{-1}F_lF_{l-1}\cdots F_1B$. That is, A is row equivalent to B.

Section 12

12.1
(a) $|A| = 22$ (b) $|A| = 0$.

12.2
$t = 0, t = 1$, or $t = 4$.

12.3
$x_1 = \frac{3-\sqrt{33}}{4}$ and $x_2 = \frac{3+\sqrt{33}}{4}$.

12.4
$|A| = 0$.

12.5
$M_{23} = -96,\ C_{23} = 96$.

12.6
$\lambda = \pm 1$.

12.7
(a) -123 (b) -123.

12.8
-240

12.9
$|A| = 6$.

12.10
$|A| = 1$.

12.11
(a) $\lambda = 3$ or $\lambda = 2$ (b) $\lambda = 2$ or $\lambda = 6$.

Section 13

13.1
$|A| = -4$.

13.2
(a)

$$\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = -6$$

(b)

$$\begin{vmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{vmatrix} = 72$$

(c)

$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

(d)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 18$$

13.3
The determinant is 0 since the first and the fifth rows are proportional.

13.4
$|A| = \frac{3}{4}$.

13.5
The determinant is -5.

13.6
The determinant is -1.

13.7
The determinant is 1.

13.8
The determinant is 6.

13.9
$(b-c)(c-a)(a-b)$.

13.10
The determinant is 0.

Section 14

14.1
The proof is by induction on $n \geq 1$. The equality is valid for $n = 1$. Suppose that it is valid up to n. Then $|A^{n+1}| = |A^n A| = |A^n||A| = |A|^n|A| = |A|^{n+1}$.

14.2
Since A is skew-symmetric, $A^T = -A$. Taking the determinant of both sides we find $|A| = |A^T| = |-A| = (-1)^n|A| = -|A|$ since n is odd. Thus, $2|A| = 0$ and therefore $|A| = 0$.

14.3
Taking the determinant of both sides of the equality $A^T A = I_n$ to obtain $|A^T||A| = 1$ or $|A|^2 = 1$ since $|A^T| = |A|$. It follows that $|A| = \pm 1$.

14.4
Taking the determinant of both sides to obtain $|A^2| = |A|$ or $|A|(|A|-1) = 0$. Hence, either A is singular or $|A| = 1$.

14.5
The coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

has determinant $|A| = 0$. By Theorem 14.2, the system has a nontrivial solution.

14.6
Finding the determinant we get $|A| = 2(c+2)(c-3)$. The determinant is 0 if $c = -2$ or $c = 3$.

14.7
$|A^3 B^{-1} A^T B^2| = |A|^3|B|^{-1}|A||B|^2 = |A|^4|B| = 80$.

14.8
We have $|AB| = |A||B| = |B||A| = |BA|$.

14.9

We have $|A + B^T| = |(A + B^T)^T| = |A^T + B|$.

14.10

Let $A = (a_{ij})$ be a triangular matrix. By Theorem 14.2, A is nonsingular if and only if $|A| \neq 0$ and this is equivalent to $a_{11}a_{22}\cdots a_{nn} \neq 0$.

Section 15

15.1

(a) $adj(A) = \begin{bmatrix} -18 & 17 & -6 \\ -6 & -10 & -2 \\ -10 & -1 & 28 \end{bmatrix}$. (b) $|A| = 94$.

15.2

Suppose first that A is invertible. Then $adj(A) = A^{-1}|A|$ so that $|adj(A)| = ||A|A^{-1}| = |A|^n|A^{-1}| = \frac{|A|^n}{|A|} = |A|^{n-1}$. If A is singular then $adj(A)$ is singular. To see this, suppose there exists a square matrix B such that $Badj(A) = adj(A)B = I_n$. Then $A = AI_n = A(adj(A)B) = (Aadj(A))B = 0$ and this leads to $adj(A) = 0$ a contradiction to the fact that $adj(A)$ is non-singular. Thus, $adj(A)$ is singular and consequently $|adj(A)| = 0 = |A|^{n-1}$.

15.3

$$adj(A) = |A|A^{-1} = \begin{bmatrix} -\frac{1}{7} & 0 & -\frac{1}{21} \\ 0 & -\frac{2}{21} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{21} & \frac{1}{21} \end{bmatrix}.$$

15.4

$|A^{-1} + adj(A)| = \frac{3^n}{2}$.

15.5

The equality is valid for $\alpha = 0$. So suppose that $\alpha \neq 0$. Then $adj(\alpha A) = |\alpha A|(\alpha A)^{-1} = (\alpha)^n|A|\frac{1}{\alpha}A^{-1} = (\alpha)^{n-1}|A|A^{-1} = (\alpha)^{n-1}adj(A)$.

15.6

(a) $|A| = 1(21-20) - 2(14-4) + 3(10-3) = 2$.

(b) The matrix of cofactors of A is

$$\begin{bmatrix} 1 & -10 & 7 \\ 1 & 4 & -3 \\ -1 & 2 & -1 \end{bmatrix}$$

The adjoint is the transpose of this cofactors matrix

$$adj(A) = \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & 2 \\ 7 & -3 & -1 \end{bmatrix}$$

(c)

$$A^{-1} = \frac{adj(A)}{|A|} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -5 & 2 & 1 \\ \frac{7}{2} & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \blacksquare$$

15.7

Suppose that $A^T = A$. Then $(adj(A))^T = (|A|A^{-1})^T = |A|(A^{-1})^T = |A|(A^T)^{-1} = |A|A^{-1} = adj(A)$.

15.8

Suppose that $A = (a_{ij})$ is a lower triangular invertible matrix. Then $a_{ij} = 0$ if $i < j$. Thus, $C_{ij} = 0$ if $i > j$ since in this case C_{ij} is the determinant of a lower triangular matrix with at least one zero on the diagonal. Hence, $adj(A)$ is lower triangular.

15.9

Suppose that A is a lower trinagular invertible matrix. Then $adj(A)$ is also a lower triangular matrix. Hence, $A^{-1} = \frac{adj(A)}{|A|}$ is a lower triangular matrix.

15.10

(a) If A has integer entries then $adj(A)$ has integer entries. If $|A| = 1$ then $A^{-1} = adj(A)$ has integer entries.
(b) Since $|A| = 1$, A is invertible and $x = A^{-1}b$. By (a), A^{-1} has integer entries. Since b has integer entries, $A^{-1}b$ has integer entries.

Section 16

16.1
$x_1 = \frac{|A_1|}{|A|} = -\frac{10}{11}, x_2 = \frac{|A_2|}{|A|} = \frac{18}{11}, x_3 = \frac{|A_3|}{|A|} = \frac{38}{11}.$

16.2
$x_1 = \frac{|A_1|}{|A|} = -\frac{3}{4}, x_2 = \frac{|A_2|}{|A|} = \frac{83}{8}, x_3 = \frac{|A_3|}{|A|} = \frac{21}{8}.$

16.3
$x_1 = \frac{|A_1|}{|A|} = -1, x_2 = \frac{|A_2|}{|A|} = 3, x_3 = \frac{|A_3|}{|A|} = 2.$

16.4
$x_1 = \frac{|A_1|}{|A|} = \frac{212}{187}, x_2 = \frac{|A_2|}{|A|} = \frac{273}{187}, x_3 = \frac{|A_3|}{|A|} = \frac{107}{187}.$

16.5
$x_1 = \frac{|A_1|}{|A|} = 4, x_2 = \frac{|A_2|}{|A|} = -1, x_3 = \frac{|A_3|}{|A|} = -\frac{1}{3}.$

16.6
$x_1 = \frac{|A_1|}{|A|} = 2, x_2 = \frac{|A_2|}{|A|} = -1, x_3 = \frac{|A_3|}{|A|} = 4.$

16.7
$x_1 = \frac{|A_1|}{|A|} = 2, x_2 = \frac{|A_2|}{|A|} = -1, x_3 = \frac{|A_3|}{|A|} = 3.$

16.8
$x_1 = \frac{|A_1|}{|A|} = 4, x_2 = \frac{|A_2|}{|A|} = 1, x_3 = \frac{|A_3|}{|A|} = -2.$

16.9
$x_1 = \frac{|A_1|}{|A|} = 5, x_2 = \frac{|A_2|}{|A|} = 2, x_3 = \frac{|A_3|}{|A|} = 2.$

16.10
$x_1 = \frac{|A_1|}{|A|} = 1, x_2 = \frac{|A_2|}{|A|} = 4, x_3 = \frac{|A_3|}{|A|} = 3.$

Section 17

17.1
We know from calculus that if f, g are differentiable functions on $[a, b]$ and $\alpha \in \mathbb{R}$ then $\alpha f + g$ is also differentiable on $[a, b]$. Hence, $D([a, b])$ is a subspace of $F([a, b]$.

17.2
Let $x, y \in S$ and $\alpha \in \mathbb{R}$. Then $A(\alpha x + y) = \alpha Ax + Ay = \alpha \times \mathbf{0} + \mathbf{0} = \mathbf{0}$. Thus, $\alpha x + y \in S$ so that S is a subspace of $\mathbb{R}^n$.

17.3
Since $\mathbf{P}$ is a subset of the vector space of all functions defined on $\mathbb{R}$, it suffices to show that $\mathbf{P}$ is a subspace. Indeed, the sum of two polynomials is again a polynomial and the scalar multiplication by a polynomial is also a polynomial.

17.4
The proof is based on the properties of the vector space $\mathbb{R}$.
(a) $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$ where we have used the fact that the addition of real numbers is commutative.
(b) $[(f + g) + h](x) = (f + g)(x) + h(x) = (f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x)) = f(x) + (g + h)(x) = [f + (g + h)](x)$.
(c) Let $\mathbf{0}$ be the zero function. Then for any $f \in F(\mathbb{R})$ we have $(f + 0)(x) = f(x) + \mathbf{0}(x) = f(x) = (\mathbf{0} + f)(x)$.
(d) $[f + (-f)](x) = f(x) + (-f(x)) = f(x) - f(x) = 0 = \mathbf{0}(x)$.
(e) $[\alpha(f + g)](x) = \alpha(f + g)(x) = \alpha f(x) + \alpha g(x) = (\alpha f + \alpha g)(x)$.
(f) $[(\alpha + \beta)f](x) = (\alpha + \beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f + \beta f)(x)$.
(g) $[\alpha(\beta f)](x) = \alpha(\beta f)(x) = (\alpha\beta)f(x) = [(\alpha\beta)f](x)$
$(h)(1f)(x) = 1f(x) = f(x)$.
Thus, $F(\mathbb{R})$ is a vector space.

17.5
Let $x \neq y$. Then $\alpha(\beta(x, y)) = \alpha(\beta y, \beta x) = (\alpha\beta x, \alpha\beta y) \neq (\alpha\beta)(x, y)$. Thus, $\mathbb{R}^2$ with the above operations is not a vector space.

17.6
Let $p, q \in U$ and $\alpha \in \mathbb{R}$. Then $\alpha p + q$ is a polynomial such that $(\alpha p + q)(3) =$

$\alpha p(3) + q(3) = 0$. That is, $\alpha p + q \in U$. This says that U is a subspace of $\mathbf{P}$.

17.7
Let $p(x) = a_0 + a_1x + \cdots + a_nx^n, q(x) = b_0 + b_1x + \cdots + b_nx^n$, and $\alpha \in R$. Then $(\alpha p + q)(x)(\alpha a_0 b_0) + (\alpha a_1 b_1)x + \cdots + (\alpha a_n b_n)x^n \in P_n$. Thus, P_n is a subspace of $\mathbf{P}$.

17.8
$(-1, 0) \in S$ but $-2(-1, 0) = (2, 0) \notin S$ so S is not a vector space.

17.9
Since for any continuous functions f and g and any scalar α the function $\alpha f + g$ is continuous, $C([a, b])$ is a subspace of $F([a, b])$ and hence a vector space.

17.10
Indeed, $\alpha(a, b, a+b) + (a', b', a'+b') = (\alpha(a+a'), \alpha(b+b'), \alpha(a+b+a'+b'))$.

17.11
Using the properties of vector spaces we have $v = v + 0 = v + (u + (-u)) = (v + u) + (-u) = (w + u) + (-u) = w + (u + (-u)) = w + 0 = w$.

17.12
(a) Let $u, v \in H \cap K$ and $\alpha \in R$. Then $u, v \in H$ and $u, v \in K$. Since H and K are subspaces, $\alpha u + v \in H$ and $\alpha u + v \in K$ that is $\alpha u + v \in H \cap K$. This shows that $H \cap K$ is a subspace.
(b) One can easily check that $H = \{(x, 0) : x \in \mathbb{R}\}$ and $K = \{(0, y) : y \in \mathbb{R}\}$ are subspaces of $\mathbb{R}^2$. The vector $(1, 0)$ belongs to H and the vector $(0, 1)$ belongs to K. But $(1, 0) + (0, 1) = (1, 1) \notin H \cup K$. It follows that $H \cup K$ is not a subspace of $\mathbb{R}^2$.
(c) If $H \subset K$ then $H \cup K = K$, a subspace of V. Similarly, if $K \subset H$ then $H \cup K = H$, again a subspace of V.

Section 18

18.1
Let U be a subspace of V containing the vectors $v_1, v_2, \cdots, v_n$. Let $x \in W$. Then $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ for some scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$. Since U is a subspace, $x \in U$. This gives $x \in U$ and consequently $W \subset U$.

18.2
Indeed, $3p_1(x) - p_2(x) + 2p_3(x) = 0$.

18.3
The equation $\vec{u} = \alpha\vec{v_1} + \beta\vec{v_2} + \gamma\vec{v_3}$ gives the system

$$\left\{\begin{array}{ccccccc} 2\alpha & + & \beta & + & 3\gamma & = & -9 \\ \alpha & - & \beta & + & 2\gamma & = & -7 \\ 4\alpha & + & 3\beta & + & 5\gamma & = & -15 \end{array}\right.$$

Solving this system (details omitted) we find $\alpha = -2, \beta = 1$ and $\gamma = -2$.

18.4
(a) Indeed, this follows because the coefficient matrix

$$A = \left[\begin{array}{ccc} 2 & 2 & 2 \\ 0 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

of the system $Ax = b$ is invertible for all $b \in \mathbb{R}^3$ ($|A| = -6$).
(b) This follows from the fact that the coefficient matrix with rows the vectors $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$ is singular.

18.5
Indeed, every $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ can be written as

$$a\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] + b\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] + c\left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right] + d\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]\}.$$

18.6

Suppose that $\alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \alpha_3\vec{v_3} = \vec{0}$. This leads to the system

$$\left\{\begin{array}{rcrcrcl} 2\alpha_1 & + & \alpha_2 & + & 7\alpha_3 & = & 0 \\ -\alpha_1 & + & 2\alpha_2 & - & \alpha_3 & = & 0 \\ & & 5\alpha_2 & + & 5\alpha_3 & = & 0 \\ 3\alpha_1 & - & \alpha_2 & + & 8\alpha_3 & = & 0 \end{array}\right.$$

The augmented matrix of this system is

$$\left[\begin{array}{ccccc} 2 & -1 & 0 & 3 & 0 \\ 1 & 2 & 5 & -1 & 0 \\ 7 & -1 & 5 & 8 & 0 \end{array}\right]$$

The reduction of this matrix to row-echelon form is carried out as follows.

Step 1: $r_1 \leftarrow r_1 - 2r_2$ and $r_3 \leftarrow r_3 - 7r_2$

$$\left[\begin{array}{ccccc} 0 & -5 & -10 & 6 & 0 \\ 1 & 2 & 5 & -1 & 0 \\ 0 & -15 & -30 & 15 & 0 \end{array}\right]$$

Step 2: $r_1 \leftrightarrow r_2$

$$\left[\begin{array}{ccccc} 1 & 2 & 5 & -1 & 0 \\ 0 & -5 & -10 & 6 & 0 \\ 0 & -15 & -30 & 15 & 0 \end{array}\right]$$

Step 2: $r_3 \leftarrow r_3 - 3r_2$

$$\left[\begin{array}{ccccc} 1 & 2 & 5 & -1 & 0 \\ 0 & -5 & -10 & 6 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right]$$

The system has a nontrivial solution so that $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly dependent.

18.7
Suppose that $\alpha(4, -1, 2) + \beta(-4, 10, 2) = (0, 0, 0)$ this leads to the system

$$\left\{\begin{array}{rcrcl} 4\alpha_1 & - & 4\alpha_2 & = & 0 \\ -\alpha_1 & + & 10\alpha_2 & = & 0 \\ 2\alpha_2 & + & 2\alpha_2 & = & 0 \end{array}\right.$$

This system has only the trivial solution so that the given vectors are linearly independent.

18.8
Suppose that $\{u, v\}$ is linearly dependent. Then there exist scalars α and β not both zero such that $\alpha u + \beta v = 0$. If $\alpha \neq 0$ then $u = -\frac{\beta}{\alpha}v$, i.e. u is a scalar multiple of v. A similar argument if $\beta \neq 0$.
Conversely, suppose that $u = \lambda v$ then $1u + (-\lambda)v = 0$. This shows that $\{u, v\}$ is linearly dependent.

18.9
Suppose that $\alpha f(x) + \beta g(x) + \gamma h(x) = 0$ for all $x \in \mathbb{R}$. Then this leads to the system

$$\left(\begin{array}{ccc} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{array}\right)\left(\begin{array}{c} \alpha \\ \beta \\ \gamma \end{array}\right)$$

Thus $\{f(x), g(x), h(x)\}$ is linearly independent if and only if the coefficient matrix of the above system is invertible and this is equivalent to $w(x) \neq 0$.

18.10
Indeed,

$$w(x) = \left|\begin{array}{ccc} e^x & xe^x & x^2e^x \\ e^x & e^x + xe^x & 2xe^x + x^2e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2e^x \end{array}\right| = 2e^x \neq 0.$$

18.11
We have already shown that

$$M_{22} = span\left\{M_1 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right], M_2 = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right], M_3 = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right], M_4\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]\right\}$$

Now, if $\alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3 + \alpha_4 M_4 = \mathbf{0}$ then

$$\left[\begin{array}{cc} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{array}\right] = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$$

and this shows that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Hence, $\{M_1, M_2, M_3, M_4\}$ is a basis for M_{22}.

Section 19

19.1
$\lambda = -3$ and $\lambda = 1$.

19.2
$\lambda = 3$ and $\lambda = -1$.

19.3
$\lambda = 3$ and $\lambda = -1$.

19.4
$\lambda = -8$.

19.5
Let $\mathbf{x}$ be an eigenvector of A corresponding to the nonzero eigenvalue λ. Then $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying both sides of this equality by A^{-1} and then dividing the resulting equality by λ to obtain $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$. That is, $\mathbf{x}$ is an eigenvector of A^{-1} corresponding to the eigenvalue $\frac{1}{\lambda}$.

19.6
Let $\mathbf{x}$ be an eigenvector of A corresponding to the eigenvalue λ. Then $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying both sides by A to obtain $A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. Now, multiplying this equality by A to obtain $A^3\mathbf{x} = \lambda^3\mathbf{x}$. Continuing in this manner, we find $A^m\mathbf{x} = \lambda^m\mathbf{x}$.

19.7
Suppose that $D = P^{-1}AP$. Then $D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}AP^2$. Thus, by induction on k one finds that $D^k = P^{-1}A^kP$.

19.8
The characteristic equation of I_n is $(\lambda - 1)^n = 0$. Hence, $\lambda = 1$ is the only eigenvalue of I_n.

19.9
(a) If λ is an eigenvalue of A then there is a nozero vector x such that $Ax = \lambda x$. By Exercise 19.6, λ^k is an eigenvalue of A^k and $A^kx = \lambda^kx$. But $A^k = \mathbf{0}$ so $\lambda^kx = \mathbf{0}$ and since $x \neq \mathbf{0}$ we must have $\lambda = 0$.

(b) Since $p(\lambda)$ is of degree n and 0 is the only eigenvalue of A, then $p(\lambda) = \lambda^n$.

19.10

Since λ is an eigenvalue of A with corresponding eigenvector x, we have $Ax = \lambda x$. Postmultiply B by P^{-1} to obtain $BP^{-1} = P^{-1}A$. Hence, $BP^{-1}x = P^{-1}Ax = \lambda P^{-1}x$. This says that λ is an eigenvalue of B with corresponding eigenvector $P^{-1}x$.

19.11

The characteristic polynomial is of degree n. The Fundamental Theorem of Algebra asserts that such a polynomial has exactly n roots. A root in this case can be either a complex number or a real number. But if a root is complex then its conjugate is also a root. Since n is odd then there must be at least one real root.

19.12

The characterisitc polynomial of A is

$$p(\lambda) = \left|\begin{array}{cccc} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ a_0 & a_1 & a_2 & \lambda + a_3 \end{array}\right|$$

Expanding this determinant along the first row we find

$$\begin{aligned} p(\lambda) &= \lambda \left|\begin{array}{ccc} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ a_1 & a_2 & \lambda + a_3 \end{array}\right| + \left|\begin{array}{ccc} 0 & -1 & 0 \\ 0 & \lambda & -1 \\ a_0 & a_2 & \lambda + a_3 \end{array}\right| \\ &= \lambda[\lambda(\lambda^2 + a_3\lambda + a_2) + a_1] + a_0 \\ &= \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \end{aligned}$$

Section 20

20.1

$$V^{-3} = \left\{ \left[\begin{array}{c} -2t-s \\ s \\ t \end{array} \right] : s,t \in \mathbb{R} \right\} = span \left\{ \left[\begin{array}{c} -2 \\ 0 \\ 1 \end{array} \right], \left[\begin{array}{c} -1 \\ 1 \\ 0 \end{array} \right] \right\}$$

20.2

$$V^{3} = \left\{ \left[\begin{array}{c} \frac{1}{2}s \\ s \end{array} \right] : s \in \mathbb{R} \right\} = span \left\{ \left[\begin{array}{c} \frac{1}{2} \\ 1 \end{array} \right] \right\}$$

and

$$V^{-1} = \left\{ \left[\begin{array}{c} 0 \\ s \end{array} \right] : s \in \mathbb{R} \right\} = span \left\{ \left[\begin{array}{c} 0 \\ 1 \end{array} \right] \right\}.$$

20.3

$$V^{3} = \left\{ \left(\begin{array}{c} -5s \\ -6s \\ s \end{array} \right) : s \in \mathbb{R} \right\} = span \left\{ \left(\begin{array}{c} -5 \\ -6 \\ 1 \end{array} \right) \right\}$$

and

$$V^{-1} = \left\{ \left(\begin{array}{c} -s \\ 2s \\ s \end{array} \right) : s \in \mathbb{R} \right\} = span \left\{ \left(\begin{array}{c} -1 \\ 2 \\ 1 \end{array} \right) \right\}.$$

20.4

$$V^{-8} = \left\{ \left[\begin{array}{c} -\frac{1}{6}s \\ -\frac{1}{6}s \\ s \end{array} \right] : s \in \mathbb{R} \right\} = span \left\{ \left[\begin{array}{c} -\frac{1}{6} \\ -\frac{1}{6} \\ 1 \end{array} \right] \right\}.$$

20.5

$$V^{1} = \left\{ \left[\begin{array}{c} s \\ s \\ s \end{array} \right] : s \in \mathbb{R} \right\} = span \left\{ \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] \right\}.$$

$$V^{2} = \left\{ \left[\begin{array}{c} \frac{2}{3}s \\ s \\ s \end{array} \right] : s \in \mathbb{R} \right\} = span \left\{ \left[\begin{array}{c} \frac{2}{3} \\ 1 \\ 1 \end{array} \right] \right\}.$$

and

$$V^3 = \left\{ \begin{bmatrix} \frac{1}{4}s \\ \frac{3}{4}s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 1 \end{bmatrix} \right\}.$$

20.6

The eigenspace corresponding to $\lambda = 1$ is

$$V^1 = \left\{ \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

and

$$V^2 = \left\{ \begin{bmatrix} -s \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\} = span \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

20.7

$$V^1 = \left\{ \begin{bmatrix} s \\ -s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

and

$$V^{-2} = \left\{ \begin{bmatrix} -s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

20.8

$$V^1 = \left\{ \begin{bmatrix} -s \\ s \\ -s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$$V^{-1} = \left\{ \begin{bmatrix} s \\ -s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

$$V^2 = \left\{ \begin{bmatrix} -s \\ 0 \\ -s \\ s \end{bmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

and

$$V^{-2} = \left\{ \begin{bmatrix} 0 \\ -s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\} = span \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

20.9
Algebraic multiplicity of $\lambda = 1$ is equal to the geometric multiplicity of 1.

20.10
The matrix is non-defective.

Section 21

21.1
(a) Suppose that $A \sim B$ and let P be an invertible matrix such that $B = P^{-1}AP$. Taking the transpose of both sides we obtain $B^T = (P^T)^{-1}A^TP^T$; that is, $A^T \sim B^T$.
(b) Suppose that A and B are invertible and $B = P^{-1}AP$. Taking the inverse of both sides we obtain $B^{-1} = P^{-1}A^{-1}P$. Hence $A^{-1} \sim B^{-1}$.

21.2
Suppose that A is an $n \times n$ invertible matrix. Then $BA = A^{-1}(AB)A$. That is $AB \sim BA$.

21.3
The eigenvalues of A are $\lambda = 4, \lambda = 2 + \sqrt{3}$ and $\lambda = 2 - \sqrt{3}$. Hence, by Theorem 21.2 A is diagonalizable.

21.4
The eigenspaces of A are

$$V^{-1} = \left\{ \left[\begin{array}{c} -s \\ 2s \\ s \end{array} \right] : s \in \mathbb{R} \right\} = span \left\{ \left[\begin{array}{c} -1 \\ 2 \\ 1 \end{array} \right] \right\}$$

and

$$V^{3} = \left\{ \left[\begin{array}{c} -5s \\ -6s \\ s \end{array} \right] : s \in \mathbb{R} \right\} = span \left\{ \left[\begin{array}{c} -5 \\ -6 \\ 1 \end{array} \right] \right\}$$

Since there are only two eigenvectors, A A is not diagonalizable.

21.5
The characteristic equation of the matrix A is

$$\left| \begin{array}{cc} \lambda - 3 & 0 \\ 0 & \lambda - 3 \end{array} \right| = 0$$

Expanding the determinant and simplifying we obtain

$$(\lambda - 3)^2 = 0.$$

The only eigenvalue of A is $\lambda = 3$. By letting $P = I_n$ and $D = A$ we see that $D = P^{-1}AP$, i.e. A is diagonalizable.

21.6
Suppose that A is diagonalizable. Then there exist matrices P and D such that $D = P^{-1}AP$, with D diagonal. Taking the transpose of both sides to obtain $D = D^T = P^TA^T(P^{-1})T = Q^{-1}A^TQ$ with $Q = (P^{-1})^T = (P^T)^{-1}$. Hence, A^T is diagonalizable. Similar argument for the converse.

21.7
Suppose that $A \sim B$. Then there exists an invertible matrix P such that $B = P^{-1}AP$. Suppose first that A is diagonalizable. Then there exist an invertible matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$. Hence, $B = P^{-1}QDQ^{-1}$ and this implies $D = (P^{-1}Q)^{-1}B(P^{-1}Q)$. That is, B is diagonalizable. For the converse, repeat the same argument using $A = (P^{-1})^{-1}BP^{-1}$.

21.8
Consider the matrices

$$A = \left[\begin{array}{cc} 2 & 1 \\ 0 & -1 \end{array}\right], B = \left[\begin{array}{cc} -1 & 0 \\ 0 & 2 \end{array}\right]$$

The matrix A has the eigenvalues $\lambda = 2$ and $\lambda = -1$ so by Theorem 21.2, A is diagonalizable. Similar argument for the matrix B. Let $C = A + B$ then

$$C = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]$$

This matrix has only one eigenvalue $\lambda = 1$ with corresponding eigenspace (details omitted)

$$V^1 = span\left\{\left[\begin{array}{c} 1 \\ 0 \end{array}\right]\right\}.$$

Hence, there is only one eigenvector of C and by Theorem 21.1, C is not diagonalizable.

21.9

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

21.10

$$P = \begin{bmatrix} -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Section 23

23.1

Given $[x_1, y_1]^T$ and $[x_2, y_2]^T$ is $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$ we find

$$\begin{aligned} T_E\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right] + \left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right) =& T_E\left(\left[\begin{array}{c} x_1 + x_2 \\ y_1 + y_2 \end{array}\right]\right) = \left[\begin{array}{c} y_1 + y_2 \\ x_1 + x_2 \end{array}\right] \\ =& \left[\begin{array}{c} y_1 \\ x_1 \end{array}\right] + \left[\begin{array}{c} y_2 \\ x_2 \end{array}\right] = T_E\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]\right) + T_E\left(\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right) \end{aligned}$$

and

$$\begin{aligned} T_E\left(\alpha\left[\begin{array}{c} x \\ y \end{array}\right]\right) =& T_E\left(\left[\begin{array}{c} \alpha x x \\ \alpha y \end{array}\right]\right) = \left[\begin{array}{c} \alpha y \\ \alpha x \end{array}\right] \\ =& \alpha\left[\begin{array}{c} y \\ x \end{array}\right] = \alpha T_E\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) \end{aligned}$$

Hence, T_E is linear.

23.2

Given $[x_1, y_1]^T$ and $[x_2, y_2]^T$ is $\mathbb{R}^2$ and $\beta \in \mathbb{R}$ we find

$$\begin{aligned} T_F\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right] + \left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right) =& T_F\left(\left[\begin{array}{c} x_1 + x_2 \\ y_1 + y_2 \end{array}\right]\right) \\ =& \left[\begin{array}{c} \alpha(x_1 + x_2) \\ y_1 + y_2 \end{array}\right] = \left[\begin{array}{c} \alpha x_1 \\ y_1 \end{array}\right] + \left[\begin{array}{c} \alpha x_2 \\ y_2 \end{array}\right] \\ =& T_F\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]\right) + T_F\left(\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right) \end{aligned}$$

and

$$\begin{aligned} T_F\left(\beta\left[\begin{array}{c} x \\ y \end{array}\right]\right) =& T_F\left(\left[\begin{array}{c} \beta x \\ \beta y \end{array}\right]\right) \\ =& \left[\begin{array}{c} \beta \alpha x \\ \beta y \end{array}\right] = \beta\left[\begin{array}{c} \alpha x \\ y \end{array}\right] = \beta T_F\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) \end{aligned}$$

Hence, T_F is linear.

23.3
Given $[x_1, y_1]^T$ and $[x_2, y_2]^T$ is $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$ we find

$$\begin{aligned}
T_G\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right] + \left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right) =& T_F\left(\left[\begin{array}{c} x_1 + x_2 \\ y_1 + y_2 \end{array}\right]\right) \\
=& \left[\begin{array}{c} x_1 + x_2 + y_1 + y_2 \\ y_1 + y_2 \end{array}\right] = \left[\begin{array}{c} x_1 + y_1 \\ y_1 \end{array}\right] + \left[\begin{array}{c} x_2 + y_2 \\ y_2 \end{array}\right] \\
=& T_G\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]\right) + T_G\left(\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right)
\end{aligned}$$

and

$$\begin{aligned}
T_G\left(\alpha\left[\begin{array}{c} x \\ y \end{array}\right]\right) =& T_G\left(\left[\begin{array}{c} \alpha x \\ \alpha y \end{array}\right]\right) \\
=& \left[\begin{array}{c} \alpha(x + y) \\ \alpha y \end{array}\right] = \alpha\left[\begin{array}{c} x + y \\ y \end{array}\right] = \alpha T_G\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right)
\end{aligned}$$

Hence, T_G is linear.

23.4
Let $[x_1, y_1]^T, [x_2, y_2]^T \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}
T\left(\alpha\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right] + \left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]\right) =& T\left(\left[\begin{array}{c} \alpha x_1 + x_2 \\ \alpha y_1 + y_2 \end{array}\right]\right) \\
=& \left[\begin{array}{c} \alpha x_1 + x_2 + \alpha y_1 + y_2 \\ \alpha x_1 + x_2 - 2\alpha y_1 - 2y_2 \\ 3\alpha x_1 + 3x_2 \end{array}\right] \\
=& \alpha\left[\begin{array}{c} x_1 + y_1 \\ x_1 - 2y_1 \\ 3x_1 \end{array}\right] + \left[\begin{array}{c} x_2 + y_2 \\ x_2 - 2y_2 \\ 3x_2 \end{array}\right] \\
=& \alpha T\left(\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right]\right) + T\left(\left[\begin{array}{c} x_2 \\ y_2 \end{array}\right]\right)
\end{aligned}$$

Hence, T is a linear transformation.

23.5

(a) Let $p, q \in P_n$ and $\alpha \in \mathbb{R}$ then

$$\begin{aligned} D[\alpha p(x)+q(x)] =&(\alpha p(x)+q(x))' \\ =&\alpha p'(x)+q'(x)=\alpha D[p(x)]+D[q(x)] \end{aligned}$$

Thus, D is a linear transformation.
(b) Let $p, q \in P_n$ and $\alpha \in \mathbb{R}$ then

$$\begin{aligned} I[\alpha p(x)+q(x)] =& \int_0^x (\alpha p(t)+q(t))dt \\ =&\alpha \int_0^x p(t)dt + \int_0^x q(t)dt = \alpha I[p(x)]+I[q(x)] \end{aligned}$$

Hence, I is a linear transformation.

23.6

Suppose that $\left[\begin{array}{c} -1 \\ 1 \\ 0 \end{array}\right] = \alpha \left[\begin{array}{c} 3 \\ -1 \\ 2 \end{array}\right] + \beta \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right]$. This leads to a linear system in the unknowns α and β. Solving this system we find $\alpha = -1$ and $\beta = 2$. Since T is linear, we have

$$T\left(\left[\begin{array}{c} -1 \\ 1 \\ 0 \end{array}\right]\right) = -T\left(\left[\begin{array}{c} 3 \\ -1 \\ 2 \end{array}\right]\right) + 2T\left(\left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right]\right) = -5+4=-1.$$

23.7

Let $[x_1, y_1, z_1]^T \in \mathbb{R}^3, [x_2, y_2, z_2]^T \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} T\left(\alpha\left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right] + \left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right]\right) =&T\left(\left[\begin{array}{c} \alpha x_1 + y_1 \\ \alpha x_2 + y_2 \\ \alpha z_1 + z_2 \end{array}\right]\right) \\ =&\left[\begin{array}{c} \alpha x_1 + x_2 \\ \alpha y_1 + y_2 \end{array}\right] = \alpha\left[\begin{array}{c} x_1 \\ y_1 \end{array}\right] + \left[\begin{array}{c} x_2 \\ y_2 \end{array}\right] \\ =&\alpha T\left(\alpha\left[\begin{array}{c} x_1 \\ y_1 \\ z_1 \end{array}\right]\right) + T\left(\alpha\left[\begin{array}{c} x_2 \\ y_2 \\ z_2 \end{array}\right]\right) \end{aligned}$$

Hence, T is a linear transformation.

23.8
Since $|A+B| \neq |A| + |B|$ in general, the given transformation is not linear.

23.9
Let $u_1, u_1 \in U$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}(T_2 \circ T_1)(\alpha u_1 + u_2) =& T_2(T_1(\alpha u_1 + u_2))\\ =& T_2(\alpha T_1(u_1) + T_1(u_2))\\ =& \alpha T_2(T_1(u_1)) + T_2(T_1(u_2))\\ =& \alpha (T_2 \circ T_1)(u_1) + (T_2 \circ T_1)(u_2).\end{aligned}$$

23.10
Consider the system in the unknowns $T(v)$ and $T(v_1)$

$$\left\{\begin{array}{ccccc} T(v) & - & 3T(v_1) & = & w \\ 2T(v) & - & 2T(v_1) & = & w_1 \end{array}\right.$$

Solving this system to find $T(v) = \frac{1}{5}(3w_1 - w)$ and $T(v_1) = \frac{1}{5}(w_1 - 2w)$.

Section 24

24.1

We first show that T is linear. Indeed, let $X, Y \in M_{mn}$ and $\alpha \in \mathbb{R}$. Then $T(\alpha X+Y) = A(\alpha X+Y) = \alpha AX + AY = \alpha T(X)+T(Y)$. Thus, T is linear. Next, we show that T is one-one. Let $X \in ker(T)$. Then $AX = \mathbf{0}$. Since A is invertible, $X = \mathbf{0}$. This shows that $ker(T) = \{0\}$ and thus T is one-one. Finally, we show that T is onto. Indeed, if $B \in R(T)$ then $T(A^{-1}B) = B$. This shows that T is onto.

24.2

Suppose that $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$. Then $\alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n) = T(0) = 0$. Since the vectors $T(v_1), T(v_2), \cdots, T(v_n)$ are linearly independent, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. This shows that the vectors $v_1, v_2, \cdots, v_n$ are linearly independent.

24.3

Since $\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] \in \mathrm{ker}(T)$, by Theorem 21.2, T is not one-one.

24.4

(a) Let $A, B \in M_{nn}$ and $\alpha \in \mathbb{R}$. Then $T(\alpha A+B) = (\alpha A + B - (\alpha A+B)^T = \alpha(A - A^T) + (B - B^T) = \alpha T(A) + T(B)$. Thus, T is linear.

(b) Let $A \in ker(T)$. Then $T(A) = \mathbf{0}$. That is $A^T = A$. This shows that A is symmetric. Conversely, if A is symmetric then $T(A) = \mathbf{0}$. It hollows that $ker(T) = \{A \in M_{nn} : A \textit{ is symmetric}\}$. Now, if $B \in R(T)$ and A is such that $T(A) = B$ then $A - A^T = B$. But then $A^T - A = B^T$. Hence, $B^T = -B$,i.e. B is skew-symmetric. Conversely, if B is skew-symmetric then $B \in R(T)$ since $T(\frac{1}{2}B) = \frac{1}{2}(B - B^T) = B$. We conclude that $R(T) = \{B \in M_{nn} : B \textit{ is } skew-symmetric\}$.

24.5

Suppose that T is one-one. Then $ker(T) = \{0\}$ and therefore $dim(ker(T)) = 0$. By Theorem 24.4, $dim(R(T)) = dimV$. The converse is similar .

24.6

If $A \in ker(T)$ then $T(A) = \mathbf{0} = A^T$. This implies that $A = \mathbf{0}$ and consequently $ker(T) = \{\mathbf{0}\}$. So T is one-one. Now suppose that $A \in M_{mn}$. Then $T(A^T) = A$ and $A^T \in M_{nn}$. This shows that T is onto. It follows that T is an isomorphism.

24.7
Let $\left[\begin{array}{c} x \\ y \end{array}\right] \in Ker(T)$. Then

$$T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right) = \left[\begin{array}{c} x+y \\ x+2y \\ y \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right].$$

This implies that $x = y = 0$ so that T is one-to-one.

24.8
$$ker(T) = \text{span}\left\{\left[\begin{array}{c} 1 \\ -2 \\ 1 \end{array}\right]\right\}.$$

24.9
Let $X = \left[\begin{array}{cc} x & y \\ z & w \end{array}\right] \in ker(T)$. This leads to the system

$$\left\{\begin{array}{rcrcrcrcl} & & 3y & - & 2x & & & = & 0 \\ 2x & + & 3y & & & - & 3w & = & 0 \\ -3x & & & - & 3z & + & 3w & = & 0 \\ & - & 3y & + & 2z & & & = & 0. \end{array}\right.$$

Solving, we find

$$X = \left[\begin{array}{cc} -z+w & \frac{2}{3}z \\ z & w \end{array}\right] = z\left[\begin{array}{cc} -1 & \frac{2}{3} \\ 1 & 0 \end{array}\right] + w\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right].$$

Hence,

$$ker(T) = \text{span}\left\{\left[\begin{array}{cc} -1 & \frac{2}{3} \\ 1 & 0 \end{array}\right], \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right\}.$$

Thus, $nullity(T) = 2$ and $rank(T) = 4 - 2 = 2$.

24.10
$$kert(T) = R(T) = \left\{\left[\begin{array}{c} 0 \\ a \end{array}\right] : a \in \mathbb{R}\right\}.$$

Section 25

25.1

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

25.2

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

25.3

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

25.4

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

25.5
$$\begin{bmatrix} 9 \\ 5 \\ 5 \end{bmatrix}.$$

25.6

$$[T]_S = \begin{bmatrix} 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 6 & 6 & 0 \\ 0 & 0 & 0 & 9 & 12 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

25.7
(a) $(TS)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 14x \\ 16x - 17y \end{bmatrix}.$
(b)
$$[TS]_{S'} = [T]_{S'}[S]_{S'} = \begin{bmatrix} 2 & 4 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 16 & = 17 \end{bmatrix}.$$

25.8

$$[T]_S^{S'} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

25.9

$$\begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix}.$$

25.10

$$[T]_S = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$