9 Solving Quasi-Linear First Order PDE via the Method of Characteristics

In this section we develop a method for finding the general solution of a quasi-linear first order partial differential equation

\[ a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u). \]  

(9.1)

This method is called the **method of characteristics** or **Lagrange’s method**. This method consists of transforming the PDE to a system of ODEs which can be solved and the found solution is transformed into a solution for the original PDE.

The method of characteristics relies on a geometrical argument. A visualization of a solution is an integral surface with equation \( z = u(x, y) \). An alternative representation of this integral surface is

\[ F(x, y, z) = u(x, y) - z = 0. \]

That is, an integral surface is a level surface of the function \( F(x, y, z) \).

Now, recall from vector calculus that the gradient vector to a level surface at the point \( (x, y, z) \) is a normal vector to the surface at that point. That is, the gradient is a vector normal to the tangent plane to the surface at the point \( (x, y, z) \). Thus, the normal vector to the surface \( F(x, y, z) = 0 \) is given by

\[ \vec{n} = \nabla F = F_x \vec{i} + F_y \vec{j} + F_z \vec{k} = u_x \vec{i} + u_y \vec{j} - \vec{k}. \]

Because of the negative \( z \)-component, the vector \( \vec{n} \) is pointing downward. Now, equation (9.1) can be written as the dot product

\[ (a(x, y, u), b(x, y, u), c(x, y, u)) \cdot (u_x, u_y, -1) = 0 \]

or

\[ \vec{v} \cdot \vec{n} = 0 \]

where \( \vec{v} = a(x, y, u)\vec{i} + b(x, y, u)\vec{j} + c(x, y, u)\vec{k} \). Thus, \( \vec{n} \) is normal to \( \vec{v} \). Since \( \vec{n} \) is normal to the surface \( F(x, y, z) = 0 \), the vector \( \vec{v} \) must be tangential to the surface \( F(x, y, z) = 0 \) and hence must lie in the tangent plane to the surface at every point. Thus, to find a solution to (9.1) we need to find an integral surface such that the surface is tangent to the vector \( \vec{v} \) at each of its point.
The required surface can be found as the union of integral curves, that is, curves that are tangent to \( \vec{v} \) at every point on the curve. If an integral curve has a parametrization
\[
\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + u(t)\vec{k}
\]
then the integral curve (i.e. the characteristic) is a solution to the ODE system
\[
\frac{dx}{dt} = a(x, y, u), \quad \frac{dy}{dt} = b(x, y, u), \quad \frac{du}{dt} = c(x, y, u) \tag{9.2}
\]
or in differential form
\[
\frac{dx}{a(x, y, u)} = \frac{dy}{b(x, y, u)} = \frac{du}{c(x, y, u)}. \tag{9.3}
\]
Equations (9.2) or (9.3) are called characteristic equations. Note that \( u(t) = u(x(t), y(t)) \) gives the values of \( u \) along a characteristic. Thus, along a characteristic, the PDE (9.1) degenerates to an ODE.

**Example 9.1**
Find the general solution of the PDE \( yuu_x + xuu_y = xy \).

**Solution.**
The characteristic equations are \( \frac{dx}{yu} = \frac{du}{xu} = \frac{dy}{xy} \). Using the first two fractions we find \( x^2 - y^2 = k_1 \). Using the last two fractions we find \( u^2 - y^2 = f(x^2 - y^2) \). Hence, the general solution can be written as \( u^2 = y^2 + f(x^2 - y^2) \), where \( f \) is an arbitrary differentiable single variable function.

**Example 9.2**
Find the general solution of the PDE \( x(y^2 - u^2)u_x - y(u^2 + x^2)y_y = (x^2 + y^2)u \).

**Solution.**
The characteristic equations are \( \frac{dx}{x(y^2 - u^2)} = \frac{du}{-y(u^2 + x^2)} = \frac{dy}{(x^2 + y^2)u} \). Using a property of proportions we can write
\[
\frac{x dx + y dy + u du}{x^2(y^2 - u^2) - y^2(u^2 + x^2) + u^2(x^2 + y^2)} = \frac{du}{(x^2 + y^2)u}.
\]
That is
\[
\frac{x dx + y dy + u du}{0} = \frac{du}{(x^2 + y^2)u}.\]
or
\[ xdx + ydy + udu = 0. \]
Hence, we find \( x^2 + y^2 + u^2 = k_1 \). Also,
\[ \frac{dx}{x} - \frac{dy}{y} = \frac{du}{u} \]
\[ \frac{dx}{y^2 - u^2 + u^2 + x^2} = \frac{du}{(x^2 + y^2)u} \]
or
\[ \frac{dx}{x} - \frac{dy}{y} = \frac{du}{u}. \]
Hence, we find \( \frac{yu}{x} = k_2 \). The general solution is given by
\[ u(x, y) = \frac{x}{y} f(x^2 + y^2 + u^2) \]
where \( f \) is an arbitrary differentiable single variable function.
Practice Problem

Problem 9.1
Find the general solution of the PDE \( \ln (y + u)u_x + u_y = -1 \).

Problem 9.2
Find the general solution of the PDE \( x(y - u)u_x + y(u - x)u_y = u(x - y) \).

Problem 9.3
Find the general solution of the PDE \( u(u^2 + xy)(xu_x - yu_y) = x^4 \).

Problem 9.4
Find the general solution of the PDE \( (y + xu)u_x - (x + yu)u_y = x^2 - y^2 \).

Problem 9.5
Find the general solution of the PDE \( (y^2 + u^2)u_x - xyu_y + xu = 0 \).

Problem 9.6
Find the general solution of the PDE \( u_t + uu_x = x \).

Problem 9.7
Find the general solution of the PDE \( (y - u)u_x + (u - x)u_y = x - y \).

Problem 9.8
Solve
\[
x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u.
\]

Problem 9.9
Solve
\[
\sqrt{1 - x^2}u_x + u_y = 0.
\]

Problem 9.10
Solve
\[
u(x + y)u_x + u(x - y)u_y = x^2 + y^2.
\]