# A Second Course in Elementary Differential Equations

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# 28 Calculus of Matrix-Valued Functions of a Real Variable

In establishing the existence result for second and higher order linear differential equations one transforms the equation into a linear system and tries to solve such a system. This procedure requires the use of concepts such as the derivative of a matrix whose entries are functions of $t$, the integral of a matrix, and the exponential matrix function. Thus, techniques from matrix theory play an important role in dealing with systems of differential equations. The present section introduces the necessary background in the calculus of matrix functions.

**Matrix-Valued Functions of a Real Variable**

A **matrix A of dimension** $m \times n$ is a rectangular array of the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where the $a_{ij}$'s are the **entries** of the matrix, $m$ is the number of rows, $n$ is the number of columns. The **zero matrix 0** is the matrix whose entries are all 0. The $n \times n$ **identity matrix** $\mathbf{I}_n$ is a square matrix whose main diagonal consists of $1's$ and the off diagonal entries are all 0. A matrix **A** can be represented with the following compact notation $\mathbf{A} = (a_{ij})$. The entry $a_{ij}$ is located in the ith row and jth column.

**Example 28.1**

Consider the matrix

$$\mathbf{A}(t) = \begin{bmatrix} -5 & 0 & 1 \\ 10 & -2 & 0 \\ -5 & 2 & -7 \end{bmatrix}$$

Find $a_{22},\ a_{32}$, and $a_{23}$.

**Solution.**

The entry $a_{22}$ is in the second row and second column so that $a_{22} = -2$. Similarly, $a_{32} = 2$ and $a_{23} = 0$ ■

An $m \times n$ array whose entries are functions of a real variable defined on a common interval is called a **matrix function.** Thus, the matrix

$$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix}$$

is a $3 \times 3$ matrix function whereas the matrix

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

is a $3 \times 1$ matrix function also known as a **vector-valued function.**
We will denote an $m \times n$ matrix function by $\mathbf{A}(t) = (a_{ij}(t))$ where $a_{ij}(t)$ is the entry in the ith row and jth coloumn.

**Arithmetic of Matrix Functions**
All the familiar rules of matrix arithmetic hold for matrix functions as well.

(i) **Equality:** Two $m \times n$ matrices $\mathbf{A}(t) = (a_{ij}(t))$ and $\mathbf{B}(t) = (b_{ij}(t))$ are said to be equal if and only if $a_{ij}(t) = b_{ij}(t)$ for all $1 \le i \le m$ and $1 \le j \le n$. That is, two matrices are equal if and only if all corresponding elements are equal. Notice that the matrices must be of the same dimension.

**Example 28.2**
Solve the following matrix equation for $a, b, c,$ and $d$

$$\begin{bmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{bmatrix} = \begin{pmatrix} 8 & 1 \\ 7 & 6 \end{pmatrix}$$

**Solution.**
Equating corresponding entries we get the system

$$\begin{cases} a - b = 8 \\ b + c = 1 \\ c + 3d = 7 \\ 2a - 4d = 6 \end{cases}$$

Adding the first two equations to obtain $a + c = 9$. Adding 4 times the third equation to 3 times the last equation to obtain $6a + 4c = 46$ or $3a + 2c = 23$.

Solving the two equations in $a$ and $c$ one finds $a = 5$ and $c = 4$. Hence, $b = -3$ and $d = 1$ ∎

(ii) **Addition:** If $\mathbf{A}(t) = (a_{ij}(t))$ and $\mathbf{B}(t) = (b_{ij}(t)$ are $m \times n$ matrices then the sum is a new $m \times n$ matrix obtained by adding corresponding elements

$$(\mathbf{A} + \mathbf{B})(t) = \mathbf{A}(t) + \mathbf{B}(t) = (a_{ij}(t) + b_{ij}(t))$$

Matrices of different dimensions cannot be added.

(iii) **Subtraction:** Let $\mathbf{A}(t) = (a_{ij}(t))$ and $\mathbf{B}(t) = (b_{ij}(t))$ be two $m \times n$ matrices. Then the difference $(\mathbf{A} - \mathbf{B})(\mathbf{t})$ is the new matrix obtained by subtracting corresponding elements,that is

$$(\mathbf{A} - \mathbf{B})(t) = \mathbf{A}(t) - \mathbf{B}(t) = (a_{ij}(t) - b_{ij}(t))$$

(iv) **Scalar Multiplication:** If $\alpha$ is a real number and $\mathbf{A}(t) = (a_{ij}(t))$ is an $m \times n$ matrix then $(\alpha\mathbf{A})(t)$ is the $m \times n$ matrix obtained by multiplying the entries of $\mathbf{A}$ by the number $\alpha$; that is,

$$(\alpha\mathbf{A})(t) = \alpha\mathbf{A}(t) = (\alpha a_{ij}(t))$$

(v) **Matrix Multiplication:** If $\mathbf{A}$(t) is an $m \times n$ matrix and $\mathbf{B}$(t) is an $n \times p$ matrix then the matrix $\mathbf{AB}(t)$ is the $m \times p$ matrix

$$\mathbf{AB}(t) = (c_{ij}(t))$$

where

$$c_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t) b_{kj}(t)$$

That is the $c_{ij}$ entry is obtained by multiplying componentwise the ith row of $\mathbf{A}$(t) by the jth column of $\mathbf{B}$(t). It is important to realize that the order of the multiplicands is significant, in other words $\mathbf{AB}$(t) is not necessarily equal to $\mathbf{BA}$(t). In mathematical terminology matrix multiplication is not commutative.

**Example 28.3**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

Show that $\mathbf{AB} \neq \mathbf{BA}$. Hence, matrix multiplication is not commutative.

**Solution.**
Using the definition of matrix multiplication we find

$$\mathbf{AB} = \begin{bmatrix} -4 & 7 \\ 0 & 5 \end{bmatrix}, \mathbf{BA} = \begin{bmatrix} -1 & 2 \\ 9 & 2 \end{bmatrix}$$

Hence, $\mathbf{AB} \neq \mathbf{BA}$ ■
(vi) **Inverse:** An $n \times n$ matrix $\mathbf{A}(\mathrm{t})$ is said to be **invertible** if and only if there is an $n \times n$ matrix $\mathbf{B}(\mathrm{t})$ such that $\mathbf{AB}(t) = \mathbf{BA}(t) = \mathbf{I}$ where $\mathbf{I}$ is the matrix whose main diagonal consists of the number 1 and 0 elsewhere. We denote the **inverse** of $\mathbf{A}(\mathrm{t})$ by $\mathbf{A^{-1}}(t)$.

**Example 28.4**
Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

given that $ad - bc \neq 0$. The quantity $ad - bc$ is called the **determinant** of $\mathbf{A}$ and is denoted by $det\mathbf{A}$

**Solution.**
Suppose that

$$\mathbf{A}^{-1} = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

Then

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This implies that

$$\begin{bmatrix} ax + cy & bx + dy \\ az + ct & bz + dt \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence,

$$\begin{array}{rcl} ax + cy & = & 1 \\ bx + dy & = & 0 \\ az + ct & = & 0 \\ bz + dt & = & 1 \end{array}$$

Applying the method of elimination to the first two equations we find

$$x = \frac{d}{ad-bc} \text{ and } y = \frac{-b}{ad-bc}$$

Applying the method of elimination to the last two equations we find

$$z = \frac{-c}{ad-bc} \text{ and } t = \frac{a}{ad-bc}$$

Hence,

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right] \blacksquare$$

**Norm of a Vector Function**

The norm of a vector function will be needed in the coming sections. In one dimension a norm is known as the absolute value. In multidimenesion, we define the norm of a vector function $\mathbf{x}$ with components $x_1, x_2, \cdots, x_n$ by

$$||\mathbf{x}|| = |x_1| + |x_2| + \cdots + |x_n|.$$

From this definition one notices the following properties:
(i) If $||\mathbf{x}|| = 0$ then $|x_1| + |x_2| + \cdots + |x_n| = 0$ and this implies that $|x_1| = |x_2| = \cdots = |x_n| = 0$. Hence, $\mathbf{x} = \mathbf{0}$.
(ii) If $\alpha$ is a scalar then $||\alpha\mathbf{x}|| = |\alpha x_1| + |\alpha x_2| + \cdots + |\alpha x_n| = |\alpha|(|x_1| + |x_2| + \cdots + |x_n|) = |\alpha|||\mathbf{x}||$.
(iii) If $\mathbf{x}$ is vector function with components $x_1, x_2, \cdots, x_n$ and $\mathbf{y}$ with components $y_1, y_2, \cdots, y_n$ then

$$\begin{array}{rcl} ||\mathbf{x}+\mathbf{y}|| & = & |x_1+y_1| + |x_2+y_2| + \cdots + |x_n+y_n| \\ & \leq & (|x_1| + |x_2| + \cdots + |x_n|) + (|y_1| + |y_2| + \cdots + |y_n|) \\ & = & ||\mathbf{x}|| + ||\mathbf{y}|| \end{array}$$

**Limits of Matrix Functions**

If $\mathbf{A}(t) = (a_{ij}(t))$ is an $m \times n$ matrix such that $\lim_{t\to t_0} a_{ij}(t) = L_{ij}$ exists for all $1 \leq i \leq m$ and $1 \leq j \leq n$ then we define

$$\lim_{t\to t_0} \mathbf{A}(t) = (L_{ij})$$

**Example 28.5**

Suppose that

$$\mathbf{A}(t) = \left[\begin{array}{cc} t^2 - 5t & t^3 \\ 2t & 3 \end{array}\right]$$

Find $\lim_{t\to 1} \mathbf{A}(t)$.

**Solution.**

$$\lim_{t\to 1}\mathbf{A}(t) = \left[\begin{array}{cc} \lim_{t\to 1}(t^2-5t) & \lim_{t\to 1} t^3 \\ \lim_{t\to 1} 2t & \lim_{t\to 1} 3 \end{array}\right] = \left[\begin{array}{cc} -4 & 1 \\ 2 & 3 \end{array}\right] \blacksquare$$

If one or more of the component function limits does not exist, then the limit of the matrix does not exist. For example, if

$$\mathbf{A}(t) = \left[\begin{array}{cc} t & t^{-1} \\ 0 & e^t \end{array}\right]$$

then $\lim_{t\to 0}\mathbf{A}$ does not exist since $\lim_{t\to 0}\frac{1}{t}$ does not exist.
We say that $\mathbf{A}$(t) is **continuous** at $t = t_0$ if

$$\lim_{t\to t_0}\mathbf{A}(t) = \mathbf{A}(t_0)$$

**Example 28.6**
Show that the matrix

$$\mathbf{A}(t) = \left[\begin{array}{cc} t & t^{-1} \\ 0 & e^t \end{array}\right]$$

is continuous at $t = 1$.

**Solution.**
Since

$$\lim_{t\to 1}\mathbf{A}(t) = \left[\begin{array}{cc} 2 & 1/2 \\ 0 & e^2 \end{array}\right] = \mathbf{A}(1)$$

we have $\mathbf{A}$(t) is continuous at $t = 1$ $\blacksquare$
Most properties of limits for functions of a single variable are also valid for limits of matrix functions.

**Matrix Differentiation**
Let $\mathbf{A}$(t) be an $m \times n$ matrix such that each entry is a differentiable function of $t$. We define the **derivative** of $\mathbf{A}$(t) to be

$$\mathbf{A}'(t) = \lim_{h\to 0}\frac{\mathbf{A}(t+h)-\mathbf{A}(t)}{h}$$

provided that the limit exists.

**Example 28.7**
Let

$$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}$$

where the entries $a_{11}, a_{12}, a_{21}$, and $a_{22}$ are differentiable. Find $\mathbf{A}'(t)$.

**Solution.**
We have

$$\begin{array}{rcl} \mathbf{A}'(t) & = & \lim_{h\to 0} \frac{\mathbf{A}(t+h)-\mathbf{A}(t)}{h} \\ & = & \begin{bmatrix} \lim_{h\to 0} \frac{a_{11}(t+h)-a_{11}(t)}{h} & \lim_{h\to 0} \frac{a_{12}(t+h)-a_{12}(t)}{h} \\ \lim_{h\to 0} \frac{a_{21}(t+h)-a_{21}(t)}{h} & \lim_{h\to 0} \frac{a_{22}(t+h)-a_{22}(t)}{h} \end{bmatrix} \\ & = & \begin{bmatrix} a'_{11}(t) & a'_{12}(t) \\ a'_{21}(t) & a'_{22}(t) \end{bmatrix} \blacksquare \end{array}$$

It follows from the previous example that the derivative of a matrix function is the matrix of derivatives of its component functions. From this fact one can check easily the following two properties of differentiation:
(i) If $\mathbf{A}$(t) and $\mathbf{B}$(t) are two $m \times n$ matrices with both of them differentiable then the matrix $(\mathbf{A} + \mathbf{B})$(t) is also differentiable and

$$(\mathbf{A} + \mathbf{B})'(t) = \mathbf{A}'(t) + \mathbf{B}'(t)$$

(ii) If $\mathbf{A}$(t) is an $m \times n$ differentiable matrix and $\mathbf{B}$(t) is an $n \times p$ differentiable matrix then the product matrix $\mathbf{AB}$(t) is also differentiable and

$$(\mathbf{AB})'(t) = \mathbf{A}'(t)\mathbf{B}(t) + \mathbf{A}(t)\mathbf{B}'(t)$$

**Example 28.8**
Write the system

$$\begin{array}{rcl} y_1' & = & a_{11}(t)y_1(t) + a_{12}(t)y_2(t) + a_{13}(t)y_3(t) + g_1(t) \\ y_2' & = & a_{21}(t)y_1(t) + a_{22}(t)y_2(t) + a_{23}(t)y_3(t) + g_2(t) \\ y_3' & = & a_{11}(t)y_1(t) + a_{12}(t)y_2(t) + a_{13}(t)y_3(t) + g_3(t) \end{array}$$

in matrix form.

**Solution.**
Let

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}, \ \mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13} \\ a_{21}(t) & a_{22}(t) & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \ \mathbf{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{bmatrix}$$

Then the given system can be written in the matrix form

$$\mathbf{y}'(t) = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{g}(t) \ \blacksquare$$

**Matrix Integration:**
Since the derivative of a matrix function is a matrix of derivatives, it should not be surprising that antiderivatives of a matrix function can be evaluated by performing the corresponding antidifferentiation operations upon each component of the matrix function. That is, if $\mathbf{A}$(t) is the $m \times n$ matrix

$$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{m1}(t) & a_{m2}(t) & \cdots & a_{mn}(t) \end{bmatrix}$$

then

$$\int \mathbf{A}(t)dt = \begin{bmatrix} \int a_{11}(t)dt & \int a_{12}(t)dt & \cdots & \int a_{1n}(t)dt \\ \int a_{21}(t)dt & \int a_{22}(t)dt & \cdots & \int a_{2n}(t)dt \\ \vdots & & & \vdots \\ \int a_{m1}(t)dt & \int a_{m2}(t)dt & \cdots & \int a_{mn}(t)dt \end{bmatrix}$$

**Example 28.9**
Determine the matrix function $\mathbf{A}$(t) if

$$\mathbf{A}'(t) = \begin{bmatrix} 2t & 1 \\ \cos t & 3t^2 \end{bmatrix}$$

**Solution.**
We have

$$\mathbf{A}(t) = \begin{bmatrix} t^2 + c_{11} & t + c_{12} \\ \sin t + c_{21} & t^3 + c_{22} \end{bmatrix} = \begin{bmatrix} t^2 & t \\ \sin t & t^3 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \ \blacksquare$$

Finally, we conclude this section by showing that

$$||\int_{t_0}^{t} \mathbf{x}(s)ds|| \leq \int_{t_0}^{t} ||\mathbf{x}(s)||ds$$

To see this,

$$\begin{aligned}
||\int_{t_0}^{t} \mathbf{x}(s)ds|| &= \left|\left| \begin{bmatrix} \int_{t_0}^{t} x_1(s)ds \\ \int_{t_0}^{t} x_2(s)ds \\ \vdots \\ \int_{t_0}^{t} x_n(s)ds \end{bmatrix} \right|\right| \\
&= |\int_{t_0}^{t} x_1(s)ds| + |\int_{t_0}^{t} x_2(s)ds| + \cdots |\int_{t_0}^{t} x_n(s)ds| \\
&\leq \int_{t_0}^{t} |x_1(s)|ds + \int_{t_0}^{t} |x_2(s)|ds + \cdots + \int_{t_0}^{t} |x_n(s)|ds \\
&= \int_{t_0}^{t} (|x_1| + |x_2| + \cdots + |x_n|)ds = \int_{t_0}^{t} ||\mathbf{x}(s)||ds
\end{aligned}$$

## Practice Problems

**Problem 28.1**
Consider the following matrices

$$\mathbf{A}(t) = \begin{bmatrix} t-1 & t^2 \\ 2 & 2t+1 \end{bmatrix}, \ \mathbf{B}(t) = \begin{bmatrix} t & -1 \\ 0 & t+2 \end{bmatrix}, \ \mathbf{c}(t) = \begin{bmatrix} t+1 \\ -1 \end{bmatrix}$$

(a) Find 2**A**(t) - 3t**B**(t)
(b) Find **A**(t)**B**(t) - **B**(t)**A**(t)
(c) Find **A**(t)**c**(t)
(d) Find det(**B**(t)**A**(t))

**Problem 28.2**
Determine all values $t$ such that $\mathbf{A}(t)$ is invertible and, for those t-values, find $\mathbf{A}^{-1}(t)$.

$$\mathbf{A}(t) = \begin{bmatrix} t+1 & t \\ t & t+1 \end{bmatrix}$$

**Problem 28.3**
Determine all values $t$ such that $\mathbf{A}(t)$ is invertible and, for those t-values, find $\mathbf{A}^{-1}(t)$.

$$\mathbf{A}(t) = \begin{bmatrix} \sin t & -\cos t \\ \sin t & \cos t \end{bmatrix}$$

**Problem 28.4**
Find

$$\lim_{t\to 0} \begin{bmatrix} \frac{\sin t}{t} & t\cos t & \frac{3}{t+1} \\ e^{3t} & \sec t & \frac{2t}{t^2-1} \end{bmatrix}$$

**Problem 28.5**
Find

$$\lim_{t\to 0} \begin{bmatrix} te^{-t} & \tan t \\ t^2-2 & e^{\sin t} \end{bmatrix}$$

**Problem 28.6**
Find $\mathbf{A}'(t)$ and $\mathbf{A}''(t)$ if

$$\mathbf{A}(t) = \begin{bmatrix} \sin t & 3t \\ t^2+2 & 5 \end{bmatrix}$$

**Problem 28.7**
Express the system

$$\begin{array}{rcl} y_1' & = & t^2 y_1 + 3y_2 + \sec t \\ y_2' & = & (\sin t) y_1 + t y_2 - 5 \end{array}$$

in the matrix form

$$\mathbf{y}'(t) = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{g}(t)$$

**Problem 28.8**
Determine $\mathbf{A}$(t) where

$$\mathbf{A}'(t) = \left[ \begin{array}{cc} 2t & 1 \\ \cos t & 3t^2 \end{array} \right], \quad \mathbf{A}(0) = \left[ \begin{array}{cc} 2 & 5 \\ 1 & -2 \end{array} \right]$$

**Problem 28.9**
Determine $\mathbf{A}$(t) where

$$\mathbf{A}''(t) = \left[ \begin{array}{cc} 1 & t \\ 0 & 0 \end{array} \right], \quad \mathbf{A}(0) = \left[ \begin{array}{cc} 1 & 1 \\ -2 & 1 \end{array} \right], \quad \mathbf{A}'(0) = \left[ \begin{array}{cc} -1 & 2 \\ -2 & 3 \end{array} \right]$$

**Problem 28.10**
Calculate $\mathbf{A}(t) = \int_0^t \mathbf{B}(s)ds$ where

$$\mathbf{B}(s) = \left[ \begin{array}{cc} e^s & 6s \\ \cos 2\pi s & \sin 2\pi s \end{array} \right]$$

**Problem 28.11**
Construct a $2 \times 2$ a nonconstant matrix function $\mathbf{A}$(t) such that $\mathbf{A}^2(t)$ is a constant matrix.

**Problem 28.12**
(a) Construct a $2 \times 2$ differentiable matrix function $\mathbf{A}$(t) such that

$$\frac{d}{dt}\mathbf{A}^2(t) \neq 2\mathbf{A}\frac{d}{dt}\mathbf{A}(t)$$

That is, the power rule is not true for matrix functions.
(b) What is the correct formula relating $\mathbf{A}^2(t)$ to $\mathbf{A}$(t) and $\mathbf{A}$'(t)?

**Problem 28.13**
Transform the following third-order equation

$$y''' - 3ty' + (\sin 2t)y = 7e^{-t}$$

into a first order system of the form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t)$$

**Problem 28.14**
By introducing new variables $x_1$ and $x_2$, write $y'' - 2y + 1 = t$ as a system of two first order linear equations of the form $\mathbf{x}' + \mathbf{A}\mathbf{x} = \mathbf{b}$

**Problem 28.15**
Write the differential equation $y'' + 4y' + 4y = 0$ as a first order system.

**Problem 28.16**
Write the differential equation $y'' + ky' + (t-1)y = 0$ as a first order system.

**Problem 28.17**
Change the following second-order equations to a first-order system.

$$y'' - 5y' + ty = 3t^2, \quad y(0) = 0, \; y'(0) = 1$$

**Problem 28.18**
Consider the following system of first-order linear equations.

$$\mathbf{x}' = \begin{bmatrix} 3 & \\ 2 & 1 \\ -1 & \end{bmatrix} \cdot \mathbf{x}$$

Find the second-order linear differential equation that $\mathbf{x}$ satisfies.

**The Determinant of a Matrix**
The determinant of a matrix function is the same as the determinant with constant entries. So we will introduce the definition of determinant of a matrix with constant entries.
A **permutation** of the set $S = \{1, 2, \dots, n\}$ is an arrangement of the elements of $S$ in some order without omissions or repetitions. We write $\sigma = (\sigma(1)\sigma(2)\cdots\sigma(n))$. In terms of functions, a permutation is a one-to-one function from $S$ onto $S$.

Let $S_n$ denote the set of all permutations on $S$. How many permutations are there in $S_n$? We have $n$ positions to be filled by $n$ numbers. For the first position, there are $n$ possibilities. For the second there are $n-1$ possibilities, etc. Thus, according to the multiplication rule of counting there are

$$n(n-1)(n-2)\dots 2.1 = n!$$

permutations.
Is there a way to list all the permutations of $S_n$? The answer is yes and one can find the permutations by using a **permutation tree** which we describe in the following example

**Problem 28.19**
List all the permutations of $S = \{1, 2, 3, 4\}$.

An **inversion** is said to occur whenever a larger integer precedes a smaller one. If the number of inversions is even (resp. odd) then the permutation is said to be **even** (**resp. odd**). We define the **sign** of a permutation to be a function $sgn$ with domain $S_n$ and range $\{-1, 1\}$ such that $sgn(\sigma) = -1$ if $\sigma$ is odd and $sgn(\sigma) = +1$ if $\sigma$ is even. For example, the permutation in $S_6$ defined by $\sigma(1) = 3, \sigma(2) = 6, \sigma(3) = 4, \sigma(5) = 2, \sigma(6) = 1$ is an even permuatation since the inversions are (6,1),(6,3),(6,4),(6,5),(6,2),(3,2),(4,2), and (5,2).
Let $A$ be an $n \times n$ matrix. An **elementary product** from $A$ is a product of $n$ entries from $A$, no two of which come from the same row or same column.

**Problem 28.20**
List all elementary products from the matrices
(a)

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

(b)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Let $A$ be an $n \times n$ matrix. Consider an elementary product of entries of $A$. For the first factor, there are $n$ possibilities for an entry from the first row. Once selected, there are $n-1$ possibilities for an entry from the second row for the second factor. Continuing, we find that there are $n!$ elementary products. They are the products of the form $a_{1\sigma(1)}a_{2\sigma(2)}\dots a_{n\sigma(n)}$, where $\sigma$ is a permutation of $\{1, 2, \dots, n\}$, i.e. a member of $S_n$.

Let $A$ be an $n \times n$ matrix. Then we define the **determinant** of $A$ to be the number

$$det(A) = \sum sgn(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\dots a_{n\sigma(n)}$$

where the sum is over all permutations $\sigma$ of $\{1, 2, \dots, n\}$.

**Problem 28.21**
Find $det(A)$ if
(a)

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

(b)

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The following theorem is of practical use. It provides a technique for evaluating determinants by greatly reducing the labor involved.

**Theorem 28.1**
Let $\mathbf{A}$ be an $n \times n$ matrix.
(a) Let $\mathbf{B}$ be the matrix obtained from $\mathbf{A}$ by multiplying a row or a column by a scalar $c$. Then $det(\mathbf{B}) = c det \mathbf{A}$.
(b) Let $\mathbf{B}$ be the matrix obtained from $\mathbf{A}$ by interchanging two rows or two columns of $\mathbf{A}$. Then $det(\mathbf{B}) = -det(\mathbf{A})$.
(c) If $\mathbf{A}$ has two identical rows or columns then its determinant is zero.
(d) Let $\mathbf{B}$ be the matrix obtained from $\mathbf{A}$ by adding $c$ times a row (or a column) to another row (column). Then $det(\mathbf{B}) = det(\mathbf{A})$.
(e) The determinant of the product of two $n \times n$ matrices is the product of their determinant.
(g) If $\mathbf{B}$ is the matrix whose columns are the rows of $\mathbf{A}$ then $\det(\mathbf{B}) = \det(\mathbf{A})$.

The proof of this theorem can be found in any textbook in elementary linear algebra.

# 29 nth Order Linear Differential Equations:Existence and Uniqueness

In the following three sections we carry the basic theory of second order linear differential equations to nth order linear differential equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t)$$

where the functions $p_0,\ p_1,\ \cdots,\ p_{n-1}$ and $g(t)$ are continuous functions for $a < t < b$.
If $g(t)$ is not identically zero, then this equation is said to be **nonhomogeneous**; if $g(t)$ is identically zero, then this equation is called **homogeneous.**

**Existence and Uniqueness of Solutions**
We begin by discussing the existence of a unique solution to the initial value problem

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t)$$

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ \cdots,\ y^{(n-1)}(t_0) = y_0^{(n-1)},\ a < t_0 < b$$

The following theorem is a generalization to Theorems 3.2 and 15.1

**Theorem 29.1**
The nonhomogeneous nth order linear differential equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t) \qquad (1)$$

with initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ \cdots,\ y^{(n-1)}(t_0) = y_0^{(n-1)},\ a < t_0 < b \qquad (2)$$

has a unique solution in $a < t < b$.

**Proof.**
**Existence:** The existence of a local solution is obtained here by transforming the problem into a first order system. This is done by introducing the variables

$$x_1 = y,\ x_2 = y',\ \cdots,\ x_n = y^{(n-1)}.$$

In this case, we have

$$\begin{array}{rcl} x_1' & = & x_2 \\ x_2' & = & x_3 \\ \vdots & = & \vdots \\ x_{n-1}' & = & x_n \\ x_n' & = & -p_{n-1}(t)x_n - \cdots + p_1(t)x_2 - p_0(t)x_1 + g(t) \end{array}$$

Thus, we can write the problem as a system:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}' + \begin{bmatrix} 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \\ p_0 & p_1 & p_2 & p_3 & \cdots & p_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}$$

or in compact form

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t), \quad \mathbf{x}(t_0) = \mathbf{y}_0 \tag{3}$$

where

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & -p_2 & -p_3 & \cdots & -p_{n-1} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \ \mathbf{b}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}, \ \mathbf{y}_0 = \begin{bmatrix} y_0 \\ y_0' \\ \vdots \\ y_0^{(n-1)} \end{bmatrix}$$

Note that if $y(t)$ is a solution of (1) then the vector-valued function

$$\mathbf{x}(t) = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

is a solution to (3). Conversely, if the vector

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

is a solution of (3) then $x_1' = x_2,\ x_1'' = x_3, \cdots, x_1^{(n-1)} = x_n$. Hence, $x_1^{(n)} = x_n' = -p_{n-1}(t)x_n - p_{n-2}(t)x_{n-1} - \cdots - p_0(t)x_1 + g(t)$ or

$$x_1^{(n)} + p_{n-1}(t)x_1^{(n-1)} + p_{n-2}(t)x_1^{(n-2)} + \cdots + p_0(t)x_1 = g(t)$$

which means that $x_1$ is a solution to (1).
Next, we start by reformulating (3) as an equivalent integral equation. Integration of both sides of (3) yields

$$\int_{t_0}^{t} \mathbf{x}'(s)ds = \int_{t_0}^{t} [\mathbf{A}(s)\mathbf{x}(s) + \mathbf{b}(s)]ds \tag{4}$$

Applying the Fundamental Theorem of Calculus to the left side of (4) yields

$$\mathbf{x}(\mathbf{t}) = \mathbf{x}(t_0) + \int_{t_0}^{t} [\mathbf{A}(s)\mathbf{x}(s) + \mathbf{b}(s)]ds \tag{5}$$

Thus, a solution of (5) is also a solution to (3) and vice versa.
To prove the existence and uniqueness, we shall use again the method of successive approximation as described in Theorem 8.1.
Letting

$$\mathbf{y}_0 = \begin{bmatrix} y_0 \\ y_0' \\ \vdots \\ y_0^{(n-1)} \end{bmatrix}$$

we can introduce Picard's iterations defined recursively as follows:

$$\begin{array}{rcl} \mathbf{y}_0(t) & \equiv & \mathbf{y}_0 \\ \mathbf{y}_1(t) & = & \mathbf{y}_0 + \int_{t_0}^{t} [\mathbf{A}(s)\mathbf{y}_0(s) + \mathbf{b}(s)]ds \\ \\ \mathbf{y}_2(t) & = & \mathbf{y}_0 + \int_{t_0}^{t} [\mathbf{A}(s)\mathbf{y}_1(s) + \mathbf{b}(s)]ds \\ \vdots & & \vdots \\ \mathbf{y}_N(t) & = & \mathbf{y}_0 + \int_{t_0}^{t} [\mathbf{A}(s)\mathbf{y}_{N-1}(s) + \mathbf{b}(s)]ds \end{array}$$

Let

$$\mathbf{y}_N(t) = \begin{bmatrix} y_{1,N} \\ y_{2,N} \\ \vdots \\ y_{n,N} \end{bmatrix}$$

For $i = 1, 2, \cdots, n$, we are going to show that the sequence $\{y_{i,N}(t)\}$ converges uniformly to a function $y_i(t)$ such that $\mathbf{y}(t)$ (with components $y_1, y_2, \cdots, y_n$)is a solution to (5) and hence a solution to (3).

Let $[c, d]$ be a closed interval containing $t_0$ and contained in $(a, b)$. Then there exist constants $k_0, k_1, \cdots, k_{n-1}$ such that

$$\max_{c \le t \le d} |p_0(t)| \le k_0, \ \max_{c \le t \le d} |p_1(t)| \le k_1, \cdots, \max_{c \le t \le d} |p_{n-1}(t)| \le k_{n-1}.$$

This implies that

$$\begin{array}{rcl} ||\mathbf{A}(t)\mathbf{x}(t)|| & = & |x_2| + |x_3| + \cdots + |x_{n-1} + |p_0||x_1| + |p_1||x_2| + \cdots + |p_{n-1}||x_n| \\ & \le & k_0|x_1| + (1 + k_1)|x_2| + \cdots + (1 + k_{n-2})|x_{n-1}| + k_{n-1}|x_n| \le K||\mathbf{x}|| \end{array}$$

for all $c \le t \le d$, where we define

$$||\mathbf{y}|| = |y_1| + |y_2| + \cdots + |y_n|$$

and where

$$K = k_0 + (1 + k_1) + \cdots + (1 + k_{n-2}) + k_{n-1}.$$

It follows that for $1 \le i \le n$

$$\begin{array}{rcl} |y_{i,N} - y_{i,N-1}| & \le & ||\mathbf{y}_N - \mathbf{y}_{N-1}|| \le \int_{t_0}^t ||\mathbf{A}(s) \cdot (\mathbf{y}_{N-1} - \mathbf{y}_{N-2})||ds \\ & \le & K \int_{t_0}^t ||\mathbf{y}_{N-1} - \mathbf{y}_{N-2}||ds \end{array}$$

But

$$\begin{array}{rcl} ||\mathbf{y}_1 - \mathbf{y_0}|| & \le & \int_{t_0}^t ||\mathbf{A}(s) \cdot \mathbf{y}_0 + \mathbf{b}(s)||ds \\ & \le & M(t - t_0) \end{array}$$

where

$$M = K||\mathbf{y}_0|| + \max_{c \le t \le d} ||\mathbf{b}(t)||$$

An easy induction yields that

$$||\mathbf{y}_{N+1} - \mathbf{y}_N|| \le MK^N \frac{(t - t_0)^{N+1}}{N!} \le MK^N \frac{(b - a)^{N+1}}{N!}.$$

Since

$$\sum_{N=0}^{\infty} MK^N \frac{(b-a)^{N+1}}{N!} = M(b-a)(e^{K(b-a)} - 1)$$

by Weierstrass M-test we conclude that the series $\sum_{N=0}^{\infty}[y_{i,N} - y_{i,N-1}]$ converges uniformly for all $c \le t \le d$. But

$$y_{i,N}(t) = \sum_{k=0}^{N-1} [y_{i,k+1}(t) - y_{i,k}(t)] + y_{i,0}$$

Thus, the sequence $\{y_{i,N}\}$ converges uniformly to a function $y_i(t)$ for all $c \le t \le d$.
The function $y_i(t)$ is a continuous function (a uniform limit of a sequence of continuous function is continuous). Also we can interchange the order of taking limits and integration for such sequences. Therefore

$$\begin{array}{rcc} \mathbf{y}(t) & = & \lim_{N\to\infty} \mathbf{y}_N(t) \\ & = & \mathbf{y}_0 + \lim_{N\to\infty} \int_{t_0}^{t} (\mathbf{A}(s)\mathbf{y}_{N-1} + \mathbf{b}(s))ds \\ & = & \mathbf{y}_0 + \int_{t_0}^{t} \lim_{N\to\infty} (\mathbf{A}(s)\mathbf{y}_{N-1} + \mathbf{b}(s))ds \\ & = & \mathbf{y}_0 + \int_{t_0}^{t} (\mathbf{A}(s)\mathbf{y} + \mathbf{b}(s))ds \end{array}$$

This shows that $\mathbf{y}(t)$ is a solution to the integral equation (5) and therefore a solution to (3).

**Uniqueness:**
The uniqueness follows from Gronwall Inequality (See Problem 8.11). Suppose that $\mathbf{y}(t)$ and $\mathbf{z}(t)$ are two solutions to the initial value problem, it follows that for all $a < t < b$ we have

$$||\mathbf{y}(t) - \mathbf{z}(t)|| \le \int_{t_0}^{t} K||\mathbf{y}(s) - \mathbf{z}(s)||ds$$

Letting $u(t) = ||\mathbf{y}(t) - \mathbf{z}(t)||$ we have

$$u(t) \le \int_{t_0}^{t} Ku(s)ds$$

so that by Gronwall's inequality $u(t) \equiv 0$ and therefore $\mathbf{y}(t) = \mathbf{z}(t)$ for all $a < t < b$. This completes a proof of the theorem ■

**Example 29.1**
Find the largest interval where

$$(t^2 - 16)y^{(4)} + 2y'' + t^2 y = \sec t,\ y(3) = 1,\ y'(3) = 3,\ y''(3) = -1$$

is guaranteed to have a unique solution.

**Solution.**
We first put it into standard form

$$y^{(4)} + \frac{2}{t^2 - 16}y'' + \frac{t^2}{t^2 - 16}y = \frac{sect}{t^2 - 16}$$

The coefficient functions are continuous for all $t \neq \pm 4$ and $t \neq (2n+1)\frac{\pi}{2}$. Since $t_0 = 3$, the largest interval where the given initial value problem is guaranteed to have a unique solution is the ineterval $\frac{\pi}{2} < t < 4$ ■

## Practice Problems

For Problems 29.1 - 29.3, use Theorem 29.1 to find the largest interval $a < t < b$ in which a unique solution is guaranteed to exist.

**Problem 29.1**

$$y''' - \frac{1}{t^2 - 9}y'' + \ln(t+1)y' + (\cos t)y = 0,\ y(0) = 1,\ y'(0) = 3,\ y''(0) = 0$$

**Problem 29.2**

$$y''' + \frac{1}{t+1}y' + (\tan t)y = 0,\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 2$$

**Problem 29.3**

$$y'' - \frac{1}{t^2 + 9}y'' + \ln(t^2+1)y' + (\cos t)y = 0,\ y(0) = 1,\ y'(0) = 3,\ y''(0) = 0$$

**Problem 29.4**
Determine the value(s) of $r$ so that $y(t) = e^{rt}$ is a solution to the differential equation

$$y''' - 2y'' - y' + 2y = 0$$

**Problem 29.5**
Transform the following third-order equation

$$y''' - 3ty' + (\sin 2t)y = 7e^{-t}$$

into a first order system of the form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t)$$

# 30 The General Solution of nth Order Linear Homogeneous Equations

In this section we consider the question of solving the homogeneous equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0 \qquad (6)$$

where $p_0(t),\ p_1(t), \cdots,\ p_{n-1}(t)$ are continuous functions in the interval $a < t < b$.
The next theorem shows that any linear combination of solutions to the homogeneous equation is also a solution.
In what follows and for the simplicity of arguments we introduce the function $L$ defined by

$$L[y] = y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y$$

**Theorem 30.1** *(Linearity)*
If $y_1$ and $y_2$ are $n$ times differentiable and $\alpha_1$ and $\alpha_2$ are scalars then $L$ satisfies the property

$$L[\alpha_1 y_1 + \alpha_2 y_2] = \alpha_1 L[y_1] + \alpha_2 L[y_2]$$

**Proof.**
Indeed, we have

$$\begin{array}{rcc} L[\alpha_1 y_1 + \alpha_2 y_2] & = & (\alpha_1 y_1 + \alpha_2 y_2)^{(n)} + p_{n-1}(t)(\alpha_1 y_1 + \alpha_2 y_2)^{(n-1)} + \cdots \\ & + & p_0(t)(\alpha_1 y_1 + \alpha_2 y_2) \\ & = & (\alpha_1 y_1^{(n)} + \alpha_1 p_{n-1}(t)y_1^{(n-1)} + \cdots + \alpha_1 p_1(t)y_1' + \alpha_1 p_0(t)y_1) \\ & + & (\alpha_2 y_2^{(n)} + \alpha_2 p_{n-1}(t)y_2^{(n-1)} + \cdots + \alpha_2 p_1(t)y_2' + \alpha_2 p_0(t)y_2) \\ & = & \alpha_1(y_1^{(n)} + p_{n-1}(t)y_1^{(n-1)} + \cdots + p_1(t)y_1' + p_0(t)y_1) \\ & + & \alpha_2(y_2^{(n)} + p_{n-1}(t)y_2^{(n-1)} + \cdots + p_1(t)y_2' + p_0(t)y_2) \\ & = & \alpha_1 L[y_1] + \alpha_2 L[y_2] \ \blacksquare \end{array}$$

The above property applies for any number of functions.
An important consequence of this theorem is the following result.

**Corollary 30.1** *(Principle of Superposition)*
If $y_1,\ y_2, \cdots, y_r$ satisfy the homogeneous equation (6) and if $\alpha_1,\ \alpha_2, \cdots,\ \alpha_r$ are any numbers, then

$$y(t) = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_r y_r$$

also satisfies the homogeneous equation (6).

**Proof.**
Since $y_1,\ y_2, \cdots,\ y_r$ are solutions to (6), $L[y_1] = L[y_2] = \cdots = L[y_r] = 0$. Now, using the linearity property of $L$ we have

$$\begin{array}{rcl} L[\alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_r y_r] & = & \alpha_1 L[y_1] + \alpha_2 L[y_2] + \cdots + \alpha_r L[y_r] \\ & = & 0 + 0 + \cdots + 0 = 0 \blacksquare \end{array}$$

The principle of superposition states that if $y_1,\ y_2, \cdots,\ y_r$ are solutions to (6) then any linear combination is also a solution. The next question that we consider is the question of existence of $n$ solutions $y_1, y_2, \cdots, y_n$ of equation (6) such that every solution to (6) can be written as a linear combination of these functions. We call such a set a functions a **fundamental set of solutions.** Note that the number of solutions comprising a fundamental set is equal to the order of the differential equation. Also, note that the general solution to (6) is then given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t).$$

Next, we consider a criterion for testing $n$ solutions for a fundamental set. For that we first introduce the following definition:
For $n$ functions $y_1, y_2, \cdots, y_n$, we define the **Wronskian** of these functions to be the determinant

$$W(t) = \left| \begin{array}{cccc} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ y_1''(t) & y_2''(t) & \cdots & y_n''(t) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{array} \right|$$

**Theorem 30.2** *(Criterion for identifying fundamental sets)*
Let $y_1(t), y_2(t), \cdots, y_n(t)$ be $n$ solutions to the homogeneous equation (6). If there is a $a < t_0 < b$ such that $W(t_0) \neq 0$ then $\{y_1, y_2, \cdots,\ y_n\}$ is a fundamental set of solutions.

**Proof.**
We need to show that if $y(t)$ is a solution to (6) then we can write $y(t)$ as a linear combination of $y_1, y_2(t), \cdots, y_n(t)$. That is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n.$$

So the problem reduces to finding the constants $c_1,\ c_2, \cdots,\ c_n$. These are found by solving the following linear system of $n$ equations in the unknowns $c_1, c_2, \cdots, c_n$:

$$\begin{array}{rcl} c_1y_1(t_0) + c_2y_2(t_0) + \cdots + c_ny_n(t_0) & = & y(t_0) \\ c_1y_1'(t_0) + c_2y_2'(t_0) + \cdots + c_ny_n'(t_0) & = & y'(t_0) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & = & \ldots \\ c_1y_1^{(n-1)}(t) + c_2y_2^{(n-1)}(t) + \cdots + c_ny_n^{(n-1)}(t) & = & y^{(n-1)}(t_0) \end{array}$$

Solving this system using Cramer's rule we find

$$c_i = \frac{W_i(t_0)}{W(t_0)}, \qquad 1 \le i \le n$$

where

$$W_i(t_0) = \left| \begin{array}{ccccccc} y_1(t_0) & y_2(t_0) & \cdots & y(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y'(t_0) & \cdots & y_n'(t_0) \\ y_1''(t_0) & y_2''(t_0) & \cdots & y''(t_0) & \cdots & y_n''(t_0) \\ \vdots & \vdots & \cdots & \vdots & & \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{array} \right|$$

That is, $W_i(t_0)$ is the determinant of $W$ with the ith column being replaced by the right-hand column of the above system. Note that $c_1,\ c_2, \cdots, c_n$ exist since $W(t_0) \neq 0$ ■

As a first application to this result, we establish the existence of fundamental sets

**Theorem 30.3**

The linear homogeneous differential equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0$$

where $p_{n-1}(t), \cdots, p_1(t), p_0(t)$ are continuous functions in $a < t < b$ has a fundamental set $\{y_1, y_2, \cdots, y_n\}$.

**Proof.**

Pick a $t_0$ in the interval $a < t < b$ and consider the $n$ initial value problems

$$y^{(n)}+p_{n-1}(t)y^{(n-1)}+\cdots+p_1(t)y'+p_0(t)y = 0,\ y(t_0) = 1, y'(t_0) = 0,\ y''(t_0) = 0, \cdots, y^{(n-1)}(t_0) = 0$$

$$y^{(n)}+p_{n-1}(t)y^{(n-1)}+\cdots+p_1(t)y'+p_0(t)y=0,\ y(t_0)=0, y'(t_0)=1,\ y''(t_0)=0,\cdots,y^{(n-1)}(t_0)=0$$
$$y^{(n)}+p_{n-1}(t)y^{(n-1)}+\cdots+p_1(t)y'+p_0(t)y=0,\ y(t_0)=0, y'(t_0)=0,\ y''(t_0)=1,\cdots,y^{(n-1)}(t_0)=0$$
$$\vdots$$
$$y^{(n)}+p_{n-1}(t)y^{(n-1)}+\cdots+p_1(t)y'+p_0(t)y=0,\ y(t_0)=0, y'(t_0)=0,\ y''(t_0)=0,\cdots,y^{(n-1)}(t_0)=1$$

Then by Theorem 27.1, we can find unique solutions $\{y_1, y_2, \cdots, y_n\}$. This set is a fundamental set by the previous theorem since

$$W(t) = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \neq 0 \blacksquare$$

Theorem 30.2 says that if one can find $a < t_0 < b$ such that $W(t_0) \neq 0$ then the set $\{y_1, y_2, \cdots, y_n\}$ is a fundamental set of solutions. The following theorem shows that the condition $W(t_0) \neq 0$ implies that $W(t) \neq 0$ for all $t$ in the interval $(a, b)$. That is, the theorem tells us that we can choose our test point $t_0$ on the basis of convenience-any test point $t_0$ will do.

**Theorem 30.4** *(Abel's)*
Let $y_1(t), y_2(t), \cdots, y_n$ be $n$ solutions to equation (6). Then
(1) $W(t)$ satisfies the differential equation $W'(t) + p_{n-1}(t)W(t) = 0$;
(2) If $t_0$ is any point in $(a, b)$ then

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p_{n-1}(s)ds}$$

Thus, if $W(t_0) \neq 0$ then $W(t) \neq 0$ for all $a < t < b$.

**Proof.**
(1) By introducing the variables $x_1 = y,\ x_2 = y', x_3 = y'', \cdots, x_n = y^{(n-1)}$ we can write the differential equation as a first order system in the form

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

where

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & -p_2 & -p_3 & \cdots & -p_{n-1} \end{bmatrix}, \ \mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

We will show in Section 33 that for any linear system of the form

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

we have

$$W'(t) = (a_{11} + a_{22} + \cdots + a_{nn})W(t)$$

In our case

$$a_{11} + a_{22} + \cdots + a_{nn} = -p_{n-1}(t)$$

so that

$$W'(t) + p_{n-1}(t)W(t) = 0$$

(2) The previous differential equation can be solved by the method of integrating factor to obtain

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p_{n-1}(s)ds} \quad \blacksquare$$

**Example 30.1**
Use the Abel's formula to find the Wronskian of the DE: $ty''' + 2y'' - t^3y' + e^{t^2}y = 0$

**Solution.**
The original equation can be written as

$$y''' + \frac{2}{t}y'' - t^2y' + \frac{e^{t^2}}{t}y = 0$$

By Abel's formula the Wronskian is

$$W(t) = Ce^{-\int \frac{2}{t}dt} = \frac{C}{t^2} \ \blacksquare$$

**Example 30.2**
Consider the linear system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Show that for any two solutions $Y_1$ and $Y_2$ we have

$$W'(t) = (a_{11} + a_{22})W(t).$$

**Solution.**
Suppose that

$$Y_1 = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

are solutions to the given system. Then we have

$$\begin{array}{rcl} W'(t) & = & \frac{d}{dt}\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \\ & = & \begin{vmatrix} u_1' & v_1' \\ u_2 & v_2 \end{vmatrix} + \begin{vmatrix} u_1 & v_1 \\ u_2' & v_2' \end{vmatrix} \end{array}$$

But

$$\begin{vmatrix} u_1' & v_1' \\ u_2 & v_2 \end{vmatrix} = \begin{vmatrix} a_{11}u_1 + a_{12}u_2 & a_{11}v_1 + a_{12}v_2 \\ u_2 & v_2 \end{vmatrix} = a_{11}W(t)$$

and

$$\begin{vmatrix} u_1 & v_1 \\ u_2' & v_2' \end{vmatrix} = \begin{vmatrix} u_1 & v_1 \\ a_{21}u_1 + a_{22}u_2 & a_{21}v_1 + a_{22}v_2 \end{vmatrix} = a_{22}W(t)$$

It follows that

$$W'(t) = (a_{11} + a_{22})W(t) \blacksquare$$

We end this section by showing that the converse of Theorem 30.2 is also true.

**Theorem 30.5**
If $\{y_1, y_2, \cdots, y_n\}$ is a fundamental set of solutions to

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0$$

where $p_{n-1}(t), \cdots, p_1(t), p_0(t)$ are continuous functions in $a < t < b$ then $W(t) \neq 0$ for all $a < t < b$.

**Proof.**
Let $t_0$ be any point in $(a, b)$. By Theorem 27.1, there is a unique solution $y(t)$ to the initial value problem

$$y^{(n)}+p_{n-1}(t)y^{(n-1)}+\cdots+p_1(t)y'+p_0(t)y = 0,\ y(t_0) = 1,\ y'(t_0) = 0, \cdots, y^{(n-1)}(t_0) = 0$$

Since $\{y_1, y_2, \cdots, y_n\}$ is a fundamental set, there exist unique constants $c_1, c_2, \cdots, c_n$ such that

$$\begin{array}{cccccc} c_1y_1(t) + c_2y_2(t) & \cdots & c_ny_n(t) & = & y(t) \\ c_1y_1'(t) + c_2y_2'(t) & \cdots & c_ny_n'(t) & = & y'(t) \\ \vdots & & & & \\ c_1y_1^{(n-1)}(t) + c_2y_2^{(n-1)}(t) & \cdots & c_ny_n^{(n-1)}(t) & = & y^{(n-1)}(t) \end{array}$$

for all $a < t < b$. In particular for $t = t_0$ we obtain the system

$$\begin{array}{cccccc} c_1y_1(t) + c_2y_2(t) & \cdots & c_ny_n(t) & = & 1 \\ c_1y_1'(t) + c_2y_2'(t) & \cdots & c_ny_n'(t) & = & 0 \\ \vdots & & & & \\ c_1y_1^{(n-1)}(t) + c_2y_2^{(n-1)}(t) & \cdots & c_ny_n^{(n-1)}(t) & = & 0 \end{array}$$

This system has a unique solution $(c_1, c_2, \cdots, c_n)$ where

$$c_i = \frac{W_i}{W(t_0)}$$

and $W_i$ is the determinant $W$ with the ith column replaced by the column

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Note that for $c_1, c_2, \cdots, c_n$ to exist we must have $W(t_0) \neq 0$. By Abel's theorem we conclude that $W(t) \neq 0$ for all $a < t < b$ ■

## Practice Problems

In Problems 30.1 - 30.3, show that the given solutions form a fundamental set for the differential equation by computing the Wronskian.

**Problem 30.1**

$$y''' - y' = 0,\ y_1(t) = 1,\ y_2(t) = e^t,\ y_3(t) = e^{-t}$$

**Problem 30.2**

$$y^{(4)} + y'' = 0,\ y_1(t) = 1,\ y_2(t) = t,\ y_3(t) = \cos t,\ y_4(t) = \sin t$$

**Problem 30.3**

$$t^2y''' + ty'' - y' = 0,\ y_1(t) = 1,\ y_2(t) = \ln t,\ y_3(t) = t^2$$

Use the fact that the solutions given in Problems 30.1 - 30.3 for a fundamental set of solutions to solve the following initial value problems.

**Problem 30.4**

$$y''' - y' = 0,\ y(0) = 3,\ y'(0) = -3,\ y''(0) = 1$$

**Problem 30.5**

$$y^{(4)} + y'' = 0,\ y(\tfrac{|pi}{2}) = 2 + \pi,\ y'(\tfrac{\pi}{2}) = 3,\ y''(\tfrac{\pi}{2}) = -3,\ y'''(\tfrac{\pi}{2}) = 1.$$

**Problem 30.6**

$$t^2y''' + ty'' - y' = 0, ,\ y(1) = 1,\ y'(1) = 2,\ y''(1) = -6$$

**Problem 30.7**
In each question below, show that the Wronskian determinant $W(t)$ behaves as predicted by Abel's Theorem. That is, for the given value of $t_0$, show that

$$W(t) = W(t_0)e^{-\int_{t_0}^{t} p_{n-1}(s)ds}$$

(a) $W(t)$ found in Problem 30.1 and $t_0 = -1$.
(b) $W(t)$ found in Problem 30.2 and $t_0 = 1$.
(c) $W(t)$ found in Problem 30.3 and $t_0 = 2$.

**Problem 30.8**
Determine $W(t)$ for the differential equation $y''' + (\sin t)y'' + (\cos t)y' + 2y = 0$ such that $W(1) = 0$.

**Problem 30.9**
Determine $W(t)$ for the differential equation $t^3y''' - 2y = 0$ such that $W(1) = 3$.

**Problem 30.10**
Consider the initial value problem

$$y''' - y' = 0,\ y(0) = \alpha,\ y'(0) = \beta,\ y''(0) = 4.$$

The general solution of the differential equation is $y(t) = c_1 + c_2e^t + c_3e^{-t}$.
(a) For what values of $\alpha$ and $\beta$ will $\lim_{t\to\infty} y(t) = 0$?
(b) For what values $\alpha$ and $\beta$ will the solution $y(t)$ be bounded for $t \geq 0$, i.e., $|y(t)| \leq M$ for all $t \geq 0$ and for some $M > 0$? Will any values of $\alpha$ and $\beta$ produce a solution $y(t)$ that is bounded for all real number $t$?

**Problem 30.11**
Consider the differential equation $y''' + p_2(t)y'' + p_1(t)y' = 0$ on the interval $-1 < t < 1$. Suppose it is known that the coefficient functions $p_2(t)$ and $p_1(t)$ are both continuous on $-1 < t < 1$. Is it possible that $y(t) = c_1 + c_2t^2 + c_3t^4$ is the general solution for some functions $p_1(t)$ and $p_2(t)$ continuous on $-1 < t < 1$?
(a) Answer this question by considering only the Wronskian of the functions $1, t^2, t^4$ on the given interval.
(b) Explicitly determine functions $p_1(t)$ and $p_2(t)$ such that $y(t) = c_1 + c_2t^2 + c_3t^4$ is the general solution of the differential equation. Use this information, in turn, to provide an alternative answer to the question.

**Problem 30.12**
(a) Find the general solution to $y''' = 0$.
(b) Using the general solution in part (a), construct a fundamental set $\{y_1(t), y_2(t), y_3(t)\}$ satisfying the following conditions

$$\begin{array}{lll} y_1(1) = 1, & y_1'(1) = 0, & y_1''(1) = 0. \\ y_2(1) = 0, & y_1'(1) = 1, & y_1''(1) = 0. \\ y_1(1) = 0, & y_1'(1) = 0, & y_1''(1) = 1. \end{array}$$

## 31 Fundamental Sets and Linear Independence

In Section 30 we established the existence of fundamental sets. There remain two questions that we would like to answer. The first one is about the number of fundamental sets. That is how many fundamental sets are there. It turns out that there are more than one. In this case, our second question is about how these sets are related. In this section we turn our attention to these questions.

We start this section by the following observation. Suppose that the Wronskian of $n$ solutions $\{y_1, y_2, \cdots, y_n\}$ to the nth order linear homogeneous differential equation is zero. In terms of linear algebra, this means that one of the columns of $W$ can be written as a linear combination of the remaining columns. For the sake of argument, suppose that the last column is a linear combination of the remaining columns:

$$\begin{bmatrix} y_n \\ y_n' \\ \vdots \\ y_n^{(n-1)} \end{bmatrix} = c_1 \begin{bmatrix} y_1 \\ y_1' \\ \vdots \\ y_1^{(n-1)} \end{bmatrix} + c_2 \begin{bmatrix} y_2 \\ y_2' \\ \vdots \\ y_2^{(n-1)} \end{bmatrix} + \cdots + c_{n-1} \begin{bmatrix} y_{n-1} \\ y_{n-1}' \\ \vdots \\ y_{n-1}^{(n-1)} \end{bmatrix}$$

This implies that

$$y_n(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_{n-1} y_{n-1}(t).$$

Such a relationship among functions merit a name:

We say that the functions $f_1, f_2, \cdots, f_m$ are **linearly dependent on an interval I** if at least one of them can be expressed as a linear combination of the others on I; equivalently, they are linearly dependent if there exist constants $c_1, c_2, \cdots, c_m$ not all zero such that

$$c_1 f_1(t) + c_2 f_2(t) + \cdots + c_m f_m(t) = 0 \tag{7}$$

for all $t$ in I. A set of functions that is not linearly dependent is said to be **linearly independent.** This means that a sum of the form (7) implies that $c_1 = c_2 = \cdots = c_m = 0$.

**Example 31.1**

Show that the functions $f_1(t) = e^t$, $f_2(t) = e^{-2t}$, and $f_3(t) = 3e^t - 2e^{-2t}$ are linearly dependent on $(-\infty, \infty)$.

**Solution.**
Since $f_3(t) = 3f_1(t) - 2f_2(t)$, the given functions are linearly dependent on $(-\infty, \infty)$ ■

The concept of linear independence can be used to test a fundamental set of solutions to the equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0 \tag{8}$$

**Theorem 31.1**
The solution set $\{y_1, y_2, \cdots, y_n\}$ is a fundamental set of solutions to

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0$$

where $p_{n-1}(t), \cdots, p_1(t), p_0(t)$ are continuous functions in $a < t < b$ if and only if the functions $y_1,\ y_2, \cdots, y_n$ are linearly independent.

**Proof.**
Suppose first that $\{y_1, y_2, \cdots, y_n\}$ is a fundamental set of solutions. Then by Theorem 28.5 there is $a < t_0 < b$ such that $W(t_0) \neq 0$. Suppose that

$$c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) = 0$$

for all $a < t < b$. By repeated differentiation of the previous equation we find

$$\begin{array}{rcl} c_1y_1'(t) + c_2y_2'(t) + \cdots + c_ny_n'(t) & = & 0 \\ c_1y_1''(t) + c_2y_2''(t) + \cdots + c_ny_n''(t) & = & 0 \\ \vdots & & \\ c_1y_1^{(n-1)}(t) + c_2y_2^{(n-1)}(t) + \cdots + c_ny_n^{(n-1)}(t) & = & 0 \end{array}$$

Thus, one finds $c_1,\ c_2, \cdots, c_n$ by solving the system

$$\begin{array}{rcl} c_1y_1(t_0) + c_2y_2(t_0) + \cdots + c_ny_n(t_0) & = & 0 \\ c_1y_1'(t_0) + c_2y_2'(t_0) + \cdots + c_ny_n'(t_0) & = & 0 \\ c_1y_1''(t_0) + c_2y_2''(t_0) + \cdots + c_ny_n''(t_0) & = & 0 \\ \vdots & & \\ c_1y_1^{(n-1)}(t_0) + c_2y_2^{(n-1)}(t_0) + \cdots + c_ny_n^{(n-1)}(t_0) & = & 0 \end{array}$$

Solving this system using Cramer's rule one finds

$$c_1 = c_2 = \cdots, c_n = \frac{0}{W(t_0)} = 0$$

Thus, $y_1(t), y_2(t), \cdots, y_n(t)$ are linearly independent.
Conversely, suppose that $\{y_1, y_2, \cdots, y_n\}$ is a linearly independent set. Suppose that $\{y_1, y_2, \cdots, y_n\}$ is not a fundamental set of solutions. Then by Theorem 30.2, $W(t) = 0$ for all $a < t < b$. Choose any $a < t_0 < b$. Then $W(t_0) = 0$. But this says that the matrix

$$\left[\begin{array}{cccc} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{array}\right]$$

is not invertible which means that there exist $c_1,\ c_2, \cdots, c_n$ not all zero such that

$$\begin{array}{rcl} c_1y_1(t_0) + c_2y_2(t_0) + \cdots + c_ny_n(t_0) & = & 0 \\ c_1y_1'(t_0) + c_2y_2'(t_0) + \cdots + c_ny_n'(t_0) & = & 0 \\ c_1y_1''(t_0) + c_2y_2''(t_0) + \cdots + c_ny_n''(t_0) & = & 0 \\ \vdots & & \\ c_1y_1^{(n-1)}(t_0) + c_2y_2^{(n-1)}(t_0) + \cdots + c_ny_n^{(n-1)}(t_0) & = & 0 \end{array}$$

Now, let $y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)$ for all $a < t < b$. Then $y(t)$ is a solution to the differential equation and $y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0$. But the zero function also is a solution to the initial value problem. By the existence and uniqueness theorem (i.e, Theorem 29.1) we must have $c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) = 0$ for all $a < t < b$ with $c_1,\ c_2, \cdots,\ c_n$ not all equal to 0. But this means that $y_1,\ y_2, \cdots, y_n$ are linearly depedent which contradicts our assumption that $y_1,\ y_2, \cdots,\ y_n$ are linearly independent ■

**Remark 31.1**
The fact that $\{y_1, y_2, \cdots, y_n\}$ are solutions is very critical. That is, if $y_1, y_2, \cdots, y_n$ are merely differentiable functions then it is possible for them to be linearly independent and yet have a vanishing Wronskian. See Section 17.

Next, we will show how to generate new fundamental sets from a given one and therefore establishing the fact that a linear homogeneous differential equation has many fundamental sets of solutions. We also show how different fundamental sets are related to each other. For this, let us start with a fundamental set $\{y_1, y_2, \cdots, y_n\}$ of solutions to (8). If $\{\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_n\}$ are $n$ solutions then they can be written as linear combinations of the $\{y_1, y_2, \cdots, y_n\}$.

That is,

$$\begin{array}{rcl} a_{11}y_1 + a_{21}y_2 + \cdots + a_{n1}y_n & = & \overline{y}_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{n2}y_n & = & \overline{y}_2 \\ \vdots & & \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{nn}y_n & = & \overline{y}_n \end{array}$$

or in matrix form as

$$\left[ \begin{array}{cccc} \overline{y}_1 & \overline{y}_2 & \cdots & \overline{y}_n \end{array} \right] = \left[ \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array} \right] \left[ \begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{array} \right]$$

**Theorem 31.2**
$\{\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_n\}$ is a fundamental set if and only if $\det(\mathbf{A}) \neq 0$ where $\mathbf{A}$ is the coefficient matrix of the above matrix equation.

**Proof.**
By differentiating (n-1) times the system

$$\begin{array}{rcl} a_{11}y_1 + a_{21}y_2 + \cdots + a_{n1}y_n & = & \overline{y}_1 \\ a_{12}y_1 + a_{22}y_2 + \cdots + a_{n2}y_n & = & \overline{y}_2 \\ \vdots & & \\ a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{nn}y_n & = & \overline{y}_n \end{array}$$

one can easily check that

$$\left[ \begin{array}{cccc} \overline{y}_1 & \overline{y}_2 & \cdots & \overline{y}_n \\ \overline{y}'_1 & \overline{y}'_2 & \cdots & \overline{y}'_n \\ \vdots & \vdots & \cdots & \vdots \\ \overline{y}_1^{(n-1)} & \overline{y}_2^{(n-1)} & \cdots & \overline{y}_n^{(n-1)} \end{array} \right] = \left[ \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{array} \right] \left[ \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right]$$

By taking the determinant of both sides and using the fact that the determinant of a product is the product of determinants then we can write

$$\overline{W}(t) = det(\mathbf{A})W(t)$$

Since $W(t) \neq 0$, $\overline{W}(t) \neq 0$ (i.e., $\{\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_n\}$ is a fundamental set) if and only if $det(\mathbf{A}) \neq 0$ ■

**Example 31.2**
The set $\{y_1(t), y_2(t), y_3(t)\} = \{1, e^t, e^{-t}\}$ is fundamental set of solutions to the differential equation

$$y''' - y' = 0$$

(a) Show that $\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\} = \{\cosh t, 1-\sinh t, 2+\sinh t\}$ is a solution set.
(b) Determine the coefficient matrix **A** described in the previous theorem.
(c) Determine whether this set is a fundamental set by calculating the determinant of the matrix **A**.

**Solution.**
(a) Since $\overline{y}_1' = \sinh t, \overline{y}_1'' = \cosh t$, and $\overline{y}_1'''(t) = \sinh t$ we have $\overline{y}_1''' - \overline{y}_1' = 0$ so that $\overline{y}_1$ is a solution. A similar argument holds for $\overline{y}_2$ and $\overline{y}_3$.
(a) Since $\overline{y}_1(t) = 0 \cdot 1 + \frac{1}{2} \cdot e^t + \frac{1}{2} \cdot e^{-t},\ \overline{y}_2(t) = 1 \cdot 1 - \frac{1}{2}e^t + \frac{1}{2} \cdot e^{-t},\ \overline{y}_3(t) = 2 \cdot 1 + \frac{1}{2} \cdot e^t - \frac{1}{2} \cdot e^{-t}$ we have

$$\mathbf{A} = \left[\begin{array}{ccc} 0 & 1 & 2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{array}\right]$$

(c) One can easily find that $det(\mathbf{A}) = \frac{3}{2} \neq 0$ so that $\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\}$ is a fundemantal set of solutions ■

## Practice Problems

**Problem 31.1**
Determine if the following functions are linearly independent

$$y_1(t) = e^{2t}, \quad y_2(t) = \sin(3t), \quad y_3(t) = \cos t$$

**Problem 31.2**
Determine whether the three functions : $f(t) = 2, g(t) = \sin^2 t, h(t) = \cos^2 t$, are linearly dependent or independent on $-\infty < t < \infty$

**Problem 31.3**
Determine whether the functions, $y_1(t) = 1; y_2(t) = 1 + t; y_3(t) = 1 + t + t^2$; are linearly dependent or independent. Show your work.

**Problem 31.4**
Consider the set of functions $\{y_1(t), y_2(t), y_3(t)\} = \{t^2+2t, \alpha t+1, t+\alpha\}$. For what value(s) $\alpha$ is the given set linearly depedent on the interval $-\infty < t < \infty$?

**Problem 31.5**
Determine whether the set $\{y_1(t), y_2(t), y_3(t)\} = \{t|t|+1, t^2-1, t\}$ is linearly independent or linearly dependent on the given interval
(a) $0 \le t < \infty$.
(b) $-\infty < t \le 0$.
(c) $-\infty < t < \infty$.

In Problems 31.6 - 31.7, for each differential equation, the corresponding set of functions $\{y_1(t), y_2(t), y_3(t)\}$ is a fundamental set of solutions.
(a) Determine whether the given set $\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\}$ is a solution set to the differential equation.
(b) If $\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\}$ is a solution set then find the coefficient matrix $\mathbf{A}$ such that

$$\begin{bmatrix} \overline{y}_1 \\ \overline{y}_2 \\ \overline{y}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

(c) If $\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\}$ is a solution set, determine whether it is a fundamental set by calculating the determinant of $\mathbf{A}$.

**Problem 31.6**

$$y''' + y'' = 0$$
$$\{y_1(t), y_2(t), y_3(t)\} = \{1, t, e^{-t}\}$$
$$\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\} = \{1 - 2t, t + 2, e^{-(t+2)}\}$$

**Problem 31.7**

$$t^2y''' + ty'' - y' = 0, t > 0$$
$$\{y_1(t), y_2(t), y_3(t)\} = \{t, \ln t, t^2\}$$
$$\{\overline{y}_1(t), \overline{y}_2(t), \overline{y}_3(t)\} = \{2t^2 - 1, 3, \ln(t^3)\}$$

# 32 Higher Order Homogeneous Linear Equations with Constant Coefficients

In this section we investigate how to solve the nth order linear homogeneous equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0 = 0 \tag{9}$$

The general solution is given by

$$y(t) = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

where $\{y_1, y_2, \cdots, y_n\}$ is a fundamental set of solutions.
What was done for second-order, linear homogeneous equations with constant coefficients holds, with the obvious modifications, for higher order analogs. As for the second order case, we seek solutions of the form $y(t) = e^{rt}$, where $r$ is a constant (real or complex-valued) to be found. Inserting into (9) we find

$$(r^n + a_{n-1}r^{n-1} + \cdots a_1r + a_0)e^{rt} = 0$$

We call $P(r) = r^n + a_{n-1}r^{n-1} + \cdots a_1r + a_0$ the **characteristic polynomial** and the equation

$$r^n + a_{n-1}r^{n-1} + \cdots a_1r + a_0 = 0 \tag{10}$$

the **characteristic equation.** Thus, for $y(t) = e^{rt}$ to be a solution to (9) $r$ must satisfy (11).

**Example 32.1**
Solve: $y''' - 4y'' + y' + 6y = 0$

**Solution.**
The characteristic equation is

$$r^3 - 4r^2 + r + 6 = 0$$

We can factor to find the roots of the equation. A calculator can efficiently do this, or you can use the rational root theorem to get

$$(r + 1)(r - 2)(r - 3) = 0$$

Thus, the roots are

$$r = -1, \quad r = 2, \quad r = 3$$

The Wronskian

$$\begin{vmatrix} e^{-t} & e^{2t} & e^{3t} \\ -e^{-t} & 2e^{2t} & 3e^{3t} \\ e^{-t} & 4e^{2t} & 9e^{3t} \end{vmatrix} = 12e^{4t} \neq 0$$

Hence, $\{e^{-t}, e^{2t}, e^{3t}\}$ is a fundamental set of solutions and the general solution is

$$y = c_1e^{-t} + c_2e^{2t} + c_3e^{3t} \blacksquare$$

In the previous example, the characteristic solution had three distinct roots and the corresponding set of solutions formed a fundamental set. This is always true according to the following theorem.

**Theorem 32.1**
Assume that the characteristic equation

$$r^n + a_{n-1}r^{n-1} + \cdots a_1r + a_0 = 0$$

has $n$ distinct roots $r_1, r_2, \cdots, r_n$ (real valued or complex valued). Then the set of solutions $\{e^{r_1t}, e^{r_2t}, \cdots, e^{r_nt}\}$ is a fundamental set of solution to the equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0 = 0$$

**Proof.**
For a fixed number $t_0$ we consider the Wronskian

$$W(t_0) = \begin{vmatrix} e^{r_1t_0} & e^{r_2t_0} & \cdots & e^{r_nt_0} \\ r_1e^{r_1t_0} & r_2e^{r_2t_0} & \cdots & r_ne^{r_nt_0} \\ r_1^2e^{r_1t_0} & r_2^2e^{r_2t_0} & \cdots & r_n^2e^{r_nt_0} \\ \vdots & & & \vdots \\ r_1^{n-1}e^{r_1t_0} & r_2^{n-1}e^{r_2t_0} & \cdots & r_n^{n-1}e^{r_nt_0} \end{vmatrix}$$

Now, in linear algebra one proves that if a row or a column of a matrix is multiplied by a constant then the determinant of the new matrix is the determinant of the old matrix multiplied by that constant. It follows that

$$W(t) = e^{r_1t_0}e^{r_2t_0}\cdots e^{r_nt_0} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & & & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{vmatrix}$$

The resulting determinant above is the well-known **Vandermonde determinant**. Its value is the product of all factors of the form $r_j - r_i$ where $j > i$. Since $r_j \neq r_i$ for $i \neq j$, this determinant is not zero and consequently $W(t_0) \neq 0$. This establishes that $\{e^{r_1 t}, e^{r_2 t}, \cdots, e^{r_n t}\}$ is a fundamental set of solutions ■

Next, we consider characteristic equations whose roots are not all distinct. For example, if $\alpha$ is a real root that appears $k$ times (in this case we say that $\alpha$ is a root of **multiplicity** $k$), that is, $P(r) = (r-\alpha)^k q(r)$, where $q(\alpha) \neq 0$, then the $k$ linearly independent solutions are given by

$$e^{\alpha t},\ te^{\alpha t},\ t^2 e^{\alpha t}, \cdots,\ t^{k-1}e^{\alpha t}$$

The remaining $n-k$ solutions needed to complete the fundamental set of solutions are determined by examining the roots of $q(r) = 0$.
If, on the other hand, $\alpha \pm i\beta$ are conjugate complex roots each of multiplicity $k$, that is

$$P(r) = (r - r_1)^k (r - \overline{r}_1)^k p(r)$$

where $r_1 = \alpha + i\beta$ and $p(r_1) \neq 0,\ p(\overline{r}_1) \neq 0$ then the 2k linearly independent solutions are given by

$$e^{\alpha t}\cos\beta t,\ te^{\alpha t}\cos\beta t, \cdots,\ t^{k-1}e^{\alpha t}\cos\beta t$$

and

$$e^{\alpha t}\sin\beta t,\ te^{\alpha t}\sin\beta t, \cdots,\ t^{k-1}e^{\alpha t}\sin\beta t$$

**Example 32.2**
Find the solution to

$$y^{(5)} + 4y''' = 0, \quad y(0) = 2, \quad y'(0) = 3, \quad y''(0) = 1, \quad y'''(0) = -1,\ y(4)(0) = 1$$

**Solution.**
We have the characteristic equation

$$r^5 + 4r^3 = r^3(r^2 + 4) = 0$$

Which has a root of multiplicity 3 at $r = 0$ and complex roots $r = 2i$ and $r = -2i$. We use what we have learned about repeated roots and complex roots to get the general solution. Since the multiplicity of the repeated root is 3, we have

$$y_1(t) = 1, \quad y_2(t) = t, \quad y_3(t) = t^2.$$

The complex roots give the other two solutions

$$y_4(t) = \cos(2t) \text{ and } y_5(t) = \sin(2t)$$

The general solution is

$$y(t) = c_1 + c_2 t + c_3 t^2 + c_4 \cos(2t) + c_5 \sin(2t)$$

Now Find the first four derivatives

$$\begin{array}{rcl} y'(t) & = & c_2 + 2c_3 t - 2c_4 \sin(2t) + 2c_5 \cos(2t) \\ y''(t) & = & 2c_3 - 4c_4 \cos(2t) - 4c_5 \sin(2t) \\ y'''(t) & = & 8c_4 \sin(2t) - 8c_5 \cos(2t) \\ y(4)(t) & = & 16c_4 \cos(2t) + 16c_5 \sin(2t) \end{array}$$

Next plug in the initial conditions to get

$$\begin{array}{rcl} 2 & = & c_1 + c_4 \\ 3 & = & c_2 + 2c_5 \\ 1 & = & 2c_3 - 4c_4 \\ -1 & = & 8c_5 \\ 1 & = & 16c_4 \end{array}$$

Solving these equations we find

$$c_1 = 31/16, \quad c_2 = 23/4, \quad c_3 = 5/8, \quad c_4 = 1/16, \quad c_5 = 1/8$$

The solution is

$$y(t) = \frac{31}{15} + \frac{23}{4} t + \frac{5}{8} t^2 + \frac{1}{16} \cos(2t) + \frac{1}{8} \sin(2t) \blacksquare$$

**Solving the Equation** $y^{(n)} - ay = 0$.
The characteristic equation corresponding to the differential equation $y^{(n)} - ay = 0$ is $r^n - a = 0$. The fundamental theorem of algebra asserts the existence of exactly $n$ roots (real or complex-valued). To find these roots, we write $a$ in polar form $a = |a|e^{i\alpha}$ where $\alpha = 0$ if $a > 0$ and $\alpha = \pi$ is $a < 0$ (since $e^{i\pi} = \cos\pi + i\sin\pi = -1$). Also, since $e^{i2k\pi} = 1$ for any integer $k$ then we can write

$$a = |a|e^{(\alpha + 2k\pi)i}$$

Thus, the charactersitic equation is

$$r^n = |a|e^{(\alpha + 2k\pi)i}.$$

Taking the nth root of both sides we find

$$r = |a|^{\frac{1}{n}} e^{\frac{(\alpha+2k\pi)i}{n}}.$$

The $n$ distinct roots are generated by taking $k = 0, 1, 2, \cdots, n-1$. We illustrate this in the next example.

**Example 32.3**
Find the general solution of $y^{(6)} + y = 0$.

**Solution.**
In this case the characteristic equation is $r^6 + 1 = 0$ or $r^6 = -1 = e^{i(2k+1)\pi}$. Thus, $r = e^{i\frac{(2k+1)\pi}{6}}$ where $k$ is an integer. Replacing $k$ by 0,1,2,3,4,5 we find

$$\begin{array}{rcl} r_0 &=& \frac{\sqrt{3}}{2} + \frac{i}{2} \\ r_1 &=& i \\ r_2 &=& -\frac{\sqrt{3}}{2} + \frac{i}{2} \\ r_3 &=& -\frac{\sqrt{3}}{2} - \frac{i}{2} \\ r_4 &=& -i \\ r_5 &=& \frac{\sqrt{3}}{2} - \frac{i}{2} \end{array}$$

Thus, the general solution is

$$y(t) = c_1 e^{\frac{\sqrt{3}}{2}t} \cos\frac{t}{2} + c_2 e^{\frac{\sqrt{3}}{2}t} \sin\frac{t}{2} + c_3 e^{\frac{-\sqrt{3}}{2}t} \cos\frac{t}{2} + c_4 e^{\frac{\sqrt{3}}{2}t} \sin\frac{t}{2} + c_5 \cos t + c_6 \sin t \blacksquare$$

## Practice Problems

**Problem 32.1**
Solve $y''' + y'' - y' - y = 0$

**Problem 32.2**
Find the general solution of $16y^{(4)} - 8y'' + y = 0$.

**Problem 32.3**
Solve the following constant coefficient differential equation :

$$y''' - y = 0.$$

**Problem 32.4**
Solve $y^{(4)} - 16y = 0$

**Problem 32.5**
Solve the initial-value problem

$$y''' + 3y'' + 3y' + y = 0,\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 0.$$

**Problem 32.6**
Given that $r = 1$ is a solution of $r^3 + 3r^2 - 4 = 0$, find the general solution to

$$y''' + 3y'' - 4y = 0$$

**Problem 32.7**
Given that $y_1(t) = e^{2t}$ is a solution to the homogeneous equation, find the general solution to the differential equation

$$y''' - 2y'' + y' - 2y = 0$$

**Problem 32.8**
Suppose that $y(t) = c_1 \cos t + c_2 \sin t + c_3 \cos(2t) + c_4 \sin(2t)$ is the general solution to the equation

$$y^{(4)} + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0$$

Find the constants $a_0, a_1, a_2$, and $a_3$.

**Problem 32.9**
Suppose that $y(t) = c_1 + c_2t + c_3 \cos 3t + c_4 \sin 3t$ is the general solution to the homogeneous equation

$$y^{(4)} + a_3y''' + a_2y'' + a_1y' + a_0y = 0$$

Determine the values of $a_0,\ a_1,\ a_2$, and $a_3$.

**Problem 32.10**
Suppose that $y(t) = c_1e^{-t} \sin t + c_2e^{-t} \cos t + c_3e^t \sin t + c_4e^t \cos t$ is the general solution to the homogeneous equation

$$y^{(4)} + a_3y''' + a_2y'' + a_1y' + a_0y = 0$$

Determine the values of $a_0,\ a_1,\ a_2$, and $a_3$.

**Problem 32.11**
Consider the homogeneous equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0 = 0$$

Suppose that $y_1(t) = t,\ y_2(t) = e^t, y_3(t) = \cos t$ are several functions belonging to a fundamental set of solutions to this equation. What is the smallest value for $n$ for which the given functions can belong to such a fundamental set? What is the fundamemtal set?

**Problem 32.12**
Consider the homogeneous equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0 = 0$$

Suppose that $y_1(t) = t^2 \sin t,\ y_2(t) = e^t \sin t$ are several functions belonging to a fundamental set of solutions to this equation. What is the smallest value for $n$ for which the given functions can belong to such a fundamental set? What is the fundamemtal set?

**Problem 32.13**
Consider the homogeneous equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0 = 0$$

Suppose that $y_1(t) = t^2,\ y_2(t) = e^{2t}$ are several functions belonging to a fundamental set of solutions to this equation. What is the smallest value for $n$ for which the given functions can belong to such a fundamental set? What is the fundamemtal set?

# 33 Non Homogeneous nth Order Linear Differential Equations

We consider again the nth order linear nonhomogeneous equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t) \tag{11}$$

where the functions $p_0,\ p_1,\ \cdots,\ p_{n-1}$ and $g(t)$ are continuous functions for $a < t < b$.
The solution structure established for second order linear nonhomogeneous equations applies as well in the nth order case.

**Theorem 33.1**
Let $\{y_1(t), y_2(t), \cdots, y_n(t)\}$ be a fundamental set of solutions to the homogeneous equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0$$

and $y_p(t)$ be a particular solution of the nonhomogeneous equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t).$$

The general solution of the nonhomogeneous equation is given by

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)$$

**Proof.**
Let $y(t)$ be any solution to equation (11). Since $y_p(t)$ is also a solution, we have

$$(y-y_p)^{(n)} + p_{n-1}(t)(y-y_p)^{(n-1)} + \cdots + p_1(t)(y-y_p)' + p_0(t)(y-y_p) =$$

$$y^{(n)}+p_{n-1}(t)y^{(n-1)}+\cdots+p_1(t)y'+p_0(t)y-(y_p^{(n)}+p_{n-1}(t)y_p^{(n-1)}+\cdots+p_1(t)y_p'+p_0(t)y_p) =$$

$$g(t) - g(t) = 0$$

Therefore $y-y_p$ is a solution to the homogeneous equation. But $\{y_1, y_2, \cdots, y_n\}$ is a fundamental set of solutions to the homogeneous equation so that there exist unique constants $c_1, c_2, \cdots, c_n$ such that $y(t)-y_p(t) = c_1y_1(t)+c_2y_2(t)+\cdots + c_ny_n(t)$. Hence,

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) \blacksquare$$

Since the sum $c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t)$ represents the general solution to the homogeneous equation, we will denote it by $y_h$ so that the general solution of (11) takes the form

$$y(t) = y_h(t) + y_p(t)$$

It follows from the above theorem that finding the general solution to nonhomogeneous equations consists of three steps:
1. Find the general solution $y_h$ of the associated homogeneous equation.
2. Find a single solution $y_p$ of the original equation.
3. Add together the solutions found in steps 1 and 2.

The superposition of solutions is valid only for homogeneous equations and not true in general for nonhomogeneous equations. (Recall the case $n = 2$ in Section 22). However, we can have a property of superposition of nonhomogeneous if one is adding two solutions of two different nonhomogeneous equations. More precisely, we have

**Theorem 33.2**
Let $y_1(t)$ be a solution of $y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g_1(t)$ and $y_2(t)$ a solution of $y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g_2(t)$. Then for any constants $c_1$ and $c_2$ the function $Y(t) = c_1y_1(t) + c_2y_2(t)$ is a solution of the equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = c_1g_1(t) + c_2g_2(t)$$

**Proof.**
We have

$$\begin{array}{rcl} L[Y] & = & c_1(y_1^{(n)} + p_{n-1}(t)y_1^{(n-1)} + \cdots + p_1(t)y_1' + p_0(t)y_1) \\ & + & c_2(y_2^{(n)} + p_{n-1}(t)y_2^{(n-1)} + \cdots + p_1(t)y_2' + p_0(t)y_2) \\ & = & c_1g_1(t) + c_2g_2(t) \blacksquare \end{array}$$

Next, we discuss methods for determining $y_p(t)$. The techinque we discuss first is known as the **method of undetermined coefficients.**
This method requires that we make an initial assumption about the form of the particular solution $y_p(t)$, but with the coefficients left unspecified, thus the name of the method. We then substitute the assumed expression into equation (11) and attempt to determine the coefficients as to satisfy that

equation.
The main advantage of this method is that it is straightforward to execute once the assumption is made as to the form of $y_p(t)$. Its major limitation is that it is useful only for equations with constant coefficients and the nonhomogeneous term $g(t)$ is restricted to a very small class of functions, namely functions of the form $e^{\alpha t}P_n(t)\cos\beta t$ or $e^{\alpha t}P_n(t)\sin\beta t$ where $P_n(t)$ is a polynomial of degree $n$.

In the following table we list examples of $g(t)$ along with the corresponding form of the particular solution.

| Form of $g(t)$ | Form of $y_p(t)$ |
|---|---|
| $P_n(t) = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_0$ | $t^r[A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0]$ |
| $P_n(t)e^{\alpha t}$ | $t^r[A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0]e^{\alpha t}$ |
| $P_n(t)e^{\alpha t}\cos\beta t$ or $P_n(t)e^{\alpha t}\sin\beta t$ | $t^re^{\alpha t}[(A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0)\cos\beta t$ |
| | $+(B_nt^n + B_{n-1}t^{n-1} + \cdots + B_1t + B_0)\sin\beta t]$ |

The number $r$ is chosen to be the smallest nonnegative integer such that no term in the assumed form is a solution of the homogeneous equation $y^{(n)}+p_{n-1}(t)y^{(n-1)}+\cdots+p_1(t)y'+p_0(t)y = 0$. The value of $r$ will be $0 \le r \le n$. Equivalently, for the three cases, $r$ is the number of times 0 is a root of the characterstic equation, $\alpha$ is a root of the characteristic equation, and $\alpha + i\beta$ is a root of the characteristic equation, respectively.

**Example 33.1**
Solve

$$y''' + y'' = \cos(2t),\ y(0) = 1,\ y'(0) = 2,\ y''(0) = 3$$

**Solution.**
We first solve the homogeneous differential equation

$$y''' + y'' = 0$$

The characteristic equation is

$$r^3 + r^2 = 0$$

Factoring gives

$$r^2(r + 1) = 0$$

Solving we find $r = 0$ (repeated twice) and $r = -1$. The homogeneous solution is

$$y_h(t) = c_1 + c_2 t + c_3 e^{-t}$$

The trial function generated by $g(t) = \cos(2t)$ is

$$y_p(t) = A\cos(2t) + B\sin(2t)$$

Then

$$\begin{array}{rcl} y_p' &=& -2A\cos(2t) + 2B\sin(2t) \\ y_p'' &=& -4A\sin(2t) - 4B\cos(2t) \\ y_p''' &=& -8A\cos(2t) + 8B\sin(2t) \end{array}$$

Plugging back into the original differential equation gives

$$[-8A\cos(2t) + 8B\sin(2t)] + [-4A\sin(2t) - 4B\cos(2t)] = \cos(2t)$$

Combining like terms gives

$$(-8A - 4B)\cos(2t) + (8B - 4A)\sin(2t) = \cos(2t)$$

Equating coefficients gives

$$\begin{array}{rcl} -8A - 4B &=& 1 \\ -4A + 8B &=& 0 \end{array}$$

Solving we find $A = -0.1$ and $B = -0.05$. The general solution is thus

$$y(t) = c_1 + c_2 t + c_3 e^{-t} - 0.1\cos(2t) - 0.05\sin(2t)$$

Now take derivatives to get

$$\begin{array}{rcl} y' &=& c_2 - c_3 e^{-t} + 0.2\sin(2t) - 0.1\cos(2t) \\ y'' &=& c_3 e^{-t} + 0.4\cos(2t) + 0.2\sin(2t) \end{array}$$

Plug in the initial values to get

$$\begin{array}{rcl} c_1 + c_3 &=& 1.1 \\ c_2 - c_3 &=& 2.1 \\ c_3 &=& 2.6 \end{array}$$

Solving we find $c_1 = -3.6,\ c_2 = 4.7,\ c_3 = 2.6$. The final solution is

$$y(t) = -3.6 + 4.7t + 2.6e^{-t} - 0.1\cos(2t) - 0.05\sin(2t) \ \blacksquare$$

Finally, we discuss a second method for finding a particular solution to a nonhomogeneous differential equation known as the **method of variation of parameters**. This method has no prior conditions to be satisfied by either $p_{n-1}(t), \cdots, p_1(t), p_0(t)$, or $g(t)$. Therefore, it may sound more general than the method of undetermined coefficients.
The basic assumption underlying the method is that we know a fundamental set of solutions $\{y_1, y_2, \cdots, y_n\}$. The homogeneous solution is then

$$y_h(t) = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

Then the constants $c_1, c_2, \cdots, c_n$ are being replaced by functions $u_1, u_2, \cdots, u_n$ so that the particular solution assumes the form

$$y_p(t) = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n \tag{12}$$

We find $u_1, u_2, \cdots, u_n$ by solving a system of $n$ equations with the $n$ unknowns $u_1, u_2, \cdots, u_n$. We obtain the system by first imposing the $n-1$ constraints

$$\begin{aligned} y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' &= 0 \\ y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' &= 0 \\ &\vdots \\ y_1^{(n-2)} u_1' + y_2^{(n-2)} u_2' + \cdots + y_n^{(n-2)} u_n' &= 0 \end{aligned} \tag{13}$$

This choice of constraints is made to make the successive derivatives of $y_p(t)$ have the following simple forms

$$\begin{aligned} y_p' &= y_1' u_1 + y_2' u_2 + \cdots + y_n' u_n \\ y_p'' &= y_1'' u_1 + y_2'' u_2 + \cdots + y_n'' u_n \\ &\vdots \\ y_p^{(n-1)} &= y_1^{(n-1)} u_1 + y_2^{(n-1)} u_2 + \cdots + y_n^{(n-1)} u_n \end{aligned}$$

Substituting (12) into (11), using (13) and the fact that each of the functions $y_1, y_2, \cdots, y_n$ is a solution of the homogeneous equation we find

$$y_1^{(n-1)} u_1' + y_2^{(n-1)} u_2' + \cdots + y_n^{(n-1)} u_n' = g(t) \tag{14}$$

Take together, equations (13) and (14) form a set of $n$ linear equations for the $n$ unknowns $u_1', u_2', \cdots, u_n'$. In matrix form that system takes the form

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g \end{bmatrix}$$

Solving this system we find

$$u_i' = \frac{W_i}{W} g$$

where $1 \leq i \leq n, W$ is the Wronskian of $\{y_1, y_2, \cdots, y_n\}$ and $W_i$ is the determinant obtained after replacing the ith column of $W$ with the column vector

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

It follows that

$$y_p(t) = y_1 \int \frac{W_1(t)}{W(t)} g(t)dt + y_2 \int \frac{W_2(t)}{W(t)} g(t)dt + \cdots + y_n \int \frac{W_n(t)}{W(t)} g(t)dt$$

**Example 33.2**
Solve

$$y''' + y' = \sec t$$

**Solution.**
We first find the homogeneous solution. The characteristic equation is

$$r^3 + r = 0 \text{ or } r(r^2 + 1) = 0$$

so that the roots are $r = 0,\ r = i,\ r = -i$.
We conclude

$$y_h(t) = c_1 + c_2 \cos t + c_3 \sin t$$

We have

$$y_p(t) = u_1 + u_2 \cos t + u_3 \sin t$$

and the Wronskian is

$$W(t) = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = 1$$

So

$$W_1(t) = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1$$

$$W_2(t) = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{vmatrix} = -\cos t$$

$$W_3(t) = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} = -\sin t$$

Hence,

$$\begin{aligned} u_1(t) &= \int \tfrac{W_1(t)}{W(t)} g(t)dt = \int \sec t dt = \ln|\sec t + \tan t| \\ u_2(t) &= \int \tfrac{W_2(t)}{W(t)} g(t)dt = \int -dt = -t \\ u_3(t) &= \int \tfrac{W_3(t)}{W(t)} g(t)dt = \int -\tfrac{\sin t}{\cos t} dt = \ln|\cos t| \end{aligned}$$

Hence, the general solution is

$$y(t) = c_1 + c_2 \cos t + c_3 \sin t + \ln|\sec t + \tan t| - t\cos t + \ln|\cos t|\sin t \blacksquare$$

## Practice Problems

**Problem 33.1**
Consider the nonhomogeneous differential equation

$$t^3y''' + at^2y'' + bty' + cy = g(t), \quad t > 0$$

Determine $a, b, c$, and $g(t)$ if the general solution is given by $y(t) = c_1t + c_2t^2 + c_3t^4 + 2\ln t$

**Problem 33.2**
Consider the nonhomogeneous differential equation

$$y''' + ay'' + by' + cy = g(t), \quad t > 0$$

Determine $a, b, c$, and $g(t)$ if the general solution is given by $y(t) = c_1 + c_2t + c_3e^{2t} + 4\sin 2t$

**Problem 33.3**
Solve

$$y^{(4)} + 4y'' = 16 + 15e^t$$

**Problem 33.4**
Solve: $y^{(4)} - 8y'' + 16y = -64e^{2t}$

**Problem 33.5**
Given that $y_1(t) = e^{2t}$ is a solution to the homogeneous equation, find the general solution to the differential equation,

$$y''' - 2y'' + y' - 2y = 12\sin 2t$$

**Problem 33.6**
Find the general solution of the equation

$$y''' - 6y'' + 12y' - 8y = \sqrt{2t}e^{2t}$$

**Problem 33.7**
(a) Verify that $\{t, t^2, t^4\}$ is a fundamental set of solutions of the differential equation

$$t^3y''' - 4t^2y'' + 8ty' - 8y = 0$$

(b) Find the general solution of

$$t^3y''' - 4t^2y'' + 8ty' - 8y = 2\sqrt{t}, \quad t > 0$$

**Problem 33.8**
(a) Verify that $\{t, t^2, t^3\}$ is a fundamental set of solutions of the differential equation

$$t^3y''' - 3t^2y'' + 6ty' - 6y = 0$$

(b) Find the general solution of by using the method of variation of parameters

$$t^3y''' - 3t^2y'' + 6ty' - 6y = t, \quad t > 0$$

**Problem 33.9**
Solve using the method of undetermined coefficients: $y''' - y' = 4 + 2\cos t$

**Problem 33.10**
Solve using the method of undetermined coefficients: $y''' - y' = -4e^t$

**Problem 33.11**
Solve using the method of undetermined coefficients: $y''' - y'' = 4e^{-2t}$

**Problem 33.12**
Solve using the method of undetermined coefficients: $y''' - 3y'' + 3y' - y = 12e^t$.

**Problem 33.13**
Solve using the method of undetermined coefficients: $y''' + y = e^t + \cos t$.

In Problems 33.14 and 33.15, answer the following two questions.
(a) Find the homogeneous general solution.
(b) Formulate an appropriate for for the particular solution suggested by the method of undetermined coefficients. You need not evaluate the undetermined coefficients.

**Problem 33.14**
$y''' - 3y'' + 3y' - y = e^t + 4e^t\cos 3t + 4$

**Problem 33.15**
$y^{(4)} + 8y'' + 16y = t\cos 2t$

Consider the nonhomogeneous differential equation

$$y''' + ay'' + by' + cy = g(t)$$

In Problems 33.16 - 33.17, the general solution of the differential equation is given, where $c_1, c_2$, and $c_3$ represent arbitrary constants. Use this information to determine the constants $a, b, c$ and the function $g(t)$.

**Problem 33.16**
$y(t) = c_1 + c_2t + c_3e^{2t} + 4\sin 2t.$

**Problem 33.17**
$y(t) = c_1 + c_2t + c_3t^2 - 2t^3$

**Problem 33.18**
Consider the nonhomogeneous differential equation

$$t^3y''' + at^2y'' + bty' + cy = g(t), \quad t > 0$$

Suppose that $y(t) = c_1t + c_2t^2 + c_3t^4 + 2\ln t$ is the general solution to the above equation. Determine the constants $a, b, c$ and the function $g(t)$

## 34 Existence and Uniqueness of Solution to Initial Value First Order Linear Systems

In this section we study the following initial-value problem

$$\begin{array}{rcl} y_1' & = & p_{11}(t)y_1 + p_{12}(t)y_2 + \cdots + p_{1n}(t)y_n + g_1(t) \\ y_2' & = & p_{21}(t)y_1 + p_{22}(t)y_2 + \cdots + p_{2n}(t)y_n + g_2(t) \\ \vdots & & \\ y_n' & = & p_{n1}(t)y_1 + p_{n2}(t)y_2 + \cdots + p_{nn}(t)y_n + g_n(t) \end{array}$$

$$y_1(t_0) = y_1^0, \; y_1(t_0) = y_2^0, \cdots, y_n(t_0) = y_n^0, \quad a < t_0 < b$$

where all the $p_{ij}(t)$ and $g_i(t)$ functions are continuous in $a < t < b$. The above system can be recast in matrix form as

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t), \quad \mathbf{y}(t_0) = \mathbf{y_0} \qquad (15)$$

where

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}, \quad \mathbf{g}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}, \quad \mathbf{y_0} = \begin{bmatrix} y_1^0 \\ y_2^0 \\ \vdots \\ y_n^0 \end{bmatrix}$$

and

$$\mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & & & \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{bmatrix}$$

We refer to differential equation in (15) as a **first order linear system**. If $\mathbf{g(t)}$ is the zero vector in $a < t < b$ then we call

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$$

a **first order homogeneous linear system**. Otherwise, we call the system a **first order nonhomogengeous linear system.**

Next we discuss the conditions required for (15) to have a unique solution. In order to establish the next theorem we state an important result from analysis.

**Theorem 34.1** *(Weierstrass M-Test)*
Assume $\{y_N(t)\}_{N=1}^{\infty}$ is a sequence of functions defined in an open interval $a < t < b$. Suppose that $\{M_N\}_{N=1}^{\infty}$ is a sequence of positive constants such that

$$|y_N(t)| \leq M_N$$

for all $a < t < b$. If $\sum_{N=1}^{\infty} M_N$ is convergent then $\sum_{N=1}^{\infty} y_N$ converges uniformly for all $a < t < b$.

**Theorem 34.2**
If the components of the matrices $\mathbf{P}(\mathrm{t})$ and $\mathbf{g}(\mathrm{t})$ are continuous in an interval $a < t < b$ then the initial value problem (15) has a unique solution on the entire interval $a < t < b$.

**Proof.**
We start by reformulating the matrix differential equation in (15) as an integral equation. Integration of both sides of (15) yields

$$\int_{t_0}^{t} \mathbf{y}'(s)ds = \int_{t_0}^{t} [\mathbf{P}(s)\mathbf{y}(s) + \mathbf{g}(s)]ds \tag{16}$$

Applying the Fundamental Theorem of Calculus to the left side of (16) yields

$$\mathbf{y}(t) = \mathbf{y_0} + \int_{t_0}^{t} [\mathbf{P}(s)\mathbf{y}(s) + \mathbf{g}(s)]ds \tag{17}$$

Thus, a solution of (17) is also a solution to (15) and vice versa.
**Existence:** To prove the existence we shall use again the method of successive approximations as described in Theorem 8.1.

$$\begin{array}{rcl}
\mathbf{y}_0(t) & \equiv & \mathbf{y}_0 \\
\mathbf{y}_1(t) & = & \mathbf{y}_0 + \int_{t_0}^{t} [\mathbf{P}(s)\mathbf{y}_0(s) + \mathbf{g}(s)]ds \\
\mathbf{y}_2(t) & = & \mathbf{y}_0 + \int_{t_0}^{t} [\mathbf{P}(s)\mathbf{y}_1(s) + \mathbf{g}(s)]ds \\
\vdots & & \vdots \\
\mathbf{y}_N(t) & = & \mathbf{y}_0 + \int_{t_0}^{t} [\mathbf{P}(s)\mathbf{y}_{N-1}(s) + \mathbf{g}(s)]ds
\end{array}$$

Write

$$\mathbf{y}_N(t) = \begin{bmatrix} y_{1,N} \\ y_{2,N} \\ \vdots \\ y_{n,N} \end{bmatrix}$$

For $i = 1, 2, \cdots, n$, we are going to show that the sequence $\{y_{i,N}(t)\}$ converges uniformly to a function $y_i(t)$ such that $\mathbf{y}(t)$ (with components $y_1, y_2, \cdots, y_n$)is a solution to (17) and hence a solution to (15).
Let $[c, d]$ be a closed interval containing $t_0$ and contained in $(a, b)$. Then by continuity there exist positive constants $k_{ij},\ 1 \leq i, j \leq n$, such that

$$\max_{c \leq t \leq d} |p_{ij}(t)| \leq k_{ij}$$

This implies that

$$\begin{array}{rcl} ||\mathbf{P}(t)\mathbf{y}(t)|| & = & |\sum_{j=1}^n p_{1j}y_j| + |\sum_{j=1}^n p_{2j}y_j| + \cdots + |\sum_{j=1}^n p_{nj}y_j| \\ & \leq & K'\sum_{j=1}^n |y_j| + K'\sum_{j=1}^n |y_j| + \cdots + K'\sum_{j=1}^n |y_j| = K||\mathbf{y}|| \end{array}$$

for all $c \leq t \leq d$, where we define

$$||\mathbf{y}|| = |y_1| + |y_2| + \cdots + |y_n|$$

and where

$$K' = \sum_{i=1}^n \sum_{j=1}^n k_{ij},\ K = nK'.$$

It follows that for $1 \leq i \leq n$

$$\begin{array}{rcl} |y_{i,N} - y_{i,N-1}| & \leq & ||\mathbf{y}_N - \mathbf{y}_{N-1}|| = ||\int_{t_0}^t [\mathbf{P}(s)(\mathbf{y}_{N-1} - \mathbf{y}_{N-2})ds|| \\ & \leq & \int_{t_0}^t ||\mathbf{P}(s)(\mathbf{y}_{N-1} - \mathbf{y}_{N-2})||ds \\ & \leq & K\int_{t_0}^t ||\mathbf{y}_{N-1} - \mathbf{y}_{N-2}||ds \end{array}$$

But

$$\begin{array}{rcl} ||\mathbf{y}_1 - \mathbf{y_0}|| & \leq & \int_{t_0}^t ||\mathbf{P}(s)\mathbf{y}_0 + \mathbf{g}(s)||ds \\ & \leq & M(t - t_0) \end{array}$$

where

$$M = K||\mathbf{y}_0|| + \max_{c \leq t \leq d} |g_1(t)| + \max_{c \leq t \leq d} |g_2(t)| + \cdots + \max_{c \leq t \leq d} |g_n(t)|.$$

An easy induction yields that for $1 \leq i \leq n$

$$|y_{i,N+1} - y_{i,N}| \leq ||\mathbf{y}_{N+1} - \mathbf{y}_N|| \leq MK^N \frac{(t - t_0)^{N+1}}{(N+1)!} \leq MK^N \frac{(b - a)^{N+1}}{(N+1)!}$$

Since

$$\sum_{N=0}^{\infty} MK^N \frac{(b-a)^{N+1}}{(N+1)!} = \frac{M}{K}(e^{K(b-a)} - 1)$$

by the Weierstrass M-test we conclude that the series $\sum_{N=0}^{\infty}[y_{i,N} - y_{i,N-1}]$ converges uniformly for all $c \leq t \leq d$. But

$$y_{i,N}(t) = \sum_{k=0}^{N-1}[y_{i,k+1}(t) - y_{i,k}(t)] + y_{i,0}.$$

Thus, the sequence $\{y_{i,N}\}$ converges uniformly to a function $y_i(t)$ for all $c \leq t \leq d$.
The function $y_i(t)$ is a continuous function (a uniform limit of a sequence of continuous functions is continuous). Also, we can interchange the order of taking limits and integration for such sequences. Therefore

$$\begin{array}{rcl} \mathbf{y}(t) & = & \lim_{N\to\infty} \mathbf{y}_N(t) \\ & = & \mathbf{y}_0 + \lim_{N\to\infty} \int_{t_0}^t (\mathbf{P}(s)\mathbf{y}_{N-1} + \mathbf{g}(s))ds \\ & = & \mathbf{y}_0 + \int_{t_0}^t \lim_{N\to\infty}(\mathbf{P}(s)\mathbf{y}_{N-1} + \mathbf{g}(s))ds \\ & = & \mathbf{y}_0 + \int_{t_0}^t (\mathbf{P}(s)\mathbf{y} + \mathbf{g}(s))ds \end{array}$$

This shows that $\mathbf{y}(t)$ is a solution to the integral equation (17) and therefore a solution to (15).

**Uniqueness:**
The uniqueness follows from Gronwall Inequality (See Problem 8.11). Suppose that $\mathbf{y}(t)$ and $\mathbf{z}(t)$ are two solutions to the initial value problem, it follows that for all $a < t < b$ we have

$$||\mathbf{y}(t) - \mathbf{z}(t)|| \leq \int_{t_0}^t K||\mathbf{y}(s) - \mathbf{z}(s)||ds$$

Letting $u(t) = ||\mathbf{y}(t) - \mathbf{z}(t)||$ we have

$$u(t) \leq \int_{t_0}^t Ku(s)ds$$

so that by Gronwall's inequality $u(t) \equiv 0$ and therefore $\mathbf{y}(t) = \mathbf{z}(t)$ for all $a < t < b$. This completes a proof of the theorem ■

**Example 34.1**
Consider the initial value problem

$$\begin{array}{rcl} y_1' &=& t^{-1}y_1 + (\tan t)y_2, \quad y_1(3) = 0 \\ y_2' &=& (\ln|t|)y_1 + e^t y_2, \quad y_2(3) = 1 \end{array}$$

Determine the largest t-interval such that a unique solution is guaranteed to exist.

**Solution.**
The function $p_{11}(t) = \frac{1}{t}$ is continuous for all $t \neq 0$. The function $p_{12}(t) = \tan t$ is continuous for all $t \neq (2n+1)\frac{\pi}{2}$ where $n$ is an integer. The function $p_{21}(t) = \ln|t|$ is continuous for all $t \neq 0$. The function $p_{22}(t) = e^t$ is continuous for all real numbers. All these functions can be continuous on the common domain $t \neq 0$ and $t \neq (2n+1)\frac{\pi}{2}$. Since $t_0 = 3$, the largest t-interval for which a unique solution is guaranteed to exist is $\frac{\pi}{2} < t < \frac{3\pi}{2}$ ■

## Practice Problems

**Problem 34.1**
Consider the initial value problem

$$\begin{array}{rcl} y_1' & = & y_1 + (\tan t)y_2 + (t+1)^{-2}, \quad y_1(0) = 0 \\ y_2' & = & (t^2-2)y_1 + 4y_2, \quad y_2(0) = 0 \end{array}$$

Determine the largest t-interval such that a unique solution is guaranteed to exist.

**Problem 34.2**
Consider the initial value problem

$$\begin{array}{rcl} (t+2)y_1' & = & 3ty_1 + 5y_2, \quad y_1(1) = 0 \\ (t-2)y_2' & = & 2y_1 + 4ty_2, \quad y_2(1) = 2 \end{array}$$

Determine the largest t-interval such that a unique solution is guaranteed to exist.

**Problem 34.3**
Verify that the functions $y_1(t) = c_1e^t \cos t + c_2e^t \sin t$ and $y_2(t) = -c_1e^t \sin t + c_2e^t \cos t$ are solutions to the linear system

$$\begin{array}{rcl} y_1' & = & y_1 + y_2 \\ y_2' & = & -y_1 + y_2 \end{array}$$

**Problem 34.4**
Consider the first order linear system

$$\begin{array}{rcl} y_1' & = & y_1 + y_2 \\ y_2' & = & -y_1 + y_2 \end{array}$$

(a) Rewrite the system in matrix form $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$ and identify the matrix $\mathbf{A}$.
(b) Rewrite the solution to this system in the form $\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$.

**Problem 34.5**
Consider the initial value problem

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t), \quad \mathbf{y}(0) = \mathbf{y}_0$$

where

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$$

(a) Verify that $\mathbf{y}(t) = c_1 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is a solution to the first order linear system.
(b) Determine $c_1$ and $c_2$ such that $\mathbf{y}$(t) solves the given initial value problem.

**Problem 34.6**
Rewrite the differential equation $(\cos t)y'' - 3ty' + \sqrt{t}y = t^2 + 1$ in the matrix form $\mathbf{y}(t) = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t)$.

**Problem 34.7**
Rewrite the differential equation $2y'' + ty + e^{3t} = y''' + (\cos t)y'$ in the matrix form $\mathbf{y}(t) = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t)$.

**Problem 34.8**
The initial value problem

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ 2\cos(2t) \end{bmatrix}, \quad \mathbf{y}(-1) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

was obtained from an initial value problem for a higher order differential equation. What is the corresponding scalar initial value problem?

**Problem 34.9**
The initial value problem

$$\mathbf{y}'(t) = \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_2 + y_3 \sin y_1 + y_3^2 \end{bmatrix}, \quad \mathbf{y}(1) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

was obtained from an initial value problem for a higher order differential equation. What is the corresponding scalar initial value problem?

**Problem 34.10**
Consider the system of differential equations

$$\begin{aligned} y'' &= tz' + y' + z \\ z'' &= y' + z' + 2ty \end{aligned}$$

Write the above system in the form

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}(t)$$

where

$$\mathbf{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ z(t) \\ z'(t) \end{bmatrix}$$

Identify $\mathbf{P}$(t) and $\mathbf{g}$(t).

**Problem 34.11**
Consider the system of differential equations

$$\begin{aligned} y'' &= 7y' + 4y - 8z + 6z' + t^2 \\ z'' &= 5z' + 2z - 6y' + 3y - \sin t \end{aligned}$$

Write the above system in the form

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}(t)$$

where

$$\mathbf{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ z(t) \\ z'(t) \end{bmatrix}$$

Identify $\mathbf{P}$(t) and $\mathbf{g}$(t).

# 35 Homogeneous First Order Linear Systems

In this section we consider the following system of $n$ homogeneous linear differential equations known as the **first order homogeneous linear system.**

$$\begin{array}{rcl} y_1' &=& p_{11}(t)y_1 + p_{12}(t)y_2 + \cdots + p_{1n}(t)y_n \\ y_2' &=& p_{21}(t)y_1 + p_{22}(t)y_2 + \cdots + p_{2n}(t)y_n \\ \vdots && \\ y_n' &=& p_{n1}(t)y_1 + p_{n2}(t)y_2 + \cdots + p_{nn}(t)y_n \end{array}$$

where the coefficient functions are all continuous in $a < t < b$. The above system can be recast in matrix form as

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) \qquad (18)$$

where

$$\mathbf{y}(t) = \left[\begin{array}{c} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{array}\right], \quad \mathbf{P}(t) = \left[\begin{array}{cccc} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & & & \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{array}\right]$$

**Example 35.1**
(a) Rewrite the given system of linear homogeneous differential equations as a homogeneous linear system of the form $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}$.

$$\begin{array}{rcl} y_1' &=& y_2 + y_3 \\ y_2' &=& -6y_1 - 3y_2 + y_3 \\ y_3' &=& -8y_1 - 2y_2 + 4y_3 \end{array}$$

(b) Verify that the vector function

$$\mathbf{y}(t) = \left[\begin{array}{c} e^t \\ -e^t \\ 2e^t \end{array}\right]$$

is a solution of $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}$.

**Solution.**
(a)

$$\left[\begin{array}{c} y_1 \\ y_2 \end{array}\right]' = \left[\begin{array}{ccc} 0 & 1 & 1 \\ -6 & -3 & 1 \\ -8 & -2 & 4 \end{array}\right] \left[\begin{array}{c} y_1 \\ y_2 \end{array}\right]$$

(b) We have

$$\mathbf{y}' = \begin{bmatrix} e^t \\ -e^t \\ 2e^t \end{bmatrix}$$

and

$$\mathbf{P}(t)\mathbf{y} = \begin{bmatrix} 0 & 1 & 1 \\ -6 & -3 & 1 \\ -8 & -2 & 4 \end{bmatrix} \begin{bmatrix} e^t \\ -e^t \\ 2e^t \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t \\ 2e^t \end{bmatrix} = \mathbf{y}' \blacksquare$$

Our first result shows that any linear combinations of solutions to (18) is again a solution.

**Theorem 35.1**
If $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_r$ are solutions to (18) then for any constants $c_1, c_2, \cdots, c_r$, the function $\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_r\mathbf{y}_r$ is also a solution.

**Proof.**
Differentiating we find

$$\begin{array}{rcl} \mathbf{y}'(t) & = & (c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_r\mathbf{y}_r)' \\ & = & c_1\mathbf{y}_1' + c_2\mathbf{y}_2' + \cdots + c_r\mathbf{y}_r' \\ & = & c_1\mathbf{P}(t)\mathbf{y}_1 + c_2\mathbf{P}(t)\mathbf{y}_2 + \cdots + c_r\mathbf{P}(t)\mathbf{y}_r \\ & = & \mathbf{P}(t)(c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_r\mathbf{y}_r) = \mathbf{P}(t)\mathbf{y} \blacksquare \end{array}$$

Next, we pose the following question: Are there solutions $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ such that every solution to (18) can be written as a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n$. We call such a set of functions a **fundamental set of solutions.** With such a set, the **general solution** is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$$

Our next question is to find a criterion for testing $n$ solutions to (18) for a fundamental set. For this purpose, writing the components of the vectors $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n$

$$\mathbf{y}_1(t) = \begin{bmatrix} y_{1,1}(t) \\ y_{2,1}(t) \\ \vdots \\ y_{n,1} \end{bmatrix}, \quad \mathbf{y}_2(t) = \begin{bmatrix} y_{1,2}(t) \\ y_{2,2}(t) \\ \vdots \\ y_{n,2} \end{bmatrix}, \quad \cdots, \mathbf{y}_n(t) = \begin{bmatrix} y_{1,n}(t) \\ y_{2,n}(t) \\ \vdots \\ y_{n,n} \end{bmatrix},$$

we define the matrix $\mathbf{\Psi}(t)$ whose columns are the vectors $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n$. That is,

$$\mathbf{\Psi}(t) = \begin{bmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n} \\ \vdots & & & \\ y_{n,1} & y_{n,2} & \cdots & y_{n,n} \end{bmatrix}$$

We call $\mathbf{\Psi}(t)$ a **solution matrix** of $\mathbf{y}' = \mathbf{P}(t)\mathbf{y}$. In this case, $\mathbf{\Psi}(t)$ is a solution to the matrix equation $\mathbf{\Psi}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)$. Indeed,

$$\begin{aligned} \mathbf{\Psi}'(t) &= [\mathbf{y}_1'(t)\ \mathbf{y}_2'(t)\ \cdots\ \mathbf{y}_n'(t)] \\ &= [\mathbf{P}(t)\mathbf{y}_1(t)\ \mathbf{P}(t)\mathbf{y}_2(t)\ \cdots\ \mathbf{P}(t)\mathbf{y}_n(t)] \\ &= \mathbf{P}(t)[\mathbf{y}_1(t)\ \mathbf{y}_2(t)\ \cdots\ \mathbf{y}_n(t)] \\ &= \mathbf{P}(t)\mathbf{\Psi}(t) \end{aligned}$$

We define the **Wronskian** of $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n$ to be the determinant of $\mathbf{\Psi}$; that is

$$W(t) = det(\mathbf{\Psi}(t)).$$

The following theorem provides a condition for the solution vectors $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n$ to form a fundamental set of solutions.

**Theorem 35.2**
Let $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ be a set of $n$ solutions to (18). If there is $a < t_0 < b$ such that $W(t_0) \neq 0$ then the set $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ forms a fundamental set of solutions.

**Solution.**
Let $\mathbf{u}(t)$ be any solution to (18). Can we find constants $c_1, c_2, \cdots, c_n$ such that

$$\mathbf{u}(t) = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n?$$

A simple matrix algebra we see that

$$\begin{aligned} c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n &= \Psi(t) = \begin{bmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n} \\ \vdots & & & \\ y_{n,1} & y_{n,2} & \cdots & y_{n,n} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\ &= \mathbf{\Psi}(t)\mathbf{c} \end{aligned}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Thus,

$$\mathbf{u}(t) = \mathbf{\Psi}(t)\mathbf{c}, \; a < t < b.$$

In particular,

$$\mathbf{u}(t_0) = \mathbf{\Psi}(t_0)\mathbf{c}.$$

Since $W(t_0) = det(\mathbf{\Psi}(t_0)) \neq 0$, the matrix $\mathbf{\Psi}(t_0)$ is invertible and as a result of this we find

$$\mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{u}(t_0) \; \blacksquare$$

When the columns of $\mathbf{\Psi}t)$ form a fundamental set of solutions of $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$ then we call $\mathbf{\Psi}(t)$ a **fundamental matrix**.

**Example 35.2**
(a) Verify the given functions are solutions of the homogeneous linear system.
(b) Compute the Wronskian of the solution set. On the basis of this calculation can you assert that the set of solutions forms a fundamental set?
(c) If the given solutions are shown in part(b) to form a fundamental set, state the general solution of the linear homogeneous system. Express the general solution as the product $\mathbf{y}(t) = \mathbf{\Psi}(t)\mathbf{c}$, where $\mathbf{\Psi}(t)$ is a square matrix whose columns are the solutions forming the fundamental set and $\mathbf{c}$ is a column vector of arbitrary constants.
(d) If the solutions are shown in part (b) to form a fundamental set, impose the given initial condition and find the unique solution of the initial value problem.

$$\mathbf{y}' = \begin{bmatrix} -21 & -10 & 2 \\ 22 & 11 & -2 \\ -110 & -50 & 11 \end{bmatrix} \mathbf{y}, \mathbf{y}(0) = \begin{bmatrix} 3 \\ -10 \\ -16 \end{bmatrix}, \mathbf{y}_1(t) = \begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix}, \quad \mathbf{y}_2(t) = \begin{bmatrix} e^t \\ 0 \\ 11e^t \end{bmatrix}$$

$$\mathbf{y}_3(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \\ 5e^{-t} \end{bmatrix}$$

**Solution.**
(a) We have

$$\mathbf{y}_1' = \begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -21 & -10 & 2 \\ 22 & 11 & -2 \\ -110 & -50 & 11 \end{bmatrix} \begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix} = \begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix} = \mathbf{y}_1'$$

Similarly,

$$\mathbf{y}_2' = \begin{bmatrix} e^t \\ 0 \\ 11e^t \end{bmatrix}$$

and

$$\begin{bmatrix} -21 & -10 & 2 \\ 22 & 11 & -2 \\ -110 & -50 & 11 \end{bmatrix} \begin{bmatrix} e^t \\ 0 \\ 11e^t \end{bmatrix} = \begin{bmatrix} e^t \\ 0 \\ 11e^t \end{bmatrix} = \mathbf{y}_2'$$

$$\mathbf{y}_3' = \begin{bmatrix} -e^{-t} \\ e^{-t} \\ -5e^{-t} \end{bmatrix}$$

and

$$\begin{bmatrix} -21 & -10 & 2 \\ 22 & 11 & -2 \\ -110 & -50 & 11 \end{bmatrix} \begin{bmatrix} e^{-t} \\ -e^{-t} \\ 5e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ e^{-t} \\ -5e^{-t} \end{bmatrix} = \mathbf{y}_3'$$

(b) The Wronskian is given by

$$W(t) = \begin{vmatrix} 5e^t & e^t & e^{-t} \\ -11e^t & 0 & -e^{-t} \\ 0 & 11e^t & 5e^{-t} \end{vmatrix} = -11e^t$$

Since $W(t) \neq 0$, the set $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ forms a fundamental set of solutions.
(c) The general solution is

$$\mathbf{y}(t) = c_2\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = \begin{bmatrix} 5e^t & e^t & e^{-t} \\ -11e^t & 0 & -e^{-t} \\ 0 & 11e^t & 5e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

(d) We have

$$\begin{bmatrix} 5 & 1 & 1 \\ -11 & 0 & -1 \\ 0 & 11 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -10 \\ -16 \end{bmatrix}$$

Solving this system using Cramer's rule we find $c_1 = 1,\ c_2 = -1,\ c_3 = -1.$ Therefore the solution to the initial value problem is

$$\mathbf{y}(t) = \begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix} - \begin{bmatrix} e^t \\ 0 \\ 11e^t \end{bmatrix} - \begin{bmatrix} e^{-t} \\ -e^{-t} \\ 5e^{-t} \end{bmatrix} = \begin{bmatrix} 4e^t - e^{-t} \\ -11e^t + e^{-t} \\ -11e^t - 5e^{-t} \end{bmatrix} \blacksquare$$

The final result of this section is Abel's theorem which states that the Wronskian of a set of solutions either vanishes nowhere or it vanishes everywhere on the interval $a < t < b$.

**Theorem 35.3** *(Abel's)*
Let $\{\mathbf{y}_1(t), \mathbf{y}_2, \cdots, \mathbf{y}_n(t)(t)\}$ be a set of solutions to (18) and let W(t) be the Wronskian of these solutions. Then W(t) satisfies the differential equation

$$W'(t) = tr(\mathbf{P}(t))W(t)$$

where

$$tr(\mathbf{P}(t)) = p_{11}(t) + p_{22}(t) + \cdots + p_{nn}(t).$$

Moreover, if $a < t_0 < b$ then

$$W(t) = W(t_0)e^{\int_{t_0}^t tr(\mathbf{P}(s))ds}$$

**Proof.**
Since $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ is a set of $n$ solutions to (18), we have

$$y'_{i,j} = \sum_{k=1}^n p_{ik}y_{k,j}, \quad 1 \le i, j \le n \tag{19}$$

Using the definition of determinant we can write

$$W(t) = \sum_\sigma sgn(\sigma)y_{1,\sigma(1)}y_{2,\sigma(2)} \cdots y_{n,\sigma(n)}$$

where the sum is taken over all one-to-one functions $\sigma$ from the set $\{1, 2, \cdots, n\}$ to itself. Taking the derivative of both sides and using the product rule we find

$$\begin{array}{rcl} W'(t) & = & (\sum_\sigma sgn(\sigma) y_{1,\sigma(1)} y_{2,\sigma(2)} \cdots y_{n,\sigma(n)})' \\ & = & \sum_\sigma sgn(\sigma) y'_{1,\sigma(1)} y_{2,\sigma(2)} \cdots y_{n,\sigma(n)} + \sum_\sigma sgn(\sigma) y_{1,\sigma(1)} y'_{2,\sigma(2)} \cdots y_{n,\sigma(n)} \\ & & + \cdots + \sum_\sigma sgn(\sigma) y_{1,\sigma(1)} y_{2,\sigma(2)} \cdots y'_{n,\sigma(n)} \\ & = & \left| \begin{array}{cccc} y'_{1,1} & y'_{1,2} & \cdots & y'_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n} \\ \vdots & & & \\ y_{n,1} & y_{n,2} & \cdots & y_{n,n} \end{array} \right| + \left| \begin{array}{cccc} y_{1,1} & y_{1,2} & \cdots & y_{1,n} \\ y'_{2,1} & y'_{2,2} & \cdots & y'_{2,n} \\ \vdots & & & \\ y_{n,1} & y_{n,2} & \cdots & y_{n,n} \end{array} \right| + \cdots + \left| \begin{array}{cccc} y_{1,1} & y_{1,2} & \cdots & y_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n} \\ \vdots & & & \\ y'_{n,1} & y'_{n,2} & \cdots & y'_{n,n} \end{array} \right| \end{array}$$

But

$$\left| \begin{array}{cccc} y'_{1,1} & y'_{1,2} & \cdots & y'_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n} \\ \vdots & & & \\ y_{n,1} & y_{n,2} & \cdots & y_{n,n} \end{array} \right| = \left| \begin{array}{cccc} \sum_{k=1}^n p_{1k} y_{k,1} & \sum_{k=1}^n p_{1k} y_{k,2} & \cdots & \sum_{k=1}^n p_{1k} y_{k,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n} \\ \vdots & & & \\ y_{n,1} & y_{n,2} & \cdots & y_{n,n} \end{array} \right|$$

We evaluate the determinant of the right-side using elementary row operations (See Theorem 26.1). We multiply the second row by $p_{12}$, the third by $p_{13}$, and so on, add these $n-1$ rows and then subtract the result from the first row. The resulting determinant is

$$\left| \begin{array}{cccc} y'_{1,1} & y'_{1,2} & \cdots & y'_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n} \\ \vdots & & & \\ y_{n,1} & y_{n,2} & \cdots & y_{n,n} \end{array} \right| = \left| \begin{array}{cccc} p_{11} y_{1,1} & p_{11} y_{1,2} & \cdots & p_{11} y_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n} \\ \vdots & & & \\ y_{n,1} & y_{n,2} & \cdots & y_{n,n} \end{array} \right| = p_{11} W(t)$$

Proceeding similarly with the other determinants we obtain

$$\begin{array}{rcl} W'(t) & = & p_{11} W(t) + p_{22} W(t) + \cdots + p_{nn} W(t) \\ & = & (p_{11} + p_{22} + \cdots + p_{nn}) W(t) \\ & = & tr(\mathbf{P}(t)) W(t) \end{array}$$

This is a first-order scalar equation for W(t), whose solution can be found by the method of integrating factor

$$W(t) = W(t_0) e^{\int_{t_0}^t tr(\mathbf{P}(s)) ds}.$$

It follows that either $W(t) = 0$ for all $a < t < b$ or $W(t) \neq 0$ for all $a < t < b$ ■

**Example 35.3**
(a) Compute the Wronskian of the solution set and verify the set is a fundamental set of solutions.
(b) Compute the trace of the coefficient matrix.
(c) Verify Abel's theorem by showing that, for the given point $t_0$, $W(t) = W(t_0)e^{\int_{t_0}^t tr(\mathbf{P}(s))ds}$.

$$\mathbf{y}' = \left[\begin{array}{cc} 9 & 5 \\ -7 & -3 \end{array}\right]\mathbf{y}, \mathbf{y}_1(t) = \left[\begin{array}{c} 5e^{2t} \\ -7e^{2t} \end{array}\right], \mathbf{y}_2(t) = \left[\begin{array}{c} e^{4t} \\ -e^{4t} \end{array}\right], \ t_0 = 0, \ -\infty < t < \infty$$

**Solution.**
(a) The Wronskian is

$$W(t) = \left|\begin{array}{cc} 5e^{2t} & e^{4t} \\ -7e^{2t} & -e^{4t} \end{array}\right| = 2e^{6t}$$

Since $W(t) \neq 0$, the set $\{\mathbf{y}_1, \mathbf{y}_2\}$ forms a fundamental set of solutions.
(b) $tr(\mathbf{P}(t)) = 9 - 3 = 6$
(c) $W(t) = 2e^{6t}$ and $W(t_0)e^{\int_{t_0}^t tr(\mathbf{P}(s))ds} = 2e^{\int_0^t 6ds} = 2e^{6t}$ ■

## Practice Problems

In Problems 35.1 - 35.3 answer the following two questions.
(a) Rewrite the given system of linear homogeneous differential equations as a homogeneous linear system of the form $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}$.
(b) Verify that the given function $\mathbf{y}(t)$ is a solution of $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}$.

**Problem 35.1**

$$\begin{aligned} y_1' &= -3y_1 - 2y_2 \\ y_2' &= 4y_1 + 3y_2 \end{aligned}$$

and

$$\mathbf{y}(t) = \begin{bmatrix} e^t + e^{-t} \\ -2e^t - e^{-t} \end{bmatrix}$$

**Problem 35.2**

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{2}{t^2}y_1 + \frac{2}{t}y_2 \end{aligned}$$

and

$$\mathbf{y}(t) = \begin{bmatrix} t^2 + 3t \\ 2t + 3 \end{bmatrix}$$

**Problem 35.3**

$$\begin{aligned} y_1' &= 2y_1 + y_2 + y_3 \\ y_2' &= y_1 + y_2 + 2y_3 \\ y_3' &= y_1 + 2y_2 + y_3 \end{aligned}$$

and

$$\mathbf{y}(t) = \begin{bmatrix} 2e^t + e^{4t} \\ -e^t + e^{4t} \\ -e^t + e^{4t} \end{bmatrix}$$

In Problems 35.4 - 35.7
(a) Verify the given functions are solutions of the homogeneous linear system.
(b) Compute the Wronskian of the solution set. On the basis of this calculation can you assert that the set of solutions forms a fundamental set?
(c) If the given solutions are shown in part(b) to form a fundamental set,

state the general solution of the linear homogeneous system. Express the general solution as the product $\mathbf{y}(t) = \mathbf{\Psi}(t)\mathbf{c}$, where $\mathbf{\Psi}(t)$ is a square matrix whose columns are the solutions forming the fundamental set and $\mathbf{c}$ is a column vector of arbitrary constants.
(d) If the solutions are shown in part (b) to form a fundamental set, impose the given initial condition and find the unique solution of the initial value problem.

**Problem 35.4**

$$\mathbf{y}' = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix}\mathbf{y}, \mathbf{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{y}_1(t) = \begin{bmatrix} 2e^{3t} - 4e^{-t} \\ 3e^{3t} - 10e^{-t} \end{bmatrix}, \mathbf{y}_2(t) = \begin{bmatrix} 4e^{3t} + 2e^{-t} \\ 6e^{3t} + 5e^{-t} \end{bmatrix}$$

**Problem 35.5**

$$\mathbf{y}' = \begin{bmatrix} -3 & -5 \\ 2 & -1 \end{bmatrix}\mathbf{y}, \mathbf{y}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \quad \mathbf{y}_1(t) = \begin{bmatrix} -5e^{-2t}\cos 3t \\ e^{-2t}(\cos 3t - 3\sin 3t) \end{bmatrix},$$

$$\mathbf{y}_2(t) = \begin{bmatrix} -5e^{-2t}\sin 3t \\ e^{-2t}(3\cos 3t + \sin 3t) \end{bmatrix}$$

**Problem 35.6**

$$\mathbf{y}' = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\mathbf{y}, \mathbf{y}(-1) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \mathbf{y}_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{y}_2(t) = \begin{bmatrix} e^{3t} \\ -2e^{3t} \end{bmatrix}$$

**Problem 35.7**

$$\mathbf{y}' = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix}\mathbf{y}, \mathbf{y}(0) = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}, \mathbf{y}_1(t) = \begin{bmatrix} e^{-2t} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_2(t) = \begin{bmatrix} 0 \\ 2e^t\cos 2t \\ -e^t\sin 2t \end{bmatrix}$$

$$\mathbf{y}_3(t) = \begin{bmatrix} 0 \\ 2e^t\sin 2t \\ e^t\cos 2t \end{bmatrix}$$

In Problems 35.8 - 35.9, the given functions are solutions of the homogeneous linear system.
(a) Compute the Wronskian of the solution set and verify the set is a fundamental set of solutions.

(b) Compute the trace of the coefficient matrix.
(c) Verify Abel's theorem by showing that, for the given point $t_0$, $W(t) = W(t_0)e^{\int_{t_0}^{t} tr(\mathbf{P}(s))ds}$.

**Problem 35.8**

$$\mathbf{y}' = \begin{bmatrix} 6 & 5 \\ -7 & -6 \end{bmatrix}\mathbf{y}, \mathbf{y}_1(t) = \begin{bmatrix} 5e^{-t} \\ -7e^{-t} \end{bmatrix}, \mathbf{y}_2(t) = \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix},\ t_0 = -1,\ -\infty < t < \infty$$

**Problem 35.9**

$$\mathbf{y}' = \begin{bmatrix} 1 & t \\ 0 & -t^{-1} \end{bmatrix}\mathbf{y}, \mathbf{y}_1(t) = \begin{bmatrix} -1 \\ t^{-1} \end{bmatrix}, \mathbf{y}_2(t) = \begin{bmatrix} e^{t} \\ 0 \end{bmatrix},\ t_0 = -1,\ t \neq 0,\ 0 < t < \infty$$

**Problem 35.10**
The functions

$$\mathbf{y}_1(t) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2(t) = \begin{bmatrix} 2e^{3t} \\ e^{3t} \end{bmatrix}$$

are known to be solutions of the homogeneous linear system $\mathbf{y}' = \mathbf{P}\mathbf{y}$, where $\mathbf{P}$ is a real $2 \times 2$ constant matrix.
(a) Verify the two solutions form a fundamental set of solutions.
(b) What is tr($\mathbf{P}$)?
(c) Show that $\mathbf{\Psi}(t)$ satisfies the homogeneous differential equation $\mathbf{\Psi}' = \mathbf{P}\mathbf{\Psi}$, where

$$\mathbf{\Psi}(t) = [\mathbf{y}_1(t)\ \ \mathbf{y}_2(t)] = \begin{bmatrix} 5 & 2e^{3t} \\ 1 & e^{3t} \end{bmatrix}$$

(d) Use the observation of part (c) to determine the matrix $\mathbf{P}$.[Hint: Compute the matrix product $\mathbf{\Psi}'(t)\mathbf{\Psi}^{-1}(t)$. It follows from part (a) that $\mathbf{\Psi}^{-1}(t)$ exists.] Are the results of parts (b) and (d) consistent?

**Problem 35.11**
The homogeneous linear system

$$\mathbf{y}' = \begin{bmatrix} 3 & 1 \\ -2 & \alpha \end{bmatrix}\mathbf{y}$$

has a fundamental set of solutions whose Wronskian is constant, $W(t) = 4,\ -\infty < t < \infty$. What is the value of $\alpha$?

# 36 First Order Linear Systems: Fundamental Sets and Linear Independence

The results presented in this section are analogous to the ones established for nth order linear homogeneous differential equations (See Section 5.3). We start by showing that fundamental sets always exist.

**Theorem 36.1**
The first-order linear homogeneous equation

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y}, \quad a < t < b$$

where the entries of $\mathbf{P}$ are all continuous in $a < t < b$, has a fundamental set of solutions.

**Proof.**
Pick a number $t_0$ such that $a < t_0 < b$. Consider the following $n$ initial value problems

$$\begin{array}{c} \mathbf{y}' = \mathbf{P}(t)\mathbf{y}, \quad \mathbf{y}(t_0) = \mathbf{e}_1 \\ \mathbf{y}' = \mathbf{P}(t)\mathbf{y}, \quad \mathbf{y}(t_0) = \mathbf{e}_2 \\ \vdots \\ \mathbf{y}' = \mathbf{P}(t)\mathbf{y}, \quad \mathbf{y}(t_0) = \mathbf{e}_n \end{array}$$

where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ \cdots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

By the existence and uniqueness theorem we find the solutions $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$. Since $W(t) = det([\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n]) = det(\mathbf{I}) = 1$ where $\mathbf{I}$ is the $n \times n$ identity matrix we see that the solution set $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ forms a fundamental set of solutions ■

Next, we establish the converse to Theorem 35.2

**Theorem 36.2**
If $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ is a fundamental set of solutions to the first order linear homogeneous system

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y}, \quad a < t < b$$

then $W(t) \neq 0$ for all $a < t < b$.

**Proof.**
It suffices to show that $W(t_0) \neq 0$ for some number $a < t_0 < b$ because by Abel's theorem this implies that $W(t) \neq 0$ for all $a < t < b$. The general solution $\mathbf{y}(t) = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$ to $\mathbf{y}' = \mathbf{P}(t)\mathbf{y}$ can be written as the matrix equation

$$\mathbf{y}(t) = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = \mathbf{\Psi}(t)\mathbf{c}, \quad a < t < b$$

where $\mathbf{\Psi}(t) = [\mathbf{y}_1\ \mathbf{y}_2 \cdots \mathbf{y}_n]$ is the fundamental matrix and

$$\mathbf{c} = \left[\begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_n \end{array}\right]$$

In particular,

$$\mathbf{y}(t_0) = \mathbf{\Psi}(t_0)\mathbf{c}.$$

This matrix equation has a unique solution for $\mathbf{c}$. This is possible only when $\mathbf{\Psi}^{-1}(t_0)$ exists which is equivalent to saying that $W(t_0) = det(\mathbf{\Psi}(t_0)) \neq 0$. This completes a proof of the theorem ■

We next extend the definition of linear dependence and independence to vector functions and show that a fundamental set of solutions is a linearly independent set of vector functions on the t-interval of existence.
We say that a set of $n \times 1$ vector functions $\{\mathbf{f}_1(t), \mathbf{f}_2(t), \cdots, \mathbf{f}_r(t)\}$, where $a < t < b$, is **linearly dependent** if one of the vector function can be written as a linear combination of the remaining functions. Equivalently, this occurs if one can find constants $k_1, k_2, \cdots, k_r$ not all zero such that

$$k_1\mathbf{f}_1(t) + k_2\mathbf{f}_2(t) + \cdots + k_r\mathbf{f}_r(t) = \mathbf{0}, \quad a < t < b.$$

If the set $\{\mathbf{f}_1(t), \mathbf{f}_2(t), \cdots, \mathbf{f}_r(t)\}$ is not linearly dependent then it is said to be **linearly independent** in $a < t < b$. Equivalently, $\{\mathbf{f}_1(t), \mathbf{f}_2(t), \cdots, \mathbf{f}_r(t)\}$ is linearly independent if and only if

$$k_1\mathbf{f}_1(t) + k_2\mathbf{f}_2(t) + \cdots + k_r\mathbf{f}_r(t) = \mathbf{0}$$

implies $k_1 = k_2 = \cdots = 0$.

**Example 36.1**
Determine whether the given functions are linearly dependent or linearly independent on the interval $-\infty < t < \infty$.

$$\mathbf{f}_1(t) = \begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix}, \ \mathbf{f}_2(t) = \begin{bmatrix} 0 \\ 1 \\ t^2 \end{bmatrix}$$

**Solution.**
Suppose that $k_1\mathbf{f}_1(t) + k_2\mathbf{f}_2(t) = \mathbf{0}$ for all $t$. This implies that for all $t$ we have

$$\begin{array}{ccc} k_1 & = & 0 \\ k_1 t + k_2 & = & 0 \\ k_2 t^2 & = & 0 \end{array}$$

Thus, $k_1 = k_2 = 0$ so that the functions $\mathbf{f}_1(t)$ and $\mathbf{f}_2(t)$ are linearly independent ■

**Theorem 36.3**
The solution set $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ is a fundamental set of solutions to

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y}$$

where the $n \times n$ matrix $\mathbf{P}(\mathrm{t})$ is continuous in $a < t < b$, if and only if the functions $\mathbf{y}_1, \ \mathbf{y}_2, \cdots, \mathbf{y}_n$ are linearly independent.

**Proof.**
Suppose first that $\{\mathbf{y}_1, \ \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ is a fundamental set of solutions. Then by Theorem 36.2 there is $a < t_0 < b$ such that $W(t_0) \neq 0$. Suppose that

$$c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t) = 0$$

for all $a < t < b$. This can be written as the matrix equation

$$\mathbf{\Psi}(t)\mathbf{c} = \mathbf{0}, \quad a < t < b$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

In particular,

$$\mathbf{\Psi}(t_0)\mathbf{c} = \mathbf{0}.$$

Since $W(t_0) = det(\mathbf{\Psi}(t_0)) \neq 0$, $\mathbf{\Psi}^{-1}(t_0)$ exists so that $\mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{\Psi}(t_0)\mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\cdot\mathbf{0} = \mathbf{0}$. Hence, $c_1 = c_2 = \cdots = c_n = 0$. Therefore, $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n$ are linearly independent.
Conversely, suppose that $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ is a linearly independent set. Suppose that $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ is not a fundamental set of solutions. Then by Theorem 35.2, $W(t) = det(\mathbf{\Psi}(t)) = 0$ for all $a < t < b$. Choose any $a < t_0 < b$. Then $W(t_0) = 0$. But this says that the matrix $\mathbf{\Psi}(t_0)$ is not invertible. In terms of matrix theory, this means that $\mathbf{\Psi}(t_0)\cdot\mathbf{c} = \mathbf{0}$ for some vector

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \neq \mathbf{0}$$

Now, let $\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t)$ for all $a < t < b$. Then $\mathbf{y}(t)$ is a solution to the differential equation and $\mathbf{y}(t_0) = \mathbf{\Psi}(t_0)\mathbf{c} = \mathbf{0}$. But the zero function also is a solution to the initial value problem. By the existence and uniqueness theorem we must have $c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t) = \mathbf{0}$ for all $a < t < b$ with $c_1,\ c_2, \cdots,\ c_n$ not all equal to 0. But this means that $\mathbf{y}_1,\ \mathbf{y}_2, \cdots, \mathbf{y}_n$ are linearly depedent which contradicts our assumption that $\mathbf{y}_1,\ \mathbf{y}_2, \cdots,\ \mathbf{y}_n$ are linearly independent ■

**Remark 36.1**
The fact that $\mathbf{y}_1,\ \mathbf{y}_2, \cdots, \mathbf{y}_n$ are solutions to $\mathbf{y}' = \mathbf{P}\mathbf{y}$ is critical in the above theorem. For example the vectors

$$\mathbf{f}_1(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{f}_2(t) = \begin{bmatrix} t \\ 2 \\ 0 \end{bmatrix},\ \mathbf{f}_3(t) = \begin{bmatrix} t^2 \\ t \\ 0 \end{bmatrix}$$

are linearly independent with $det(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) \equiv 0$.

**Example 36.2**
Consider the functions

$$\mathbf{f}_1(t) = \begin{bmatrix} e^t \\ 0 \end{bmatrix},\quad \mathbf{f}_2(t) = \begin{bmatrix} t^2 \\ t \end{bmatrix}$$

(a) Let $\mathbf{\Psi}(t) = [\mathbf{f}_1(t)\ \mathbf{f}_2(t)]$. Determine $det(\mathbf{\Psi}(t))$.
(b) Is it possible that the given functions form a fundamental set of solutions for a linear system $\mathbf{y}' = \mathbf{P}(t)\mathbf{y}$ where $\mathbf{P}(\mathrm{t})$ is continuous on a t-interval containing the point $t = 0$? Explain.
(c) Determine a matrix $\mathbf{P}(\mathrm{t})$ such that the given vector functions form a fundamental set of solutions for $\mathbf{y}' = \mathbf{P}(t)\mathbf{y}$. On what t-interval(s) is the coefficient matrix $\mathbf{P}(\mathrm{t})$ continuous?(Hint: The matrix $\mathbf{\Psi}(t)$ must satisfy $\mathbf{\Psi}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)$ and $det(\mathbf{\Psi}(t)) \neq 0$.)

**Solution.**
(a) We have

$$det(\mathbf{\Psi})(t) = \begin{vmatrix} e^t & t^2 \\ 0 & t \end{vmatrix} = te^t$$

(b) Since $det(\mathbf{\Psi})(0) = 0$, the given functions do not form a fundamental set for a linear system $\mathbf{y}' = \mathbf{P}(t)\mathbf{y}$ on any t-interval containing 0.
(c) For $\mathbf{\Psi}(t)$ to be a fundamental matrix it must satisfy the differential equation $\mathbf{\Psi}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)$ and the condition $det(\mathbf{\Psi}(t)) \neq 0$. But $det(\mathbf{\Psi}(t)) = te^t$ and this is not zero on any interval not containing zero. Thus, our coefficient matrix $\mathbf{P}(\mathrm{t})$ must be continuous on either $-\infty < t < 0$ or $0 < t < \infty$. Now, from the equation $\mathbf{\Psi}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)$ we can find $\mathbf{P}(t) = \mathbf{\Psi}'(t)\mathbf{\Psi}^{-1}(t)$. That is,

$$\begin{aligned} \mathbf{P}(t) &= \mathbf{\Psi}'(t)\mathbf{\Psi}^{-1}(t) = \tfrac{1}{te^t}\begin{bmatrix} e^t & 2t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t & -t^2 \\ 0 & e^t \end{bmatrix} \\ &= \tfrac{1}{te^t}\begin{bmatrix} te^t & (2t - t^2)e^t \\ 0 & e^t \end{bmatrix} \\ &= t^{-1}\begin{bmatrix} t & 2t - t^2 \\ 0 & 1 \end{bmatrix} \blacksquare \end{aligned}$$

Finally, we will show how to generate new fundamental sets from a given one and therefore establishing the fact that a first order linear homogeneous system has many fundamental sets of solutions. We also show how different fundamental sets are related to each other. For this, let us start with a fundamental set $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ of solutions to $\mathbf{y}' = \mathbf{P}(t)\mathbf{y}$. If $\overline{\mathbf{y}}_1, \overline{\mathbf{y}}_2, \cdots, \overline{\mathbf{y}}_n$ are $n$ solutions then they can be written as linear combinations of the $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$.

That is,

$$\begin{aligned} a_{11}\mathbf{y}_1 + a_{21}\mathbf{y}_2 + \cdots + a_{n1}\mathbf{y}_n &= \overline{\mathbf{y}}_1 \\ a_{12}\mathbf{y}_1 + a_{22}\mathbf{y}_2 + \cdots + a_{n2}\mathbf{y}_n &= \overline{\mathbf{y}}_2 \\ &\vdots \\ a_{1n}\mathbf{y}_1 + a_{2n}\mathbf{y}_2 + \cdots + a_{nn}\mathbf{y}_n &= \overline{\mathbf{y}}_n \end{aligned}$$

or in matrix form as

$$\begin{bmatrix} \overline{\mathbf{y}}_1 & \overline{\mathbf{y}}_2 & \cdots & \overline{\mathbf{y}}_n \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

That is

$$\overline{\mathbf{\Psi}}(t) = \mathbf{\Psi}(t)\mathbf{A}$$

**Theorem 36.4**
$\{\overline{\mathbf{y}}_1, \overline{\mathbf{y}}_2, \cdots, \overline{\mathbf{y}}_n\}$ is a fundamental set if and only if $\det(\mathbf{A}) \neq 0$ where $\mathbf{A}$ is the coefficient matrix of the above matrix equation.

**Proof.**
Since $\overline{\mathbf{\Psi}}(t) = \mathbf{\Psi}(t)\mathbf{A}$ and $W(t) = det(\mathbf{\Psi}(t)) \neq 0$, $\overline{W}(t) = det(\overline{\mathbf{\Psi}}(t)) \neq 0$ if and only if $det(\mathbf{A}) \neq 0$. That is, $\{\overline{\mathbf{y}}_1, \overline{\mathbf{y}}_2, \cdots, \overline{\mathbf{y}}_n\}$ is a fundamental set of solutions if and only if $\det(\mathbf{A}) \neq 0$ ■

**Example 36.3**
Let

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{y}, \ \mathbf{\Psi}(t) = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}, \ \ \overline{\mathbf{\Psi}}(t) = \begin{bmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{bmatrix}$$

(a) Verify that the matrix $\mathbf{\Psi}(t)$ is a fundamental matrix of the given linear system.
(b) Determine a constant matrix $\mathbf{A}$ such that the given matrix $\overline{\mathbf{\Psi}}(t)$ can be represented as $\overline{\mathbf{\Psi}}(t) = \mathbf{\Psi}(t)\mathbf{A}$.
(c) Use your knowledge of the matrix $\mathbf{A}$ and Theorem 36.4 to determine whether $\overline{\mathbf{\Psi}}(t)$ is also a fundamental matrix, or simply a solution matrix.

**Solution.**
(a) Since

$$\mathbf{\Psi}'(t) = \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}$$

and

$$\mathbf{P}(t)\mathbf{\Psi}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} = \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}$$

we conclude that $\mathbf{\Psi}$ is a solution matrix. To show that $\mathbf{\Psi}(t)$ is a fundamental matrix we need to verify that $det(\mathbf{\Psi}(t)) \neq 0$. Since $det(\mathbf{\Psi}(t)) = -2 \neq 0$, $\mathbf{\Psi}(t)$ is a fundamental matrix.
(b) First write

$$\overline{\mathbf{\Psi}}(t) = \begin{bmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^t - e^{-t} & e^t + e^{-t} \\ e^t + e^{-t} & e^t - e^{-t} \end{bmatrix}$$

Thus, the question is to find $a, b, c$, and $d$ such that

$$\frac{1}{2} \begin{bmatrix} e^t - e^{-t} & e^t + e^{-t} \\ e^t + e^{-t} & e^t - e^{-t} \end{bmatrix} = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ae^t + ce^{-t} & be^t + de^{-t} \\ ae^t - ce^{-t} & be^t - de^{-t} \end{bmatrix}$$

Comparing entries we find $a = 1/2,\ b = 1/2,\ c = -1/2$, and $d = 1/2$.
(c) Since $det(\mathbf{A}) = \frac{1}{2}$, $\overline{\mathbf{\Psi}}(t)$ is a fundamental matrix ■

## Practice Problems

In Problems 36.1 - 36.4, determine whether the given functions are linearly dependent or linearly independent on the interval $-\infty < t < \infty$.

**Problem 36.1**

$$\mathbf{f}_1(t) = \begin{bmatrix} t \\ 1 \end{bmatrix},\ \mathbf{f}_2(t) = \begin{bmatrix} t^2 \\ 1 \end{bmatrix}$$

**Problem 36.2**

$$\mathbf{f}_1(t) = \begin{bmatrix} e^t \\ 1 \end{bmatrix},\ \mathbf{f}_2(t) = \begin{bmatrix} e^{-t} \\ 1 \end{bmatrix},\ \mathbf{f}_3(t) = \begin{bmatrix} \frac{e^t - e^{-t}}{2} \\ 0 \end{bmatrix}$$

**Problem 36.3**

$$\mathbf{f}_1(t) = \begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix},\ \mathbf{f}_2(t) = \begin{bmatrix} 0 \\ 1 \\ t^2 \end{bmatrix},\ \mathbf{f}_3(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

**Problem 36.4**

$$\mathbf{f}_1(t) = \begin{bmatrix} 1 \\ \sin^2 t \\ 0 \end{bmatrix},\ \mathbf{f}_2(t) = \begin{bmatrix} 0 \\ 2(1 - \cos^2 t) \\ -2 \end{bmatrix},\ \mathbf{f}_3(t) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

**Problem 36.5**
Let

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\mathbf{y},\ \mathbf{\Psi}(t) = \begin{bmatrix} e^t & e^{-t} & 4e^{2t} \\ 0 & -2e^{-t} & e^{2t} \\ 0 & 0 & 3e^{2t} \end{bmatrix},\ \overline{\mathbf{\Psi}}(t) = \begin{bmatrix} e^t + e^{-t} & 4e^{2t} & e^t + 4e^{2t} \\ -2e^{-t} & e^{2t} & e^{2t} \\ 0 & 3e^{2t} & 3e^{2t} \end{bmatrix}$$

(a) Verify that the matrix $\mathbf{\Psi}(t)$ is a fundamental matrix of the given linear system.
(b) Determine a constant matrix $\mathbf{A}$ such that the given matrix $\overline{\mathbf{\Psi}}(t)$ can be represented as $\overline{\mathbf{\Psi}}(t) = \mathbf{\Psi}(t)\mathbf{A}$.
(c) Use your knowledge of the matrix $\mathbf{A}$ and Theorem 34.4 to determine whether $\overline{\mathbf{\Psi}}(t)$ is also a fundamental matrix, or simply a solution matrix.

**Problem 36.6**
Let

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{\Psi}(t) = \begin{bmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{bmatrix}$$

where the matrix $\mathbf{\Psi}(t)$ is a fundamental matrix of the given homogeneous linear system. Find a constant matrix $\mathbf{A}$ such that $\overline{\mathbf{\Psi}}(t) = \mathbf{\Psi}(t)\mathbf{A}$ with $\overline{\mathbf{\Psi}}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

# 37 Homogeneous Systems with Constant Coefficients

In this section, we consider solving linear homogeneous systems of the form $\mathbf{y}' = \mathbf{P}\mathbf{y}$ where $\mathbf{P}$ is a matrix with real-valued constants. Recall that the general solution to this system is given by

$$\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t)$$

where $\{\mathbf{y}_1(t), \mathbf{y}_2(t), \cdots, \mathbf{y}_n(t)\}$ is a fundamental set of solutions. So the problem of finding the general solution reduces to the one of finding a fundamental set of solutions.
Let's go back and look at how we solved a second order linear homogeneous equation with constant coefficients

$$y'' + ay' + by = 0 \tag{20}$$

To find the fundamental set of solutions we considered trial functions of the form $y = e^{rt}$ and find out that $r$ is a solution to the characteristic equation $r^2 + ar + b = 0$. But (20) is a first order homogeneous linear system

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{21}$$

where $y_1 = y$ and $y_2 = y'$.
Now, if $r$ is a solution to the characteristic equation $r^2 + ar + b = 0$ then one can easily check that the vector function

$$\mathbf{y} = \begin{bmatrix} e^{rt} \\ re^{rt} \end{bmatrix} = e^{rt} \begin{bmatrix} 1 \\ r \end{bmatrix}$$

is a solution to (21).
Motivated by the above discussion we will consider trial functions for the system

$$\mathbf{y}' = \mathbf{P}\mathbf{y} \tag{22}$$

of the form $\mathbf{y} = e^{rt}\mathbf{x}$ where $\mathbf{x}$ is a nonzero vector. Substituting this into (22) we find $re^{rt}\mathbf{x} = \mathbf{P}e^{rt}\mathbf{x}$. This can be written as a linear system of the form

$$(\mathbf{P} - r\mathbf{I})\mathbf{x} = \mathbf{0} \tag{23}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix.
Since system (23) has a nonzero solution $\mathbf{x}$, the matrix $\mathbf{P} - r\mathbf{I}$ cannot be invertible (otherwise $\mathbf{x}$=$\mathbf{0}$). This means that

$$p(r) = det(\mathbf{P} - r\mathbf{I}) = 0. \qquad (24)$$

We call (24) the **characteristic equation** associated to the linear system (22). Its solutions are called **eigenvalues**. A vector $\mathbf{x}$ corresponding to an eigenvalue $r$ is called an **eigenvector**. The pair $(r, \mathbf{x})$ is called an **eigenpair**. It follows that each eigenpair $(r, \mathbf{x})$ yields a solution of the form $\mathbf{y}(t) = e^{rt}\mathbf{x}$. If there are $n$ different eigenpairs then these will yield $n$ different solutions. We will show below that these $n$ different solutions form a fundamental set of solutions and therefore yield the general solution to (22). Thus, we need to address the following questions:
(1) Given an $n \times n$ matrix $\mathbf{P}$, do there always exist eigenpairs? Is it possible to find $n$ different eigenpairs and thereby form $n$ different solutions of (22)?
(2) How do we find these eigenpairs?
As pointed out earlier, the eigenvalues are solutions to equation (24). But

$$p(r) = \begin{vmatrix} a_{11} - r & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - r & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - r \end{vmatrix} = 0$$

The determinant is the sum of elementary products each having $n$ factors no two come from the same row or column. Thus, one of the term has the form $(a_{11} - r)(a_{22} - r) \cdots (a_{nn} - r)$. From this we see that $p(r)$ is a polynomial of degree $n$. We call $\mathbf{p}$(r) the **characteristic polynomial.** By the Fundamental Theorem of Algebra, the equation $p(r) = 0$ has $n$ roots, and therefore $n$ eigenvalues. These eigenvalues may be zero or nonzero, real or complex, and some of them may be repeated.
Now, for each eigenvalue $r$, we find a corresponding eigenvector by solving the linear system of $n$ equations in $n$ unknowns: $(\mathbf{P} - r\mathbf{I}) = \mathbf{0}$.

**Example 37.1**
Consider the homogeneous first order system

$$\mathbf{y}' = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \mathbf{y}$$

(a) Show that $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are eigenvectors of $\mathbf{P}$. Determine the corresponding eigenvalues.
(b) For each eigenpair found in (a), form a solution $\mathbf{y}_k$ of the system $\mathbf{y}' = \mathbf{P}\mathbf{y}$.
(c) Calculate the Wronskian and decide if the two solutions form a fundamental set.

**Solution.**
(a) Since

$$\mathbf{P}\mathbf{x}_1 = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = 2\mathbf{x}_1$$

$\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue 2. Similarly,

$$\mathbf{P}\mathbf{x}_2 = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} = 3\mathbf{x}_2$$

Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue 3.
(b) The two solutions are $\mathbf{y}_1(t) = e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{3t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
(c) The Wronskian is

$$W(t) = \begin{bmatrix} e^{2t} & -2e^{3t} \\ -e^{2t} & e^{3t} \end{bmatrix} = -e^{5t}.$$

Since $W(t) \neq 0$, the set $\{\mathbf{y}_1, \mathbf{y}_2\}$ forms a fundamental set of solutions ■

**Example 37.2**
Find the eigenvalues of the matrix $\mathbf{P} = \begin{bmatrix} 8 & 0 \\ 5 & 2 \end{bmatrix}$.

**Solution.**
The characteristic polynomial is $p(r) = \begin{vmatrix} 8-r & 0 \\ 3 & 2-r \end{vmatrix} = (8-r)(2-r)$. Thus, the eigenvalues are $r = 8$ and $r = 2$ ■

**Example 37.3**
Suppose that $r = 2$ is an eigenvalue of the matrix $\mathbf{P} = \begin{bmatrix} -4 & 3 \\ -4 & 4 \end{bmatrix}$. Find the eigenvector corresponding to this eigenvalue.

**Solution.**
We have $(\mathbf{P} - 2\mathbf{I})\mathbf{x} = \mathbf{0}$ or

$$\left[\begin{array}{cc} -6 & 3 \\ -4 & 2 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Solving this system we find $2x_1 = x_2$. Thus, an eigenvector is $\left[\begin{array}{c} 1 \\ 2 \end{array}\right]$ ■

We next list some properties of eigenvalues and eigenvectors.

**Theorem 37.1**
(a) If $(r, \mathbf{x})$ is an eigenpair then for any $\alpha \neq 0$, $(r, \alpha\mathbf{x})$ is also an eigenpair. This shows that eigenvectors are not unique.
(b) A matrix $\mathbf{P}$ can have a zero eigenvalue.
(c) A real matrix may have one or more complex eigenvalues and eigenvectors.

**Proof.**
(a) Suppose that $\mathbf{x}$ is an eigenvector corresponding to an eigenvalue $r$ of a matrix $\mathbf{P}$. Then for any nonzero constant $\alpha$ we have $\mathbf{P}(\alpha\mathbf{x}) = \alpha\mathbf{P}\mathbf{x} = r(\alpha\mathbf{x})$ with $\alpha\mathbf{x} \neq 0$. Hence, $(r, \alpha\mathbf{x})$ is an eigenpair.
(b) The characteristic equation of the matrix $\mathbf{P} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right]$ is $r(r-1) = 0$ so that $r = 0$ is an eigenvalue.
(c) The characteristic equation of the matrix $\mathbf{P} = \left[\begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array}\right]$ is $r^2 - 2r + 2 = 0$. Its roots are $r = 1 + i$ and $r = 1 - i$. For the $r = 1 + i$ we have the system

$$\left[\begin{array}{cc} -i & 1 \\ -1 & -i \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$$

A solution to this system is the vector $\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ i \end{array}\right]$. Similarly, for $r = 1 - i$ we have

$$\left[\begin{array}{cc} i & 1 \\ -1 & i \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$$

A solution to this system is the vector $\mathbf{x}_2 = \left[\begin{array}{c} 1 \\ -i \end{array}\right]$ ■

**Theorem 37.2**
Eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ corresponding to distinct eigenvalues $r_1, r_2, \cdots, r_k$ are linearly independent.

**Proof.**
Let us prove this by induction on $k$. The result is clear for $k = 1$ because eigenvectors are nonzero and a subset consisting of one nonzero vector is linearly independent. Now assume that the result holds for $k-1$ eigenvectors. Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ be eigenvectors corresponding to distinct eigenvalues $r_1, r_2, \cdots, r_k$. Assume that there is a linear combination

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0}.$$

Then we have

$$\begin{array}{rl}
c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k = \mathbf{0} & \Longrightarrow \\
\mathbf{P}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) = \mathbf{0} & \Longrightarrow \\
c_1\mathbf{P}\mathbf{x}_1 + c_2\mathbf{P}\mathbf{x}_2 + \cdots + c_k\mathbf{P}\mathbf{x}_k = \mathbf{0} & \Longrightarrow \\
c_1r_1\mathbf{x}_1 + c_2r_2\mathbf{x}_2 + \cdots + c_kr_k\mathbf{x}_k = \mathbf{0} & \Longrightarrow \\
(c_1r_1\mathbf{x}_1 + c_2r_2\mathbf{x}_2 + \cdots + c_kr_k\mathbf{x}_k) - (c_1r_k\mathbf{x}_1 + c_2r_k\mathbf{x}_2 + \cdots + c_kr_k\mathbf{x}_k) = \mathbf{0} & \Longrightarrow \\
c_1(r_1 - r_k)\mathbf{x}_1 + c_2(r_2 - r_k)\mathbf{x}_2 + \cdots + c_{k-1}(r_{k-1} - r_k)\mathbf{x}_{k-1} = \mathbf{0} &
\end{array}$$

But by the induction hypothesis, the vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{k-1}$ are linearly independent so that $c_1(r_1 - r_k) = c_2(r_2 - r_k) = \cdots = c_{k-1}(r_{k-1} - r_k) = 0$. Since the eigenvalues are all distinct, we must have $c_1 = c_2 = \cdots = c_{k-1} = 0$. In this case we are left with $c_k\mathbf{x}_k = \mathbf{0}$. Since $\mathbf{x}_k \neq \mathbf{0}$, $c_k = 0$. This shows that $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$ is linearly independent ■

The next theorem states that $n$ linearly independent eigenvectors yield a fundamental set of solutions to the equation $\mathbf{y}' = \mathbf{P}\mathbf{y}$.

**Theorem 37.3**
Consider the homogeneous system $\mathbf{y}' = \mathbf{P}\mathbf{y}, \ -\infty < t < \infty$. Suppose that $\mathbf{P}$ has eigenpairs $(r_1, \mathbf{x}_1), (r_2, \mathbf{x}_2), \cdots, (r_n, \mathbf{x}_n)$ where $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are linearly independent. Then the set of solutions

$$\{e^{r_1t}\mathbf{x}_1, e^{r_2t}\mathbf{x}_2, \cdots, e^{r_nt}\mathbf{x}_n\}$$

forms a fundamental set of solutions.

**Proof.**
We will show that the vectors $e^{r_1t}\mathbf{x}_1, e^{r_2t}\mathbf{x}_2, \cdots, e^{r_nt}\mathbf{x}_n$ are linearly independent. Suppose that

$$c_1e^{r_1t}\mathbf{x}_1 + c_2e^{r_2t}\mathbf{x}_2 + \cdots + c_ne^{r_nt}\mathbf{x}_n = \mathbf{0}$$

for all $-\infty < t < \infty$. In particular, we can replace $t$ by 0 and obtain

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}.$$

Since the vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are linearly independent, we must have $c_1 = c_2 = \cdots = c_n = 0$. This shows that $e^{r_1t}\mathbf{x}_1, e^{r_2t}\mathbf{x}_2, \cdots, e^{r_nt}\mathbf{x}_n$ are linearly independent. Since each vector is also a solution, by Theorem 34.3 the set $\{e^{r_1t}\mathbf{x}_1, e^{r_2t}\mathbf{x}_2, \cdots, e^{r_nt}\mathbf{x}_n\}$ forms a fundamental set of solutions ■

Combining Theorem 37.2 and Theorem 37.3 we obtain

**Theorem 37.4**
Consider the homogeneous system $\mathbf{y}' = \mathbf{P}\mathbf{y}, \ -\infty < t < \infty$. Suppose that $\mathbf{P}$ has $n$ eigenpairs $(r_1, \mathbf{x}_1), (r_2, \mathbf{x}_2), \cdots, (r_n, \mathbf{x}_n)$ with distinct eigenvalues. Then the set of solutions

$$\{e^{r_1t}\mathbf{x}_1, e^{r_2t}\mathbf{x}_2, \cdots, e^{r_nt}\mathbf{x}_n\}$$

forms a fundamental set of solutions.

**Proof.**
Since the eigenvalues are distinct, by Theorem 37.2 the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are linearly independent. But then by Theorem 35.3 the set of solutions

$$\{e^{r_1t}\mathbf{x}_1, e^{r_2t}\mathbf{x}_2, \cdots, e^{r_nt}\mathbf{x}_n\}$$

forms a fundamental set of solutions ■

**Example 37.4**
Solve the following initial value problem

$$\mathbf{y}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

**Solution.**
The characteristic equation is

$$\begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = (r+1)(r+3) = 0$$

Solving this quadratic equation we find $r_1 = -1$ and $r_2 = -3$. Now,

$$(\mathbf{P}+\mathbf{I})\mathbf{x} = \left[\begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]\left[\begin{array}{c} -x_1+x_2 \\ x_1-x_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Solving this system we find $x_1 = x_2$. Letting $x_1 = 1$ then $x_2 = 1$. Thus, an eigenvector is

$$\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$$

Similarly,

$$(\mathbf{P}+3\mathbf{I})\mathbf{x} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]\left[\begin{array}{c} x_1+x_2 \\ x_1+x_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Solving this system we find $x_1 = x_2$. Letting $x_1 = 1$ then $x_2 = -1$. Thus, an eigenvector is

$$\mathbf{x}_2 = \left[\begin{array}{c} 1 \\ -1 \end{array}\right]$$

By Theorem 35.4, a fundamental set of solutions is given by $\{e^{-t}\mathbf{x}_1,\ e^{-3t}\mathbf{x}_2\}$. The general solution is then

$$\mathbf{y}(t) = c_1e^{-t}\mathbf{x}_1 + c_2e^{-3t}\mathbf{x}_2.$$

Using the initial condtion we find $c_1 + c_2 = 3$ and $c_1 - c_2 = 1$. Solving this system we find $c_1 = 2$ and $c_2 = 1$. Hence, the unique solution is given by

$$\begin{array}{rcl} \mathbf{y}(t) & = & 2e^{-t}\mathbf{x}_1 + e^{-3t}\mathbf{x}_2 \\ & = & \left[\begin{array}{c} 2e^{-t}+e^{-3t} \\ 2e^{-t}-e^{-3t} \end{array}\right] \blacksquare \end{array}$$

## Practice Problems

In Problems 37.1 - 37.3, a $2 \times 2$ matrix $\mathbf{P}$ and vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are given.
(a) Decide which, if any, of the given vectors is an eigenvector of $\mathbf{P}$, and determine the corresponding eigenvalue.
(b) For the eigenpair found in part (a), form a solution $\mathbf{y}_k(t)$, where $k = 1$ or $k = 2$, of the first order system $\mathbf{y}' = \mathbf{P}\mathbf{y}$.
(c) If two solution are found in part (b), do they form a fundamental set of solutions for $\mathbf{y}' = \mathbf{P}\mathbf{y}$.

**Problem 37.1**

$$\mathbf{P} = \begin{bmatrix} 7 & -3 \\ 16 & -7 \end{bmatrix}, \ \mathbf{x}_1 = \begin{bmatrix} 3 \\ 8 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

**Problem 37.2**

$$\mathbf{P} = \begin{bmatrix} -5 & 2 \\ -18 & 7 \end{bmatrix}, \ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

**Problem 37.3**

$$\mathbf{P} = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}, \ \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

In Problems 37.4 - 37.6, an eigenvalue is given of the matrix $\mathbf{P}$. Determine a corresponding eigenvector.

**Problem 37.4**

$$\mathbf{P} = \begin{bmatrix} 5 & 3 \\ -4 & -3 \end{bmatrix}, \quad r = -1$$

**Problem 37.5**

$$\mathbf{P} = \begin{bmatrix} 1 & -7 & 3 \\ -1 & -1 & 1 \\ 4 & -4 & 0 \end{bmatrix}, \quad r = -4$$

**Problem 37.6**

$$\mathbf{P} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \\ 4 & 3 & -2 \end{bmatrix}, \quad r = 5$$

In Problems 37.7 - 37.10, Find the eigenvalues of the matrix $\mathbf{P}$.

**Problem 37.7**

$$\mathbf{P} = \begin{bmatrix} -5 & 1 \\ 0 & 4 \end{bmatrix}$$

**Problem 37.8**

$$\mathbf{P} = \begin{bmatrix} 3 & -3 \\ -6 & 6 \end{bmatrix}$$

**Problem 37.9**

$$\mathbf{P} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 2 \end{bmatrix}$$

**Problem 37.10**

$$\mathbf{P} = \begin{bmatrix} 1 & -7 & 3 \\ -1 & -1 & 1 \\ 4 & -4 & 0 \end{bmatrix}$$

In Problems 37.11 - 37.13, the matrix $\mathbf{P}$ has distinct eigenvalues. Using Theorem 37.4 determine a fundamental set of solutions of the system $\mathbf{y}' = \mathbf{P}\mathbf{y}$.

**Problem 37.11**

$$\mathbf{P} = \begin{bmatrix} -0.09 & 0.02 \\ 0.04 & -0.07 \end{bmatrix}$$

**Problem 37.12**

$$\mathbf{P} = \begin{bmatrix} 1 & 2 & 0 \\ -4 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Problem 37.13**

$$\mathbf{P} = \begin{bmatrix} 3 & 1 & 0 \\ -8 & 6 & 2 \\ -9 & -9 & 4 \end{bmatrix}$$

**Problem 37.14**
Solve the following initial value problem.

$$\mathbf{y}' = \begin{bmatrix} 5 & 3 \\ -4 & -3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

**Problem 37.15**
Solve the following initial value problem.

$$\mathbf{y}' = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

**Problem 37.16**
Find $\alpha$ so that the vector $\mathbf{x}$ is an eigenvector of $\mathbf{P}$. What is the corresponding eigenvalue?

$$\mathbf{P} = \begin{bmatrix} 2 & \alpha \\ 1 & -5 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

**Problem 37.17**
Find $\alpha$ and $\beta$ so that the vector $\mathbf{x}$ is an eigenvector of $\mathbf{P}$ corresponding the eigenvalue $r = 1$.

$$\mathbf{P} = \begin{bmatrix} \alpha & \beta \\ 2\alpha & \beta \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

## 38 Homogeneous Systems with Constant Coefficients: Complex Eigenvalues

We continue the study of finding the general solution of $\mathbf{y}' = \mathbf{P}\mathbf{y}$ where $\mathbf{P}$ is an $n \times n$ matrix with real entries. In this section, we consider the case when $\mathbf{P}$ possesses complex eigenvalues. We start with the following result.

**Theorem 38.1**
If $(r, \mathbf{x})$ is an eigenpair of $\mathbf{P}$ then $(\overline{r}, \overline{\mathbf{x}})$ is an eigenpair of $\mathbf{P}$. Thus, complex eigenvalues always occur in conjugate pairs.

**Proof.**
Write $r = \alpha + i\beta$. Then we have $\mathbf{P}\mathbf{x} = (\alpha + i\beta)\mathbf{x}$. Take the conjugate of both sides to obtain $\overline{\mathbf{P}}\overline{\mathbf{x}} = (\alpha - i\beta)\overline{\mathbf{x}}$. But $\mathbf{P}$ is a real matrix so that $\overline{\mathbf{P}} = \mathbf{P}$. Thus, $\mathbf{P}\overline{\mathbf{x}} = (\alpha - i\beta)\overline{\mathbf{x}}$. This shows that $\alpha - i\beta$ is an eigenvalue of $\mathbf{P}$ with corresponding eigenvector $\overline{\mathbf{x}}$ ■

In most applications, real-valued solutions are more meaningful then complex valued solutions. Our next task is to describe how to convert the complex solutions to $\mathbf{y}' = \mathbf{P}\mathbf{y}$ into real-valued solutions.

**Theorem 38.2**
Let $\mathbf{P}$ be a real valued $n \times n$ matrix. If $\mathbf{P}$ has complex conjugate eigenvalues $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where $\beta \neq 0$, and corresponding (complex conjugate) eigenvectors $\mathbf{x}_1 = \mathbf{a} + i\mathbf{b}$ and $\mathbf{x}_2 = \mathbf{a} - i\mathbf{b}$ then $\mathbf{y}_1 = e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t)$ and $\mathbf{y}_2 = e^{\alpha t}(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t)$ are two solutions of $\mathbf{y}' = \mathbf{P}\mathbf{y}$.(These functions are the real and imaginary parts of the two solutions, $e^{(\alpha+i\beta)t}\mathbf{x}_1$ and $e^{(\alpha-i\beta)t}\mathbf{x}_2$).

**Proof.**
By Euler's formula we have

$$\begin{aligned} e^{(\alpha+i\beta)t}\mathbf{x}_1 &= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{a} + i\mathbf{b}) \\ &= e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t) + e^{\alpha t}i(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t) \\ &= \mathbf{y}_1 + i\mathbf{y}_2 \end{aligned}$$

and

$$\begin{aligned} e^{(\alpha-i\beta)t}\mathbf{x}_2 &= e^{\alpha t}(\cos\beta t - i\sin\beta t)(\mathbf{a} - i\mathbf{b}) \\ &= e^{\alpha t}(\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t) - e^{\alpha t}i(\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t) \\ &= \mathbf{y}_1 - i\mathbf{y}_2 \end{aligned}$$

We next show that $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions to $\mathbf{y}' = \mathbf{P}\mathbf{y}$. Indeed,

$$[e^{(\alpha+i\beta)t}\mathbf{x}_1]' = \mathbf{y_1}' + i\mathbf{y_2}'$$

and

$$\mathbf{P}e^{(\alpha+i\beta)t}\mathbf{x}_1 = \mathbf{P}\mathbf{y_1} + i\mathbf{P}\mathbf{y_2}$$

Since $\mathbf{P}e^{(\alpha+i\beta)t}\mathbf{x}_1 = [e^{(\alpha+i\beta)t}\mathbf{x}_1]'$, we must have $\mathbf{P}\mathbf{y_1} = \mathbf{y}_1'$ and $\mathbf{P}\mathbf{y_2} = \mathbf{y}_2'$ ■

**Example 38.1**
Solve

$$\mathbf{y}' = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{y}$$

**Solution.**
The characteristic equation is

$$\begin{vmatrix} -\frac{1}{2} - r & 1 \\ -1 & -\frac{1}{2} - r \end{vmatrix} = (r + \frac{1}{2})^2 + 1 = 0$$

Solving this quadratic equation we find $r_1 = -\frac{1}{2} - i$ and $r_2 = -\frac{1}{2} + i$. Now,

$$(\mathbf{P} + (\frac{1}{2} + i)\mathbf{I})\mathbf{x} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} -ix_1 + x_2 \\ -x_1 - ix_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = -ix_2$. Letting $x_2 = i$ then $x_1 = 1$. Thus, an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

An eigenvector corresponding to the eigenvalue $-\frac{1}{2} + i$ is then

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

The general solution is then

$$\begin{aligned} \mathbf{y}(t) &= c_1 e^{-\frac{t}{2}} \left[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right] + c_2 e^{-\frac{t}{2}} \left[ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t \right] \\ &= \begin{bmatrix} e^{-\frac{t}{2}}(c_1 \cos t + c_2 \sin t) \\ e^{-\frac{t}{2}}(-c_1 \sin t + c_2 \cos t) \end{bmatrix} \end{aligned}$$ ■

## Practice Problems

**Problem 38.1**
Find the eigenvalues and the eigenvectors of the matrix

$$\mathbf{P} = \begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix}$$

**Problem 38.2**
Find the eigenvalues and the eigenvectors of the matrix

$$\mathbf{P} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

**Problem 38.3**
Find the eigenvalues and the eigenvectors of the matrix

$$\mathbf{P} = \begin{bmatrix} 1 & -4 & -1 \\ 3 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}$$

In Problems 36.4 - 36.6, one or more eigenvalues and corresponding eigenvectors are given for a real matrix $\mathbf{P}$. Determine a fundamental set of solutions for $\mathbf{y}' = \mathbf{P}\mathbf{y}$, where the fundamental set consists entirely of real solutions.

**Problem 38.4**
$\mathbf{P}$ is a $2 \times 2$ matrix with an eigenvalue $r = i$ and corresponding eigenvector

$$\mathbf{x} = \begin{bmatrix} -2+i \\ 5 \end{bmatrix}$$

**Problem 38.5**
$\mathbf{P}$ is a $2\times2$ matrix with an eigenvalue $r = 1+i$ and corresponding eigenvector

$$\mathbf{x} = \begin{bmatrix} -1+i \\ i \end{bmatrix}$$

**Problem 38.6**
$\mathbf{P}$ is a $4\times4$ matrix with eigenvalues $r = 1+5i$ with corresponding eigenvector

$$\mathbf{x} = \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and eigenvalue $r = 1 + 2i$ with corresponding eigenvector

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}$$

**Problem 38.7**
Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

**Problem 38.8**
Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

**Problem 38.9**
Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & -4 & -1 \\ 3 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 9 \\ 4 \end{bmatrix}$$

# 39 Homogeneous Systems with Constant Coefficients: Repeated Eigenvalues

In this section we consider the case when the characteristic equation possesses repeated roots. A major difficulty with repeated eigenvalues is that in some situations there is not enough linearly independent eigenvectors to form a fundamental set of solutions. We illustrate this in the next example.

**Example 39.1**
Solve the system

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \mathbf{y}$$

**Solution.**
The characteristic equation is

$$\begin{vmatrix} 1-r & 2 \\ 0 & 1-r \end{vmatrix} = (r-1)^2 = 0$$

and has a repeated root $r = 1$. We find an eigenvector as follows.

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It follows that $x_2 = 0$ and $x_1$ is arbitrary. Letting $x_1 = 1$ then an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This is the only eigenvector. It yields the solution

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

But we need two linearly independent solutions to form the general solution of the given system and we only have one. How do we find a second solution $\mathbf{y}_2(t)$ such that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions?
Let $\mathbf{y}$(t) be a solution. Write

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

Then we have

$$\begin{array}{rcl} y_1'(t) & = & y_1 + 2y_2 \\ y_2'(t) & = & y_2 \end{array}$$

Solving the second equation we find $y_2(t) = c_2e^t$. Substituting this into the first differential equation we find $y_1'(t) = y_1 + 2c_2e^t$. Solving this equation using the method of integrating factor we find $y_1(t) = c_1e^t + c_2te^t$. Therefore the general solution to $\mathbf{y}' = \mathbf{P}\mathbf{y}$ is

$$\mathbf{y}(t) = \left[\begin{array}{c} c_1e^t + c_2te^t \\ c_2e^t \end{array}\right] = c_1e^t\left[\begin{array}{c} 1 \\ 0 \end{array}\right] + c_2\left(e^t\left[\begin{array}{c} 0 \\ 1 \end{array}\right] + te^t\left[\begin{array}{c} 1 \\ 0 \end{array}\right]\right) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t).$$

Thus, a second solution to $\mathbf{y}' = \mathbf{P}\mathbf{y}$ is

$$\mathbf{y}_2(t) = e^t\left[\begin{array}{c} 0 \\ 1 \end{array}\right] + te^t\left[\begin{array}{c} 1 \\ 0 \end{array}\right]$$

Finally, letting $\mathbf{\Psi}(t) = [\mathbf{y}_1 \ \ \mathbf{y}_2]$ we find

$$W(0) = det(\mathbf{\Psi}(0)) = \left|\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right| = 1$$

so that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions ■

**Example 39.2**
Solve the initial value problem

$$\mathbf{y}' = \left[\begin{array}{cc} 13 & 11 \\ -11 & -9 \end{array}\right]\mathbf{y}, \quad \mathbf{y}(0) = \left[\begin{array}{c} 1 \\ 2 \end{array}\right]$$

**Solution.**
The characteristic equation is

$$\left|\begin{array}{cc} 13 - r & 11 \\ -11 & -9 - r \end{array}\right| = (r-2)^2 = 0$$

and has a repeated root $r = 2$. We find an eigenvector as follows.

$$\left[\begin{array}{cc} 11 & 11 \\ -11 & -11 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} 11x_1 + 11x_2 \\ -11x_1 - 11x_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

It follows that $x_2 = -x_1$. Letting $x_1 = 1$ then an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, one solution of $\mathbf{y}' = \mathbf{P}\mathbf{y}$ is

$$\mathbf{y}_1(t) = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}.$$

The second solution has the form

$$\mathbf{y}_2(t) = te^{2t}\mathbf{x}_1 + e^{2t}\mathbf{x}_2$$

where $\mathbf{x}_2$ is to be determined. Substituting $\mathbf{y}_2$ into the equation $\mathbf{y}' = \mathbf{P}\mathbf{y}$ we find

$$(1+2t)e^{2t}\mathbf{x}_1 + 2e^{2t}\mathbf{x}_2 = \mathbf{P}(te^{2t}\mathbf{x}_1 + e^{2t}\mathbf{x}_2).$$

We can rewrite this equation as

$$te^{2t}(\mathbf{P}\mathbf{x}_1 - 2\mathbf{x}_1) + e^{2t}(\mathbf{P}\mathbf{x}_2 - 2\mathbf{x}_2 - \mathbf{x}_1) = \mathbf{0}$$

But the set $\{e^{2t}, te^{2t}\}$ is linearly independent so that

$$\begin{aligned} \mathbf{P}\mathbf{x}_1 - 2\mathbf{x}_1 &= \mathbf{0} \\ \mathbf{P}\mathbf{x}_2 - 2\mathbf{x}_2 &= \mathbf{x}_1 \end{aligned}$$

From the second equation we find

$$\begin{bmatrix} 11 & 11 \\ -11 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11x_1 + 11x_2 \\ -11x_1 - 11x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This shows that $11x_1 + 11x_2 = 1$. Thus,

$$\mathbf{x}_2 = \frac{1}{11}\begin{bmatrix} 1 - 11x_2 \\ 11x_2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - x_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Letting $x_2 = 0$ we find

$$\mathbf{x}_2 = \frac{1}{11}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence,

$$\mathbf{y}_2(t) = te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{e^{2t}}{11}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} te^{2t} + \frac{e^{2t}}{11} \\ -te^{2t} \end{bmatrix}$$

Computing the Wronskian of the two solutions we find

$$W(0) = \begin{vmatrix} 1 & \frac{1}{11} \\ -1 & 0 \end{vmatrix} = -\frac{1}{11} \neq 0$$

Therefore, the two solutions form a fundamental set of solutions and the general solution is given by

$$\mathbf{y}(t) = \begin{bmatrix} e^{2t} & te^{2t} + \frac{e^{2t}}{11} \\ -e^{2t} & -te^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Imposing the initial condition,

$$\mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{11} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solving this system we find $c_1 = -2$ and $c_2 = 33$. Hence, the unique solution to the initial value problem is

$$\mathbf{y}(t) = \begin{bmatrix} e^{2t} + 33te^{2t} \\ 2e^{2t} - 33te^{2t} \end{bmatrix} \blacksquare$$

**Multiplicity of an Eigenvalue**
As you have seen from the discussion above, when an eigenvalue is repeated then one worries as to whether there exist enough linearly independent eigenvectors. These considerations lead to the following definitions.
Let $\mathbf{P}$ be an $n \times n$ matrix and

$$det(\mathbf{P} - r\mathbf{I}) = (r - r_1)^{n_1}(r - r_2)^{n_2} \cdots (r - r_k)^{n_k}.$$

The numbers $n_1, n_2, \cdots, n_k$ are called the **algebraic multiplicities** of the eigenvalues $r_1, r_2, \cdots, r_k$. For example, if $det(\mathbf{P} - r\mathbf{I}) = (r-2)^3(r-4)^2(r+1)$ then we say that 2 is an eigenvalue of $\mathbf{P}$ of multiplicity 3, 4 is of multiplicity 2, and $-1$ is of multiplicity 1.
We define the **geometric multiplicity** of an eigenvalue to be the number of linearly independent eigenvectors corresponding to the eigenvalue.

**Example 39.3**
Find the algebraic and geometric multiplicities of the matrix

$$\mathbf{P} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

**Solution.**
The characteristic equation is given by

$$\begin{vmatrix} 2-r & 1 & 1 & 1 \\ 0 & 2-r & 0 & 1 \\ 0 & 0 & 2-r & 1 \\ 0 & 0 & 0 & 3-r \end{vmatrix} = (2-r)\begin{vmatrix} 2-r & 0 & 1 \\ 0 & 2-r & 1 \\ 0 & 0 & 3-r \end{vmatrix} = (2-r)^3(3-r) = 0$$

Thus, $r = 2$ is an eigenvalue of algebraic multiplicity 3 and $r = 3$ is an eigenvalue of algebraic multiplicity 1.
Next, we find eigenvector(s) associated to $r = 2$. We have

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Hence, the linearly independent eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

It follows that $r = 2$ has geometric multiplicity 2.
Similarly, we find an eigenvector associated to $r = 3$.

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we find

$$\mathbf{x}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

It follows that $r = 1$ has geometric multiplicity 1 ■

All the above examples discussed thus far suggest the following theorem. For the proof, we remind the reader of the following definition: A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ forms a **basis** of $\mathbb{R}^n$ if every vector in $\mathbb{R}^n$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ and the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are linearly independent.

**Theorem 39.1**
Let $\mathbf{A}$ be an $n \times n$ matrix with eigenvalue $r_1$. Then the geometric multiplicity of $r_1$ is less than or equal to the algebraic multiplicity of $r_1$.

**Proof.**
Let $r_1$ be an eigenvalue of $\mathbf{A}$ with algebraic multiplicity $a$ and geometric multiplicity $g$. Then we have $g$ linearly independent eigenvectors $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_g\}$ with eigenvalues $r_1$. We next extend $B$ to a basis

$$\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_g, \mathbf{w}_{g+1}, \cdots, \mathbf{w}_n\}$$

of $\mathbb{R}^n$ as follows. Let $W$ be the set of all linear combinations of the vectors of $B$. If $g = n$ then $W = \mathbb{R}^n$ and we are done. If $g < n$ then $W$ is a proper subset of $\mathbb{R}^n$. Then we can find $\mathbf{w}_{g+1}$ that belongs to $\mathbb{R}^n$ but not in $W$. Then $\mathbf{w}_{g+1} \notin W$ and the set $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_g, \mathbf{w}_{g+1}\}$ is linearly independent. If $g + 1 = n$ then the set of all linear combinations of elements of $S_1$ is equal to $\mathbb{R}^n$ and we are done. If not, we can continue this extension process. In $n - g$ steps we will get a set of $n$ linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_g, \mathbf{w}_{g+1}, \cdots, \mathbf{w}_n\}$ in $\mathbb{R}^n$ which will be a basis of $\mathbb{R}^n$.
Now, let

$$\mathbf{P} = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_g\ \mathbf{w}_{g+1}\ \cdots\ \mathbf{w}_n] = [\mathbf{P}_1\ \ \mathbf{P}_2]$$

where $\mathbf{P}_1$ is the first $g$ columns and $\mathbf{P}_2$ is the last $n - g$ columns. Since the columns of $\mathbf{P}$ form a basis of $\mathbb{R}^n$ we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_g\mathbf{v}_g + c_{g+1}\mathbf{w}_{g+1} + \cdots + c_n\mathbf{w}_n = \mathbf{0}$$

which implies that $c_1 = c_2 = \cdots = c_n = 0$ and therefore $\mathbf{P}$ is an invertible matrix.
Next, write

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

where $\mathbf{B}_{11}$ is $g \times g$ matrix. Now, comparing

$$\mathbf{AP} = [r_1\mathbf{v}_1 \; r_1\mathbf{v}_2 \; \cdots \; r_1\mathbf{v}_g \; \mathbf{Aw}_{g+1} \; \cdots \; \mathbf{Aw}_n] = [r_1\mathbf{P}_1 \;\; \mathbf{AP}_2]$$

with

$$\mathbf{P}\begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = [\mathbf{P}_1\mathbf{B}_{11} + \mathbf{P}_2\mathbf{B}_{21} \;\; \mathbf{P}_1\mathbf{B}_{12} + \mathbf{P}_2\mathbf{B}_{22}]$$

we get $\mathbf{B}_{11} = r_1\mathbf{I}_g$ and $\mathbf{B}_{21} = \mathbf{0}$. Thus,

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} r_1\mathbf{I}_g & \mathbf{B}_{12} \\ \mathbf{0} & \mathbf{B}_{22} \end{bmatrix}$$

It follows that

$$\begin{aligned} det(\mathbf{A} - r\mathbf{I}_n) &= det(\mathbf{P}^{-1}(\mathbf{A} - rI_n)\mathbf{P}) = det(\mathbf{P}^{-1}\mathbf{AP} - r\mathbf{I}_{(n-g)\times(n-g)}) \\ &= det(\begin{bmatrix} (r_1 - r)\mathbf{I}_g & \mathbf{B}_{12} \\ \mathbf{0} & \mathbf{B}_{22} - r_1\mathbf{I}_{(n-g)\times(n-g)} \end{bmatrix}) \\ &= (r_1 - r)^g det(\mathbf{B}_{22} - r\mathbf{I}_{(n-g)\times(n-g)}) \end{aligned}$$

In particular, $r_1$ appears as a root of the characteristic polynomial for at least $g$ times. Since the algebraic multiplicity $a$ is the total number of times $r_1$ appears as a root, we conclude that $a \geq g$ ■

If $k_i$ is the geometric multiplicity of an eigenvalue $r_i$ of an $n \times n$ matrix $\mathbf{P}$ and $n_i$ is its algebraic multiplicity such that $k_i < n_i$ then we say that the eigenvalue $r_i$ is **defective** (it's missing some of its eigenvalues) and we call the matrix $\mathbf{P}$ a **defective matrix**. A matrix that is not defective is said to have a **full set of eigenvectors**.
There are important family of square matrices that always have a full set of eigenvectors, namely, real symmetric matrices and Hermitian matrices that we discuss next.
The **transpose** of a matrix $\mathbf{P}$, denoted by $\mathbf{P}^T$, is another matrix in which the rows and columns have been reversed. That is, $(\mathbf{P}^T)_{ij} = (\mathbf{P})_{ji}$. For example, the matrix

$$\mathbf{P} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

would have the transpose

$$\mathbf{P}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

**Theorem 39.2**
(a) If $\mathbf{P}$ and $\mathbf{Q}$ are $n \times n$ matrices then $(\mathbf{P}+\mathbf{Q})^T = \mathbf{P}^T + \mathbf{Q}^T$.
(b) If $\mathbf{P}$ is an $n \times m$ matrix and $\mathbf{Q}$ is an $m \times p$ matrix then $(\mathbf{PQ})^T = \mathbf{Q}^T\mathbf{P}^T$.

**Proof.**
(a) We have $[(\mathbf{P}+\mathbf{Q})^T]_{ij} = (\mathbf{P}+\mathbf{Q})_{ji} = (\mathbf{P})_{ji} + (\mathbf{Q})_{ji} = (\mathbf{P}^T)_{ij} + (\mathbf{Q}^T)_{ij}$.
(b) We have

$$\begin{array}{rcl} ((\mathbf{PQ})^T)_{ij} & = & (\mathbf{PQ})_{ji} = \sum_{k=1}^m (\mathbf{P})_{jk}(\mathbf{Q})_{ki} \\ & = & \sum_{k=1}^m (\mathbf{Q}^T)_{ik}(\mathbf{P}^T)_{kj} \\ & = & (\mathbf{Q}^T\mathbf{P}^T)_{ij} \blacksquare \end{array}$$

An $n \times n$ matrix $\mathbf{P}$ with real entries and with the property $\mathbf{P} = \mathbf{P}^T$ is called a **real symmetric matrix**. For example, the matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

is a real symmetric matrix.
Real symmetric matrices are a special case of a larger class of matrices, known as Hermitian matrices. An $n \times n$ matrix $\mathbf{P}$ is called **Hermitian** if $\mathbf{P} = \overline{\mathbf{P}}^T$, where $\overline{\mathbf{P}}$ is the complex conjugate of $\mathbf{P}$(The conjugate of a complex matrix is the conjugate of all its entries.) For example,

$$\mathbf{P} = \begin{bmatrix} 3 & 2+i \\ 2-i & 1 \end{bmatrix}$$

is a Hermitian matrix. Note that $\mathbf{P}^T = \overline{\mathbf{P}}^T$ when $\mathbf{P}$ is real matrix. Also note that a real symmetric matrix is a Hermitian matrix.

**Theorem 39.3**
If $\mathbf{P}$ is a real symmetric matrix or Hermitian matrix then its eigenvalues are all real.

**Proof.**
We prove the theorem for real Hermitian matrices. Suppose $r$ is an eigenvalue of $\mathbf{P}$ with corresponding eigenvector $\mathbf{x}$. We will show that $r$ is real. That is, $\overline{r} = r$. Since $\mathbf{Px} = r\mathbf{x}$, we can multiply both sides of this equation from the left by $\overline{\mathbf{x}}^T$ to obtain $\overline{\mathbf{x}}^T\mathbf{Px} = r\overline{\mathbf{x}}^T\mathbf{x}$. On the other hand, we have $\overline{\mathbf{P}}\ \overline{\mathbf{x}} = \overline{r}\ \overline{\mathbf{x}}$. Thus, $\overline{\mathbf{x}}^T\overline{\mathbf{P}}^T\mathbf{x} = \overline{r}\ \overline{\mathbf{x}}^T\mathbf{x}$. Since $\overline{\mathbf{P}}^T = \mathbf{P}$ then $r\overline{\mathbf{x}}^T\mathbf{x} = \overline{r}\ \overline{\mathbf{x}}^T\mathbf{x}$. Since $\overline{\mathbf{x}}^T\mathbf{x} = ||\mathbf{x}||_2 \neq 0$ , where $||\mathbf{x}||_2$ is the two norm of $x$,($\mathbf{x}$ is an eigenvector) we see that $\overline{r} = r$, that is, $r$ is real ■

The following theorem asserts that every Hermitian or real symmetric matrix has a full set of eigenvectors. Therefore, when we study the homogeneous linear first order system $\mathbf{y}' = \mathbf{Py}$, where $\mathbf{P}$ is an $n \times n$ a real symmetric matrix we know that all solutions forming a fundamental set are of the form $e^{rt}\mathbf{x}$, where $(r, \mathbf{x})$ is an eigenpair.

**Theorem 39.4**
If $\mathbf{P}$ is a Hermitian matrix (or a symmetric matrix) then for each eigenvalue, the algebraic multiplicity equals the geometric multiplicity.

**Proof.**
We will prove the result for real symmetric matrices. In Section 41, we will show that a real symmetric matrix has a set of $n$ linearly independent eigenvectors. So if $a_1, a_2, \cdots, a_k$ are the algebraic multiplicities with corresponding geometric multiplicities $g_1, g_2, \cdots, g_k$ then we have $a_1 + a_2 + \cdots + a_k = g_1 + g_2 + \cdots + g_k = n$. By Theorem 39.1, this happens only when $a_i = g_i$ ■

## Practice Problems

In Problems 39.1 - 39.4, we consider the initial value problem $\mathbf{y}' = \mathbf{P}\mathbf{y},\ \mathbf{y}(0) = \mathbf{y}_0$.
(a) Compute the eigenvalues and the eigenvectors of $\mathbf{P}$.
(b) Construct a fundamental set of solutions for the given differential equation. Use this fundamental set to construct a fundamental matrix $\mathbf{\Psi}(t)$.
(c) Impose the initial condition to obtain the unique solution to the initial value problem.

**Problem 39.1**

$$\mathbf{P} = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

**Problem 39.2**

$$\mathbf{P} = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

**Problem 39.3**

$$\mathbf{P} = \begin{bmatrix} -3 & -36 \\ 1 & 9 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

**Problem 39.4**

$$\mathbf{P} = \begin{bmatrix} 6 & 1 \\ -1 & 4 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

**Problem 39.5**
Consider the homogeneous linear system

$$\mathbf{y}' = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{y}$$

(a) Write the three component differential equations of $\mathbf{y}' = \mathbf{P}\mathbf{y}$ and solve these equations sequentially, first finding $y_3(t)$, then $y_2(t)$, and then $y_1(t)$.
(b) Rewrite the component solutions obtained in part (a) as a single matrix equation of the form $\mathbf{y} = \mathbf{\Psi}(t)\mathbf{c}$. Show that $\mathbf{\Psi}(t)$ is a fundamental matrix.

In Problems 39.6 - 39.8, Find the eigenvalues and eigenvectors of $\mathbf{P}$. Give the geometric and algebraic multiplicity of each eigenvalue. Does $\mathbf{P}$ have a full set of eigenvectors?

**Problem 39.6**

$$\mathbf{P} = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

**Problem 39.7**

$$\mathbf{P} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

**Problem 39.8**

$$\mathbf{P} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

**Problem 39.9**
Let $\mathbf{P}$ be a $2 \times 2$ real matrix with an eigenvalue $r_1 = a + ib$ where $b \neq 0$. Can $\mathbf{P}$ have a repeated eigenvalue? Can $\mathbf{P}$ be defective?

**Problem 39.10**
Dtermine the numbers $x$ and $y$ so that the following matrix is real and symmetric.

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & x \\ y & 2 & 2 \\ 6 & 2 & 7 \end{bmatrix}$$

**Problem 39.11**
Dtermine the numbers $x$ and $y$ so that the following matrix is Hermitian.

$$\mathbf{P} = \begin{bmatrix} 2 & x + 3i & 7 \\ 9 - 3i & 5 & 2 + yi \\ 7 & 2 + 5i & 3 \end{bmatrix}$$

**Problem 39.12**
(a) Give an example of a $2 \times 2$ matrix $\mathbf{P}$ that is not invertible but have a full set of eigenvectors.
(b) Give an example of a $2 \times 2$ matrix $\mathbf{P}$ that is invertible but does not have a full set of eigenvectors.

# 40 Nonhomogeneous First Order Linear Systems

In this section, we seek the general solution to the nonhomogeneous first order linear system

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}(t) \tag{25}$$

where the components of the $n \times n$ matrix $\mathbf{P}$(t) and the $n \times 1$ vector $\mathbf{g}$(t) are continuous on $a < t < b$.
The solution structure is similar to one for nth order linear nonhomogeneous equations and is the result of the following theorem.

**Theorem 40.1**
Let $\{\mathbf{y}_1(t), \mathbf{y}_2(t), \cdots, \mathbf{y}_n(t)\}$ be a fundamental set of solutions to the homogeneous equation $\mathbf{y}' = \mathbf{P}(t)\mathbf{y}$ and $\mathbf{y}_p(t)$ be a particular solution of the nonhomogeneous equation $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}(t)$. Then the general solution of the nonhomogeneous equation is given by

$$\mathbf{y}(t) = \mathbf{y}_p(t) + c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t)$$

**Proof.**
Let $\mathbf{y}(t)$ be any solution to equation (25). Since $\mathbf{y}_p(t)$ is also a solution, we have

$$\begin{array}{rcl} (\mathbf{y} - \mathbf{y}_p)' & = & \mathbf{y}' - \mathbf{y}_p' \\ & = & \mathbf{P}(t)\mathbf{y} + \mathbf{g}(t) - [\mathbf{P}(t)\mathbf{y}_p + \mathbf{g}(t)] \\ & = & \mathbf{g}(t) - \mathbf{g}(t) = 0 \end{array}$$

Therefore $\mathbf{y} - \mathbf{y}_p$ is a solution to the homogeneous equation. But $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ is a fundamental set of solutions to the homogeneous equation so that there exist unique constants $c_1, c_2, \cdots, c_n$ such that $\mathbf{y}(t) - \mathbf{y}_p(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t)$. Hence,

$$\mathbf{y}(t) = \mathbf{y}_p(t) + c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t) \blacksquare$$

Since the sum $c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \cdots + c_n\mathbf{y}_n(t)$ represents the general solution to the homogeneous equation then we will denote it by $\mathbf{y}_h$ so that the general solution of (25) takes the form

$$\mathbf{y}(t) = \mathbf{y}_h(t) + \mathbf{y}_p(t)$$

Superposition theorem for nth order linear nonhomogeneous equations holds as well for linear systems.

**Theorem 40.2**
Let $\mathbf{y}_1(t)$ be a solution of $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}_1(t)$ and $\mathbf{y}_2(t)$ a solution of $\mathbf{y}' = \mathbf{P}(t)\mathbf{y}+\mathbf{g}_2(t)$. Then for any constants $c_1$ and $c_2$ the function $\mathbf{y}_p(t) = c_1\mathbf{y}_1(t)+c_2\mathbf{y}_2(t)$ is a particular solution of the equation

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + c_1\mathbf{g}_1(t) + c_2\mathbf{g}_2(t), \quad a < t < b.$$

**Proof.**
We have

$$\begin{array}{rcl} \mathbf{y}_p' &=& (c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t))' \\ &=& c_1\mathbf{y}_1'(t) + c_2\mathbf{y}_2'(t) \\ &=& c_1(\mathbf{P}(t)\mathbf{y}_1 + \mathbf{g}_1(t)) + c_2(\mathbf{P}(t)\mathbf{y}_2 + \mathbf{g}_2(t)) \\ &=& \mathbf{P}(t)(c_1\mathbf{y}_1 + c_2\mathbf{y}_2) + c_1\mathbf{g}_1(t) + c_2\mathbf{g}_2(t) \\ &=& \mathbf{P}(t)\mathbf{y}_p + c_1\mathbf{g}_1(t) + c_2\mathbf{g}_2(t) \blacksquare \end{array}$$

**Example 40.1**
Consider the system

$$\mathbf{y}' = \left[\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}\right]\mathbf{y} + \left[\begin{array}{c} e^{2t} \\ -2t \end{array}\right]$$

(a) Find $\mathbf{y}_h(t)$.
(b) Find $\mathbf{y}_p(t)$.
(c) Find the general solution to the given system.

**Solution.**
(a) The characteristic equation is

$$\left|\begin{array}{cc} 1-r & 2 \\ 2 & 1-r \end{array}\right| = (r-1)^2 - 4 = 0$$

Thus, the eigenvalues are $r_1 = -1$ and $r_2 = 3$. An eigenvector corresponding to $r_1 = -1$ is found as follows

$$(\mathbf{P}+\mathbf{I})\mathbf{x}_1 = \left[\begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} 2x_1 + 2x_2 \\ 2x_1 + 2x_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Solving this system we find $x_2 = -x_1$. Letting $x_1 = 1$ we find $x_2 = -1$ and an eigenvector is

$$\mathbf{x}_1 = \left[\begin{array}{c} 1 \\ -1 \end{array}\right]$$

Similarly, for $r_2 = 3$ we have

$$(\mathbf{P} - 3\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + 2x_2 \\ 2x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_2 = x_1$. Letting $x_1 = 1$ we find $x_2 = 1$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence,

$$\mathbf{y}_h(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) To find $\mathbf{y}_p(t)$ we note first that

$$\mathbf{g}(t) = \begin{bmatrix} e^{2t} \\ -2t \end{bmatrix} = e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \mathbf{g}_1(t) + \mathbf{g}_2(t).$$

By Superposition Theorem above, we will find a particular solution to $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}_1(t)$ as well as to $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}_2(t)$. For the first equation, we use the method of undetermined coefficients. That is, we seek a solution of the form $\mathbf{u}_p(t) = e^{2t}\mathbf{a}$ where $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is a constant vector to be determined. Substituting $\mathbf{u}_p$ into the equation $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}_1(t)$ to obtain

$$2e^{2t}\mathbf{a} = \mathbf{P}(t)(e^{2t})\mathbf{a} + e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This equation reduces to

$$\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solving this system we find $\mathbf{a} = -\frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Hence,

$$\mathbf{u}_p(t) = -\frac{1}{3} e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Now, for the system $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}_2(t)$ we consider the guess function $\mathbf{v}_p(t) = t\mathbf{b} + \mathbf{c}$ where $\mathbf{b}$ and $\mathbf{c}$ are vectors whose components are to be determined. Substituting this guess into the differential equation we find

$$\mathbf{b} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} (t\mathbf{b} + \mathbf{c}) + t \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

and this reduces to

$$\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\mathbf{b} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) + t\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\mathbf{c} - \mathbf{b}\right) = \mathbf{0}$$

Since the set $\{1, t\}$ is linearly independent, the last equation implies the two systems

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solving these systems we find

$$\mathbf{b} = \frac{2}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \mathbf{c} = \frac{2}{9}\begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

Hence,

$$\mathbf{v}_p(t) = t\frac{2}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{2}{9}\begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

By the Superposition Theorem, we have

$$\mathbf{y}_p(t) = \mathbf{u}_p(t) + \mathbf{v}_p(t).$$

(c) The general solution is given by

$$\mathbf{y}(t) = c_1e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{3}e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{2t}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{2}{9}\begin{bmatrix} -4 \\ 5 \end{bmatrix} \blacksquare$$

**The Variation of Parameters Method**

Next, we consider a method for finding a particular solution to (25) and the unique solution to the initial-value problem

$$\mathbf{y}' = \mathbf{P}(t) + \mathbf{g}(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0 \tag{26}$$

To solve the above initial-value problem, we start by looking at a fundamental set $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n,\}$ of the homogeneous equation. Then we construct the fundamental matrix $\mathbf{\Psi}(t) = [\mathbf{y}_1 \ \mathbf{y}_2 \ \cdots \ \mathbf{y}_n]$. Recall that $\mathbf{\Psi}(t)$ (See Section 33) satisfies the differential equation

$$\mathbf{\Psi}' = \mathbf{P}(t)\mathbf{\Psi}, \quad a < t < b.$$

Now, since

$$\mathbf{y}(t) = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = \mathbf{\Psi}(t)\mathbf{c}$$

we "vary the parameter" and look for a solution to the initial-value problem (26) of the form $\mathbf{y} = \mathbf{\Psi}(t)\mathbf{u}(t)$, where $\mathbf{u}$ is an unknown vector to be determined. Substituting this vector function into (26) to obtain

$$\mathbf{\Psi}'(t)\mathbf{u}(t) + \mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)\mathbf{u}(t) + \mathbf{g}(t).$$

Using the fact that $\mathbf{\Psi}' = \mathbf{P}(t)\mathbf{\Psi}$ the last equation reduces to

$$\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{g}(t)$$

Since $\mathbf{\Psi}(t)$ is a fundamental matrix, $det(\mathbf{\Psi}(t)) \neq 0$ and this implies that the matrix $\mathbf{\Psi}(t)$ is invertible. Hence, we can write

$$\mathbf{u}'(t) = \mathbf{\Psi}^{-1}(t)\mathbf{g}(t).$$

Integrating both sides we find

$$\mathbf{u}(t) = \mathbf{u}(t_0) + \int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds$$

where $\mathbf{u}(t_0)$ is an arbitrary constant vector. It follows that the general solution to $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}(t)$ is given by

$$\mathbf{y}(t) = \mathbf{\Psi}(t)\mathbf{u}(t) = \mathbf{\Psi}(t)\mathbf{u}(t_0) + \mathbf{\Psi}(t)\int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds = \mathbf{y}_h(t) + \mathbf{y}_p(t)$$

where $\mathbf{y}_h(t) = \mathbf{\Psi}(t)\mathbf{u}(t_0)$ and $\mathbf{y}_p(t) = \mathbf{\Psi}(t)\int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds$. Finally, since $\mathbf{y}(t_0) = \mathbf{\Psi}(t_0)\mathbf{u}(t_0)$ we have $\mathbf{u}(t_0) = \mathbf{\Psi}^{-1}(t_0)\mathbf{y}(t_0)$ and the unique solution to the initial value problem is given by

$$\mathbf{y}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{y}(t_0) + \mathbf{\Psi}(t)\int_{t_0}^{t} \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds.$$

We refer to the last expression as the **variation of parameters** formula for the solution of the initial-value problem.

**Remark 40.1**
Consider the initial value problem

$$y' = p(t)y + g(t), \quad y(t_0) = y_0, \quad a < t < b.$$

Clearly, $\{e^{\int_{t_0}^t p(s)ds}\}$ is a fundamental set. Letting $\Psi(t) = e^{\int_{t_0}^t p(s)ds}$ in the variation of parameters formula we see that the unique solution is given by

$$y(t) = y_0 e^{\int_{t_0}^t p(s)ds} + e^{\int_{t_0}^t p(s)ds} \int_{t_0}^t e^{-\int_{t_0}^t p(s)ds} g(s)ds$$

which is nothing than the method of integrating factor.

**Example 40.2**

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

**Solution.**
We first find a fundamental matrix of the linear system $\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{y}$. The characteristic equation is

$$\begin{vmatrix} 1-r & 1 \\ 1 & 1-r \end{vmatrix} = r(r-2) = 0$$

and has eigenvalues $r_1 = 0$ and $r_2 = 2$. We find an eigenvector corresponding to $r_1 = 0$ as follows.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It follows that $x_1 = -x_2$. Letting $x_1 = 1$ then $x_2 = -1$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

An eigenvector corresponding to $r_2 = 2$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving we find $x_1 = x_2$. Letting $x_1 = 1$ we find $x_2 = 1$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, a fundamental matrix is

$$\mathbf{\Psi} = \begin{bmatrix} 1 & e^{2t} \\ -1 & e^{2t} \end{bmatrix}.$$

Therefore,

$$\mathbf{\Psi}^{-1} = 0.5 \begin{bmatrix} 1 & -1 \\ e^{-2t} & e^{-2t} \end{bmatrix}.$$

But the variation of parameters formula is

$$\mathbf{y}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(0)\mathbf{y}(0) + \mathbf{\Psi}(t)\int_0^t \mathbf{\Psi}^{-1}(s)\mathbf{g}(s)ds.$$

Thus,

$$\begin{aligned} \mathbf{y}(t) &= \begin{bmatrix} 1 & e^{2t} \\ -1 & e^{2t} \end{bmatrix} \int_0^t 0.5 \begin{bmatrix} e^{2s} \\ 1 \end{bmatrix} ds \\ &= \begin{bmatrix} 1 & e^{2t} \\ -1 & e^{2t} \end{bmatrix} (0.25) \begin{bmatrix} 1e^{2t} - 1 \\ 2t \end{bmatrix} \\ &= 0.25 \begin{bmatrix} e^{2t} - 1 + 2te^{2t} \\ -(e^{2t} - 1) + 2te^{2t} \end{bmatrix} \blacksquare \end{aligned}$$

## Practice Problems

In Problems 40.1 - 40.3, we consider the initial value problem $\mathbf{y}' = \mathbf{P}\mathbf{y} + \mathbf{g}(t),\ \mathbf{y}(t_0) = \mathbf{y}_0$.
(a) Find the eigenpairs of the matrix $\mathbf{P}$ and form the general homogeneous solution of the differential equation.
(b) Construct a particular solution by assuming a solution of the form suggested and solving for the undetermined constant vectors $\mathbf{a}$,$\mathbf{b}$, and $\mathbf{c}$.
(c) Form the general solution of the nonhomogeneous differential equation.
(d) Find the unique solution to the initial value problem.

**Problem 40.1**

$$\mathbf{y}' = \left[\begin{array}{cc} -2 & 1 \\ 1 & -2 \end{array}\right]\mathbf{y} + \left[\begin{array}{c} 1 \\ 1 \end{array}\right], \quad \mathbf{y}_0 = \left[\begin{array}{c} 3 \\ 1 \end{array}\right]$$

Try $\mathbf{y}_p(t) = \mathbf{a}$.

**Problem 40.2**

$$\mathbf{y}' = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]\mathbf{y} + \left[\begin{array}{c} t \\ -1 \end{array}\right], \quad \mathbf{y}_0 = \left[\begin{array}{c} 2 \\ -1 \end{array}\right]$$

Try $\mathbf{y}_p(t) = t\mathbf{a} + \mathbf{b}$.

**Problem 40.3**

$$\mathbf{y}' = \left[\begin{array}{cc} -3 & -2 \\ 4 & 3 \end{array}\right]\mathbf{y} + \left[\begin{array}{c} \sin t \\ 0 \end{array}\right], \quad \mathbf{y}_0 = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Try $\mathbf{y}_p(t) = (\sin t)\mathbf{a} + (\cos t)\mathbf{b}$.

**Problem 40.4**
Consider the initial value problem

$$\mathbf{y}' = \left[\begin{array}{cc} 0 & 2 \\ -2 & 0 \end{array}\right]\mathbf{y} + \mathbf{g}(t), \quad \mathbf{y}\left(\frac{\pi}{2}\right) = \mathbf{y}_0.$$

Suppose we know that

$$\mathbf{y}(t) = \left[\begin{array}{c} 1 + \sin 2t \\ e^t + \cos 2t \end{array}\right]$$

is the unique solution. Determine $\mathbf{g}(\mathrm{t})$ and $\mathbf{y}_0$.

**Problem 40.5**
Consider the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & t \\ t^2 & 1 \end{bmatrix} \mathbf{y} + \mathbf{g}(t), \quad \mathbf{y}(1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Suppose we know that

$$\mathbf{y}(t) = \begin{bmatrix} t+\alpha \\ t^2+\beta \end{bmatrix}$$

is the unique solution. Determine $\mathbf{g}$(t) and the constants $\alpha$ and $\beta$.

**Problem 40.6**
Let $\mathbf{P}$(t) be a $2 \times 2$ matrix with continuous entries. Consider the differential equation $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \mathbf{g}(t)$. Suppose that $\mathbf{y}_1(t) = \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix}$ is the solution to $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ and $\mathbf{y}_2(t) = \begin{bmatrix} e^t \\ -1 \end{bmatrix}$ is the solution to $\mathbf{y}' = \mathbf{P}(t)\mathbf{y} + \begin{bmatrix} e^t \\ -1 \end{bmatrix}$. Determine $\mathbf{P}$(t). Hint: Form the matrix equation $[\mathbf{y}_1' \ \mathbf{y}_2'] = \mathbf{P}[\mathbf{y}_1 \ \mathbf{y}_2] + [\mathbf{g}_1 \ \mathbf{g}_2]$.

**Problem 40.7**
Consider the linear system $\mathbf{y}' = \mathbf{P}\mathbf{y} + \mathbf{b}$ where $\mathbf{P}$ is a constant matrix and $\mathbf{b}$ is a constant vector. An **equilibrium solution,** $\mathbf{y}$(t), is a constant solution of the differential equation.
(a) Show that $\mathbf{y}' = \mathbf{P}\mathbf{y} + \mathbf{b}$ has a unique equilibrium solution when $\mathbf{P}$ is invertible.
(b) If the matrix $\mathbf{P}$ is not invertible, must the differential equation $\mathbf{y}' = \mathbf{P}\mathbf{y} + \mathbf{b}$ possess an equilibrium solution? If an equilibrium solution does exist in this case, is it unique?

**Problem 40.8**
Determine all the equilibrium solutions (if any).

$$\mathbf{y}' = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

**Problem 40.9**
Determine all the equilibrium solutions (if any).

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

Consider the homogeneous linear system $\mathbf{y}' = \mathbf{P}\mathbf{y}$. Recall that any associated fundamental matrix satisfies the matrix differential equation $\mathbf{\Psi}' = \mathbf{P}\mathbf{\Psi}$. In Problems 40.10 - 40.12, construct a fundamental matrix that solves the matrix initial value problem $\mathbf{\Psi}' = \mathbf{P}\mathbf{\Psi}, \quad \mathbf{\Psi}(t_0) = \mathbf{\Psi}_0$.

**Problem 40.10**

$$\mathbf{\Psi}' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{\Psi}, \quad \mathbf{\Psi}(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Problem 40.11**

$$\mathbf{\Psi}' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{\Psi}, \quad \mathbf{\Psi}(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

**Problem 40.12**

$$\mathbf{\Psi}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \mathbf{\Psi}, \quad \mathbf{\Psi}\left(\frac{\pi}{4}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In Problems 40.13 - 40.14, use the method of variation of parameters to solve the given initial value problem.

**Problem 40.13**

$$\mathbf{y}' = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

**Problem 40.14**

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

# 41 Solving First Order Linear Systems with Diagonalizable Constant Coefficients Matrix

In this section we discuss a method for solving the initial value problem

$$\mathbf{y}' = \mathbf{P}\mathbf{y} + \mathbf{g}(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0, \quad a < t < b$$

where $\mathbf{P}$ is a nondefective constant matrix and the entries of $\mathbf{g}$(t) are continuous in $a < t < b$.. This type of matrices is always diagonalizable, a concept that we will introduce and discuss below.

**Similar Matrices**
An $n \times n$ matrix $\mathbf{A}$ is said to be **similar** to an $n \times n$ matrix $\mathbf{B}$ if there is an invertible $n \times n$ matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{B}$. Note that if we let $\mathbf{R} = \mathbf{T}^{-1}$ then $\mathbf{B} = \mathbf{R}\mathbf{A}\mathbf{R}^{-1}$ so whether the inverse comes first or last does not matter. Also, note that from this definition we can write $\mathbf{A} = (\mathbf{T}^{-1})^{-1}\mathbf{B}\mathbf{T}^{-1}$ so that the matrix $\mathbf{B}$ is similar to $\mathbf{A}$. That's why, in the literature one will just say that $\mathbf{A}$ and $\mathbf{B}$ are **similar matrices.**
The first important result of this concept in the following theorem.

**Theorem 41.1**
If $\mathbf{A}$ and $\mathbf{B}$ are similar then they have the same characteristic equation and therefore the same eigenvalues.

**Proof.**
Since $\mathbf{A}$ and $\mathbf{B}$ are similar, $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ for some invertible matrix $\mathbf{T}$. From this one notices the following

$$\begin{aligned} det(\mathbf{A} - r\mathbf{I}) &= det(\mathbf{T}^{-1}(\mathbf{A} - r\mathbf{I})\mathbf{T}) = \\ &= det(\mathbf{T}^{-1}(\mathbf{A}\mathbf{T} - r\mathbf{I}) = det(\mathbf{B} - r\mathbf{I}) \end{aligned}$$

This shows that $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic equation and therefore the same eigenvalues. We point out the following equality that we used in the above discussion: $det(\mathbf{T}^{-1}\mathbf{T}) = det(\mathbf{T}^{-1})det(\mathbf{T}) = det(\mathbf{I}) = 1$ ■

The second important result is the following.

**Theorem 41.2**
Suppose that $\mathbf{W}(t)$ is a solution to the system $\mathbf{y}' = \mathbf{B}\mathbf{y}$ and $\mathbf{B}$ and $\mathbf{A}$ are similar matrices with $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. Then $\mathbf{y}(t) = \mathbf{T}\mathbf{W}(t)$ is a solution to $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

**Proof.**
Since $\mathbf{W}(t)$ is a solution to $\mathbf{y}' = \mathbf{B}\mathbf{y}$ we have $\mathbf{W}' = \mathbf{B}\mathbf{W}$. But $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ so we can write $\mathbf{W}' = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{W}$. Thus, $\mathbf{T}\mathbf{W}' = \mathbf{A}\mathbf{T}\mathbf{W}$. That is, $(\mathbf{T}\mathbf{W}(t))' = \mathbf{A}(\mathbf{T}\mathbf{W})$. But this says that $\mathbf{y}(t) = \mathbf{T}\mathbf{W}(t)$ is a solution to $\mathbf{y}' = \mathbf{A}\mathbf{y}$ ■

**Diagonalizable Matrices**
An $n \times n$ matrix $\mathbf{A}$ is said to be **diagonalizable** if there is an invertible matrix $\mathbf{T}$ such that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D} = \left[\begin{array}{ccccc} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{array}\right]$$

In other words, the matrix $\mathbf{A}$ is similar to a diagonal matrix.
Our first question regarding diagonalization is the question of whether every square matrix is diagonalizable.

**Example 41.1**
Show that the matrix

$$\mathbf{A} = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]$$

is not diagonalizable.

**Solution.**
If $\mathbf{A}$ is diagonalizable then we expect to find an invertible matrix $\mathbf{T} = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is a diagonal matrix. But

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \frac{1}{ad-bc}\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right]\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\frac{1}{ad-bc}\left[\begin{array}{cc} cd & d^2 \\ -c^2 & -cd \end{array}\right]$$

Now, if $c = 0$ then $d \neq 0$ and therefore the above product does not result in a diagonal matrix. Similar argument applies if $d = 0$. Hence, the given matrix is not diagonalizable ■

Note that the above matrix does not have a full set of eigenvectors. Indeed, the characteristic equation of the matrix $\mathbf{A}$ is

$$\left|\begin{array}{cc} -r & 1 \\ 0 & -r \end{array}\right| = 0$$

Expanding the determinant and simplifying we obtain

$$r^2 = 0.$$

The only eigenvalue of $\mathbf{A}$ is $r = 0$. Now, an eigenvector is found as follows.

$$\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{c} x_2 \\ 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

Thus, we find that $x_2 = 0$ and $x_1$ is arbitrary. Hence, an eigenvector is

$$\mathbf{x} = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$$

Since the geometric multiplicity of $r = 0$ is less than its algebraic multiplicity, $\mathbf{A}$ is defective.
So, is having a full set of eigenvectors results in the matrix to be diagonalizable? The answer to this question is provided by the following theorem.

**Theorem 41.3**
An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if it has a set of $n$ linearly independent eigenvectors.

**Proof.**
($\Longrightarrow$) : Suppose first that $\mathbf{A}$ is diagonalizable. Then there are an ivnertible matrix $\mathbf{T}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$. By Theorem 41.1, the diagonal entries of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$. Now, let $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$ be the $n$ columns of $\mathbf{T}$ so that $\mathbf{T} = [\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n]$. Since $\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{D}$ we have $\mathbf{A}\mathbf{T} = \mathbf{T}\mathbf{D}$. That is,

$$\mathbf{A}[\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n] = [\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n]\left[\begin{array}{ccccc} r_1 & 0 & 0 & \cdots & 0 \\ 0 & r_2 & 0 & \cdots & 0 \\ 0 & 0 & r_3 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & r_n \end{array}\right]$$

where $r_1, r_2, \cdots, r_n$ are the eigenvalues of $\mathbf{A}$. The above matrix equation is the same as

$$[\mathbf{A}\mathbf{c}_1 \ \mathbf{A}\mathbf{c}_2 \ \cdots \ \mathbf{A}\mathbf{c}_n] = [r_1\mathbf{c}_1 \ r_2\mathbf{c}_2 \ \cdots \ r_n\mathbf{c}_n]$$

and this equality yields

$$\mathbf{A}\mathbf{c}_1 = r_1\mathbf{c}_1, \ \mathbf{A}\mathbf{c}_2 = r_2\mathbf{c}_2, \ \cdots, \ \mathbf{A}\mathbf{c}_n = r_n\mathbf{c}_n.$$

This shows that the eigenvectors of $\mathbf{A}$ are just the columns of $\mathbf{T}$. Now, if

$$c_1\mathbf{c}_1 + c_2\mathbf{A}\mathbf{c}_2 + \cdots + c_n\mathbf{c}_n = [\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n]\mathbf{c} = \mathbf{0}$$

then the invertibility of $\mathbf{T}$ forces $\mathbf{c} = \mathbf{0}$. This shows that the columns of $\mathbf{T}$, and therefore the eigenvectors of $\mathbf{A}$, are linearly independent.
$(\Longleftarrow)$ : Now, suppose that $\mathbf{A}$ has $n$ linearly independent eigenvectors $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_n$ with corresponding eigenvalues $r_1, r_2, \cdots, r_n$. Let $\mathbf{T} = [\mathbf{c}_1 \ \mathbf{c}_2 \ \cdots \ \mathbf{c}_n]$ and $\mathbf{D}$ be the diagonal matrix with diagonal entries $r_1, r_2, \cdots, r_n$. Then $\mathbf{AT} = \mathbf{TA}$. Also, since the eigenvectors are linearly independent, $\mathbf{T}$ is invertible and therefore $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{D}$. This shows that $\mathbf{A}$ is diagonalizable ■

**Remark 41.1**
We have seen in Section 39 that real symmetric matrices and Hermitian matrices have full set of eigenvectors. According to the previous theorem these matrices are always diagonalizable.

**Solution Method of First Order Linear Systems by Uncoupling**
We finally describe a method based on matrix diagonalization for solving the initial value problem

$$\mathbf{y}' = \mathbf{P}\mathbf{y} + \mathbf{g}(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0, \quad a < t < b. \tag{27}$$

where the components of $\mathbf{g}$(t) are continuous in $a < t < b$ and the matrix $\mathbf{P}$ is a diagonalizable constant matrix, that is, there is an invertible matrix $\mathbf{T}$ such that

$$\mathbf{T}^{-1}\mathbf{P}\mathbf{T} = \mathbf{D} = \left[\begin{array}{ccccc} r_1 & 0 & 0 & \cdots & 0 \\ 0 & r_2 & 0 & \cdots & 0 \\ 0 & 0 & r_3 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & r_n \end{array}\right]$$

where $r_1, r_2, \cdots, r_n$ are the eigenvalues of $\mathbf{P}$. Thus, $\mathbf{P} = \mathbf{TDT}^{-1}$. Substituting into the differential equation we find

$$\mathbf{y}' = \mathbf{TDT}^{-1}\mathbf{y} + \mathbf{g}(t)$$

or

$$(\mathbf{T}^{-1}\mathbf{y})' = \mathbf{D}(\mathbf{T}^{-1}\mathbf{y}) + \mathbf{T}^{-1}\mathbf{g}(t).$$

Letting $\mathbf{z}(t) = \mathbf{T}^{-1}\mathbf{y}$ then the previous equation reduces to

$$\mathbf{z}'(t) = \mathbf{Dz}(t) + \mathbf{T}^{-1}\mathbf{g}(t).$$

Letting

$$\mathbf{z}(t) = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad \mathbf{T}^{-1}\mathbf{g}(t) = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}, \quad \mathbf{z}(t_0) = \mathbf{T}^{-1}\mathbf{y}_0 = \begin{bmatrix} z_1^0 \\ z_2^0 \\ \vdots \\ z_n^0 \end{bmatrix}$$

We can write

$$\begin{bmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{bmatrix} = \begin{bmatrix} r_1 z_1 \\ r_2 z_2 \\ \vdots \\ r_n z_n \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}$$

Thus, for $1 \le i \le n$ we have the scalar initial value problem

$$z_i' = r_i z_i + h_i, \quad z_i(t_0) = z_i^0.$$

Solving this equation using the method of integrating factor we find

$$z_i(t) = e^{r_i(t-t_0)} z_i^0 + \int_{t_0}^{t} e^{r_i(t-s)} h_i(s) ds, \quad 1 \le i \le n.$$

Having found the vector $\mathbf{z}$(t) we then find the solution to the original initial value problem by forming the matrix product $\mathbf{y}(t) = \mathbf{Tz}(t)$.

**Example 41.2**
Solve the following system by making the change of variables $\mathbf{y} = \mathbf{Tz}$.

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} t \\ -t+3 \end{bmatrix}$$

**Solution.**
The characteristic equation is

$$\begin{vmatrix} 1-r & 1 \\ 1 & 2-r \end{vmatrix} = r(r-3) = 0$$

Thus, the eigenvalues are $r_1 = 0$ and $r_2 = 3$. An eigenvector corresponding to $r_1 = 0$ is found as follows

$$(\mathbf{P} + 0\mathbf{I})\mathbf{x}_1 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $x_1 = -x_2$. Letting $x_2 = -1$ we find $x_1 = 1$ and an eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly, for $r_2 = 3$ we have

$$(\mathbf{P} - 3\mathbf{I})\mathbf{x}_2 = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_2 \\ 2x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we find $2x_1 = x_2$. Letting $x_1 = 1$ we find $x_2 = 2$ and an eigenvector is

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore

$$\mathbf{T} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Thus,

$$\mathbf{T}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

Letting $\mathbf{y} = \mathbf{T}\mathbf{z}$ we obtain

$$\mathbf{z}' = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{z} + \begin{bmatrix} t-1 \\ 1 \end{bmatrix}$$

That is,

$$\begin{aligned} z_1' &= t - 1 \\ z_2' &= 3z_2 + 1 \end{aligned}$$

Solving this system we find

$$\mathbf{z}(t) = \begin{bmatrix} \frac{1}{2}t^2 - t + c_1 \\ -\frac{1}{3} + c_2 e^{3t} \end{bmatrix}$$

Thus, the general solution is

$$\mathbf{y}(t) = \mathbf{T}\mathbf{z}(t) = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2}t^2 - t + c_1 \\ -\frac{1}{3} + c_2 e^{3t} \end{bmatrix} = \begin{bmatrix} 1 & e^{3t} \\ -1 & 2e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} \frac{1}{2}t^2 - t - \frac{1}{3} \\ -\frac{1}{2}t^2 + t - \frac{2}{3} \end{bmatrix} \blacksquare$$

## Practice Problems

In Problems 41.1 - 41.4, the given matrix is diagonalizable. Find matrices $\mathbf{T}$ and $\mathbf{D}$ such that $\mathbf{T}^{-1}\mathbf{P}\mathbf{T} = \mathbf{D}$.

**Problem 41.1**

$$\mathbf{P} = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$$

**Problem 41.2**

$$\mathbf{P} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

**Problem 41.3**

$$\mathbf{P} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

**Problem 41.4**

$$\mathbf{P} = \begin{bmatrix} -2 & 2 \\ 0 & 3 \end{bmatrix}$$

In Problems 41.5 - 41.6, you are given the characteristic polynomial for the matrix $\mathbf{P}$. Determine the geometric and algebraic multiplicities of each eigenvalue. If the matrix $\mathbf{P}$ is diagonalizable, find matrices $\mathbf{T}$ and $\mathbf{D}$ such that $\mathbf{T}^{-1}\mathbf{P}\mathbf{T} = \mathbf{D}$.

**Problem 41.5**

$$\mathbf{P} = \begin{bmatrix} 7 & -2 & 2 \\ 8 & -1 & 4 \\ -8 & 4 & -1 \end{bmatrix}, \quad p(r) = (r-3)^2(r+1).$$

**Problem 41.6**

$$\mathbf{P} = \begin{bmatrix} 5 & -1 & 1 \\ 14 & -3 & 6 \\ 5 & -2 & 5 \end{bmatrix}, \quad p(r) = (r-2)^2(r-3).$$

**Problem 41.7**
At leat two (and possibly more) of the following four matrices are diagonalizable. You should be able to recognize two by inspection. Choose them and give a reason for your choice.

$$(a)\left[\begin{array}{cc}5&6\\3&4\end{array}\right],\ (b)\left[\begin{array}{cc}3&6\\6&9\end{array}\right],\ (c)\left[\begin{array}{cc}3&0\\3&-4\end{array}\right],\ (d)\left[\begin{array}{cc}1&3\\-1&4\end{array}\right]$$

**Problem 41.8**
Solve the following system by making the change of variables $\mathbf{y}=\mathbf{Tz}$.

$$\mathbf{y}'=\left[\begin{array}{cc}-4&-6\\3&5\end{array}\right]\mathbf{y}+\left[\begin{array}{c}e^{2t}-2e^t\\e^{-2t}+e^t\end{array}\right]$$

**Problem 41.9**
Solve the following system by making the change of variables $\mathbf{y}=\mathbf{Tz}$.

$$\mathbf{y}'=\left[\begin{array}{cc}3&2\\1&4\end{array}\right]\mathbf{y}+\left[\begin{array}{c}4t+4\\-2t+1\end{array}\right]$$

**Problem 41.10**
Solve the following system by making the change of variables $\mathbf{x}=\mathbf{Tz}$.

$$\mathbf{x}''=\left[\begin{array}{cc}6&7\\-15&-16\end{array}\right]\mathbf{x}$$

**Problem 41.11**
Solve the following system by making the change of variables $\mathbf{x}=\mathbf{Tz}$.

$$\mathbf{x}''=\left[\begin{array}{cc}4&2\\2&1\end{array}\right]\mathbf{x}$$

# 42 Solving First Order Linear Systems Using Exponential Matrix

The matrix exponential plays an important role in solving systems of linear differential equations. In this section, we will define such a concept and study some of its important properties.
Recall from calculus the power series expansion of $e^t$ given by

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

and this series converges for all real numbers $t$.
To develop something similar with number $t$ replaced by a matrix $\mathbf{A}$ one proceeds as follows: The absolute value used for measuring the distance between numbers is now replaced by a matrix norm given by

$$||\mathbf{A}|| = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij})^2}$$

Next, we construct the sequence of partial sums

$$\begin{array}{rcl} \mathbf{S_1} & = & \mathbf{I} + \frac{\mathbf{A}}{1!} \\ \mathbf{S_2} & = & \mathbf{I} + \frac{\mathbf{A}}{1!} + \frac{\mathbf{A}}{2!} \\ \vdots & & \\ \mathbf{S_n} & = & \mathbf{I} + \frac{\mathbf{A}}{1!} + \cdots + \frac{\mathbf{A}}{n!} \end{array}$$

With little effort which we don't pursue here, it can be shown that the sequence of partial sums converges and its limit is denoted by $e^{\mathbf{A}}$. That is,

$$\lim_{n\to\infty} \mathbf{S_n} = e^{\mathbf{A}}$$

or

$$e^{\mathbf{A}} = \mathbf{I} + \frac{\mathbf{A}}{1!} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}$$

and this series converges for any square matrix $\mathbf{A}$.

**Example 42.1**
Suppose that

$$\mathbf{A}(t) = \left[\begin{array}{cc} 0 & t \\ t & 0 \end{array}\right]$$

Find $e^{\mathbf{A}}$.

**Solution.**
One can easily see that for any nonnegative odd integer $n$ we have

$$\mathbf{A}^n = \begin{bmatrix} 0 & t^n \\ t^n & 0 \end{bmatrix}$$

and for nonnegative even integer $n$

$$\mathbf{A}^n = \begin{bmatrix} t^n & 0 \\ 0 & t^n \end{bmatrix}$$

Thus,

$$e^{\mathbf{A}} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} & \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \\ \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \end{bmatrix} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \blacksquare$$

The following theorem describes some of the important properties of the exponential matrix.

**Theorem 42.1**
(i) If $\mathbf{AB} = \mathbf{BA}$ then $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$.
(ii) For any square matrix $\mathbf{A}$, $e^{\mathbf{A}}$ is invertible with $(e^{\mathbf{A}})^{-1} = e^{-\mathbf{A}}$.
(iii) For any invertible $n \times n$ matrix $\mathbf{P}$ and any $n \times n$ matrix $\mathbf{A}$

$$e^{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}} = \mathbf{P}^{-1}e^{\mathbf{A}}\mathbf{P}$$

(Thus, if $\mathbf{A}$ is similar to $\mathbf{B}$; then $e^{\mathbf{A}}$ is similar to $e^{\mathbf{B}}$).
(iv) If A has eigenvalues $r_1, r_2, \cdots, r_n$ (not necessarily distinct), then $e^{\mathbf{A}}$ has eigenvalues $e^{r_1}, e^{r_2}, \cdots, e^{r_n}$.
(v) $det(e^{\mathbf{A}}) = e^{tr(\mathbf{A})}$.
(vi) $\mathbf{A}e^{\mathbf{A}} = e^{\mathbf{A}}\mathbf{A}$.

When dealing with systems of differential equations, one has often to deal with expressions like $e^{\mathbf{P}t}$, where $\mathbf{P}(t)$ is a matrix and $t$ is a real number or real variable. With the above formula of the exponential matrix function we get

$$e^{\mathbf{P}t} = \mathbf{I} + \frac{\mathbf{P}t}{1!} + \frac{\mathbf{P}^2t^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{\mathbf{P}^n t^n}{n!}.$$

Let's find the derivative of $e^{\mathbf{P}t}$. To prove this, we calculate this derivative via the usual limit formula:

$$\frac{d}{dt}e^{\mathbf{P}t} = \lim_{h\to 0}\frac{e^{\mathbf{P}(t+h)} - e^{\mathbf{P}t}}{h}.$$

But

$$e^{\mathbf{P}(t+h)} = e^{\mathbf{P}t}e^{\mathbf{P}h}.$$

since $\mathbf{P}t$ and $\mathbf{P}h$ commute.(These matrices are scalar multiples of the same matrix **P,** and **P** commutes with itself.) Going back to the derivative, we get

$$\begin{aligned}
\frac{d}{dt}e^{\mathbf{P}t} =& \lim_{h\to 0}\frac{e^{\mathbf{P}(t+h)} - e^{\mathbf{P}t}}{h} \\
=& e^{\mathbf{P}t}\lim_{h\to 0}\frac{e^{\mathbf{P}h} - \mathbf{I}}{h} \\
=& e^{\mathbf{P}t}\lim_{h\to 0}\frac{1}{h}\left(\mathbf{P}h + \frac{1}{2!}\mathbf{P}^2h^2 + \frac{1}{3!}\mathbf{P}^3h^3 + \cdots\right) \\
=& e^{\mathbf{P}t}\mathbf{P} = \mathbf{P}e^{\mathbf{P}t}
\end{aligned}$$

Now, consider the initial value problem

$$\mathbf{y}' = \mathbf{P}\mathbf{y} + \mathbf{g}(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0, \quad a < t < b$$

where **P** is a constant square matrix and the entries of $\mathbf{g}(\mathrm{t})$ are continuous in $a < t < b$. Then one has

$$\begin{aligned}
\mathbf{y}' - \mathbf{P}\mathbf{y} =& \mathbf{g}(t) \\
\left(e^{-\mathbf{P}(t-t_0)}\mathbf{y}\right)' =& e^{-\mathbf{P}(t-t_0)}\mathbf{g}(t) \\
e^{-\mathbf{P}(t-t_0)}\mathbf{y} =& \mathbf{y}_0 + \int_{t_0}^{t} e^{-\mathbf{P}(s-t_0)}\mathbf{g}(s)ds \\
\mathbf{y}(t) =& e^{(t-t_0)\mathbf{P}}\mathbf{y}_0 + \int_{t_0}^{t} e^{(t-s)\mathbf{P}}\mathbf{g}(s)ds
\end{aligned}$$

**Remark 42.1**
The above procedure does not apply if the matrix $\mathbf{P}$ is not constant! That is, $\frac{d}{dt}e^{\mathbf{P}(t)} \neq \mathbf{P}'(t)e^{\mathbf{P}(t)}$. This is due to the fact that matrix multiplication is not commutative in general.

**Example 42.2**
Find $e^{\mathbf{P}t}$ if $\mathbf{P} = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$.

**Solution.**
Since $\mathbf{P}^2 = 0$ we find

$$e^{\mathbf{P}(t)} = \mathbf{I} + t\mathbf{P} = \begin{bmatrix} 1+6t & 9 \\ -4 & 1-6t \end{bmatrix} \blacksquare$$

## Practice Problems

**Problem 42.1**
Find $e^{\mathbf{P}(t)}$ if $\mathbf{P} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

**Problem 42.2**
Consider the linear differential system

$$\mathbf{y}' = \mathbf{P}\mathbf{y}, \quad \mathbf{P} = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

(a) Calculate $e^{\mathbf{P}t}$. Hint: Every square matrix satisfies its characteristic equation.
(b) Use the result from part (a) to find two independent solutions of the differential system. Form the general solution.

**Problem 42.3**
Show that if

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

then

$$e^{\mathbf{D}} = \begin{bmatrix} e^{d_1} & 0 & 0 \\ 0 & e^{d_2} & 0 \\ 0 & 0 & e^{d_3} \end{bmatrix}$$

**Problem 42.4**
Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(\mathbf{0}) = \mathbf{y}_0$$

**Problem 42.5**
Show that if $r$ is an eigenvalue of $\mathbf{P}$ then $e^r$ is an eigenvalue of $e^{\mathbf{P}}$.

**Problem 42.6**
Show that $det(e^{\mathbf{A}}) = e^{tr(\mathbf{A})}$. Hint: Recall that the determinant of a matrix is equal to the product of its eigenvalues and the trace is the sume of the eigenvalues. This follows from the expansion of the characteristic equation into a polynomial.

**Problem 42.7**
Prove: For any invertible $n \times n$ matrix $\mathbf{P}$ and any $n \times n$ matrix $\mathbf{A}$

$$e^{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}} = \mathbf{P}^{-1}e^{\mathbf{A}}\mathbf{P}$$

(Thus, if $\mathbf{A}$ is similar to $\mathbf{B}$; then $e^{\mathbf{A}}$ is similar to $e^{\mathbf{B}}$).

**Problem 42.8**
Prove: If $\mathbf{AB} = \mathbf{BA}$ then $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$.

**Problem 42.9**
Prove: For any square matrix $\mathbf{A}$, $e^{\mathbf{A}}$ is invertible with $(e^{\mathbf{A}})^{-1} = e^{-\mathbf{A}}$.

**Problem 42.10**
Consider the two matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Show that $\mathbf{AB} \neq \mathbf{BA}$ and $e^{\mathbf{A}+\mathbf{B}} \neq e^{\mathbf{A}}e^{\mathbf{A}}$.

# 43 The Laplace Transform: Basic Definitions and Results

Laplace transform is yet another operational tool for solving constant coefficients linear differential equations. The process of solution consists of three main steps:

- The given "hard" problem is transformed into a "simple" equation.
- This simple equation is solved by purely algebraic manipulations.
- The solution of the simple equation is transformed back to obtain the solution of the given problem.

In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration.
The above procedure can be summarized by Figure 43.1


Figure 43.1

In this section we introduce the concept of Laplace transform and discuss some of its properties.
The Laplace transform is defined in the following way. Let $f(t)$ be defined for $t \geq 0$. Then the **Laplace transform** of $f$, which is denoted by $\mathcal{L}[f(t)]$ or by $F(s)$, is defined by the following equation

$$\mathcal{L}[f(t)] = F(s) = \lim_{T\to\infty} \int_0^T f(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt$$

The integral which defined a Laplace transform is an improper integral. An improper integral may **converge** or **diverge**, depending on the integrand. When the improper integral in convergent then we say that the function $f(t)$ possesses a Laplace transform. So what types of functions possess Laplace transforms, that is, what type of functions guarantees a convergent improper integral.

**Example 43.1**
Find the Laplace transform, if it exists, of each of the following functions

$$(a)\ f(t) = e^{at} \quad (b)\ f(t) = 1 \quad (c)\ f(t) = t \quad (d)\ f(t) = e^{t^2}$$

**Solution.**
(a) Using the definition of Laplace transform we see that

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-(s-a)t}dt = \lim_{T\to\infty}\int_0^T e^{-(s-a)t}dt.$$

But

$$\int_0^T e^{-(s-a)t}dt = \begin{cases} T & \text{if } s = a \\ \frac{1-e^{-(s-a)T}}{s-a} & \text{if } s \neq a. \end{cases}$$

For the improper integral to converge we need $s > a$. In this case,

$$\mathcal{L}[e^{at}] = F(s) = \frac{1}{s-a}, \quad s > a.$$

(b) In a similar way to what was done in part (a), we find

$$\mathcal{L}[1] = \int_0^\infty e^{-st}dt = \lim_{T\to\infty}\int_0^T e^{-st}dt = \frac{1}{s}, \ s > 0.$$

(c) We have

$$\mathcal{L}[t] = \int_0^\infty te^{-st}dt = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^\infty = \frac{1}{s^2}, \ s > 0.$$

(d) Again using the definition of Laplace transform we find

$$\mathcal{L}[e^{t^2}] = \int_0^\infty e^{t^2-st}dt.$$

If $s \leq 0$ then $t^2 - st \geq 0$ so that $e^{t^2-st} \geq 1$ and this implies that $\int_0^\infty e^{t^2-st}dt \geq \int_0^\infty$. Since the integral on the right is divergent, by the comparison theorem of improper integrals (see Theorem 43.1 below) the integral on the left is also divergent. Now, if $s > 0$ then $\int_0^\infty e^{t(t-s)}dt \geq \int_s^\infty dt$. By the same reasoning the integral on the left is divergent. This shows that the function $f(t) = e^{t^2}$ does not possess a Laplace transform ■

The above example raises the question of what class or classes of functions possess a Laplace transform. Looking closely at Example 43.1(a), we notice that for $s > a$ the integral $\int_0^\infty e^{-(s-a)t}dt$ is convergent and a critical component for this convergence is the type of the function $f(t)$. To be more specific, if $f(t)$ is a continuous function such that

$$|f(t)| \leq Me^{at}, \quad t \geq C \tag{28}$$

where $M \geq 0$ and $a$ and $C$ are constants, then this condition yields

$$\int_0^\infty f(t)e^{-st}dt \leq \int_0^C f(t)e^{-st}dt + M\int_C^\infty e^{-(s-a)t}dt.$$

Since $f(t)$ is continuous in $0 \leq t \leq C$, by letting $A = \max\{|f(t)| : 0 \leq t \leq C\}$ we have

$$\int_0^C f(t)e^{-st}dt \leq A\int_0^C e^{-st}dt = A\left(\frac{1}{s} - \frac{e^{-sC}}{s}\right) < \infty.$$

On the other hand, Now, by Example 43.1(a), the integral $\int_C^\infty e^{-(s-a)t}dt$ is convergent for $s > a$. By the comparison theorem of improper integrals (see Theorem 43.1 below) the integral on the left is also convergent. That is, $f(t)$ possesses a Laplace transform.
We call a function that satisfies condition (28) a function with an **exponential order at infinity.** Graphically, this means that the graph of $f(t)$ is contained in the region bounded by the graphs of $y = Me^{at}$ and $y = -Me^{at}$ for $t \geq C$. Note also that this type of functions controls the negative exponential in the transform integral so that to keep the integral from blowing up. If $C = 0$ then we say that the function is **exponentially bounded**.

**Example 43.2**
Show that any bounded function $f(t)$ for $t \geq 0$ is exponentially bounded.

**Solution.**
Since $f(t)$ is bounded for $t \geq 0$, there is a positive constant $M$ such that $|f(t)| \leq M$ for all $t \geq 0$. But this is the same as (28) with $a = 0$ and $C = 0$. Thus, $f(t)$ has is exponentially bounded ■

Another question that comes to mind is whether it is possible to relax the condition of continuity on the function $f(t)$. Let's look at the following situation.

**Example 43.3**
Show that the square wave function whose graph is given in Figure 43.2 possesses a Laplace transform.

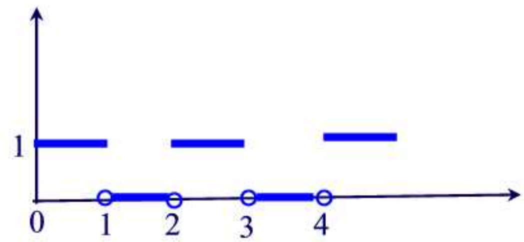



Figure 43.2

Note that the function is periodic of period 2.

**Solution.**
Since $f(t)e^{-st} \leq e^{-st}$, we have $\int_0^\infty f(t)e^{-st}dt \leq \int_0^\infty e^{-st}dt$. But the integral on the right is convergent for $s > 0$ so that the integral on the left is convergent as well. That is, $\mathcal{L}[f(t)]$ exists for $s > 0$ ■

The function of the above example belongs to a class of functions that we define next. A function is called **piecewise continuous** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (i.e. the subinterval without its endpoints) and has a finite limit at the endpoints (**jump discontinuities** and no vertical asymptotes) of each subinterval. Below is a sketch of a piecewise continuous function.

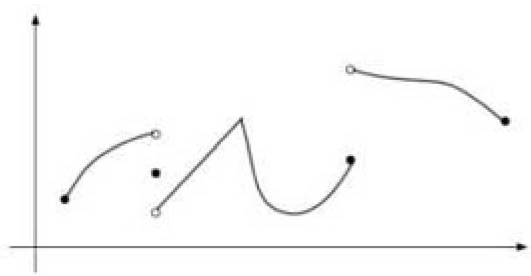


Figure 43.3

Note that a piecewise continuous function is a function that has a finite number of breaks in it and doesnt blow up to infinity anywhere. A function defined for $t \geq 0$ is said to be **piecewise continuous on the infinite interval** if it is piecewise continuous on $0 \leq t \leq T$ for all $T > 0$.

**Example 43.4**
Show that the following functions are piecewise continuous and of exponential order at infinity for $t \geq 0$

$$\text{(a) } f(t) = t^n \quad \text{(b) } f(t) = t^n \sin at$$

**Solution.**
(a) Since $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \geq \frac{t^n}{n!}$, we have $t^n \leq n!e^t$. Hence, $t^n$ is piecewise continuous and exponentially bounded.
(b) Since $|t^n \sin at| \leq n!e^t$, we have $t^n \sin at$ is piecewise continuous and exponentially bounded ∎

Next, we would like to establish the existence of the Laplace transform for all functions that are piecewise continuous and have exponential order at infinity. For that purpose we need the following comparison theorem from calculus.

**Theorem 43.1**
Suppose that $f(t)$ and $g(t)$ are both integrable functions for all $t \geq t_0$ such that $|f(t)| \leq |g(t)$ for $t \geq t_0$. If $\int_{t_0}^{\infty} g(t)dt$ is convergent, then $\int_{t_0}^{\infty} f(t)dt$ is also convergent. If, on the other hand, $\int_{t_0}^{\infty} f(t)dt$ is divergent then $\int_{t_0}^{\infty} f(t)dt$ is also divergent.

**Theorem 43.2** *(Existence)*
Suppose that $f(t)$ is piecewise continuous on $t \geq 0$ and has an exponential order at infinity with $|f(t)| \leq Me^{at}$ for $t \geq C$. Then the Laplace transform

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

exists as long as $s > a$. Note that the two conditions above are sufficient, but not necessary, for $F(s)$ to exist.

**Proof.**
The integral in the definition of $F(s)$ can be splitted into two integrals as follows

$$\int_0^{\infty} f(t)e^{-st}dt = \int_0^{C} f(t)e^{-st}dt + \int_C^{\infty} f(t)e^{-st}dt.$$

Since $f(t)$ is piecewise continuous in $0 \leq t \leq C$, it is bounded there. By letting $A = \max\{|f(t)| : 0 \leq t \leq C\}$ we have

$$\int_0^{C} f(t)e^{-st}dt \leq A\int_0^{C} e^{-st}dt = A\left(\frac{1}{s} - \frac{e^{-sC}}{s}\right) < \infty.$$

Now, by Example 43.1(a), the integral $\int_C^\infty f(t)e^{-st}dt$ is convergent for $s > a$. By Theorem 43.1 the integral on the left is also convergent. That is, $f(t)$ possesses a Laplace transform ■

In what follows, we will denote the class of all piecewise continuous functions with exponential order at infinity by $\mathcal{PE}$. The next theorem shows that any linear combination of functions in $\mathcal{PE}$ is also in $\mathcal{PE}$. The same is true for the product of two functions in $\mathcal{PE}$.

**Theorem 43.3**
Suppose that $f(t)$ and $g(t)$ are two elements of $\mathcal{PE}$ with

$$|f(t)| \leq M_1 e^{a_1 t}, \quad t \geq C_1 \quad \text{and} \quad |g(t)| \leq M_2 e^{a_1 t}, \quad t \geq C_2.$$

(i) For any constants $\alpha$ and $\beta$ the function $\alpha f(t) + \beta g(t)$ is also a member of $\mathcal{PE}$. Moreover

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)].$$

(ii) The function $h(t) = f(t)g(t)$ is an element of $\mathcal{PE}$.

**Proof.**
(i) It is easy to see that $\alpha f(t) + \beta g(t)$ is a piecewise continuous function. Now, let $C = C_1 + C_2,\ a = \max\{a_1, a_2\}$, and $M = |\alpha|M_1 + |\beta|M_2$. Then for $t \geq C$ we have

$$|\alpha f(t) + \beta g(t)| \leq |\alpha||f(t)| + |\beta||g(t)| \leq |\alpha|M_1 e^{a_1 t} + |\beta|M_2 e^{a_2 t} \leq Me^{at}.$$

This shows that $\alpha f(t) + \beta g(t)$ is of exponential order at infinity. On the other hand,

$$\begin{aligned}
\mathcal{L}[\alpha f(t) + \beta g(t)] &= \lim_{T\to\infty} \int_0^T [\alpha f(t) + \beta g(t)]dt \\
&= \alpha \lim_{T\to\infty} \int_0^T f(t)dt + \beta \lim_{T\to\infty} \int_0^T g(t)dt \\
&= \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]
\end{aligned}$$

(ii) It is clear that $h(t) = f(t)g(t)$ is a piecewise continuous function. Now, letting $C = C_1 + C_2,\ M = M_1 M_2$, and $a = a_1 + a_2$ then we see that for $t \geq C$ we have

$$|h(t)| = |f(t)||g(t)| \leq M_1 M_2 e^{(a_1+a_2)t} = Me^{at}.$$

Hence, $h(t)$ is of exponential order at infinity. By Theorem 43.2 , $\mathcal{L}[h(t)]$ exists for $s > a$ ■

We next discuss the problem of how to determine the function $f(t)$ if $F(s)$ is given. That is, how do we invert the transform. The following result on uniqueness provides a possible answer. This result establishes a one-to-one correspondence between the set $\mathcal{PE}$ and its Laplace transforms. Alternatively, the following theorem asserts that the Laplace transform of a member in $\mathcal{PE}$ is unique.

**Theorem 43.4**
Let $f(t)$ and $g(t)$ be two elements in $\mathcal{PE}$ with Laplace transforms $F(s)$ and $G(s)$ such that $F(s) = G(s)$ for some $s > a$. Then $f(t) = g(t)$ for all $t \geq 0$ where both functions are continuous.

The standard techniques used to prove this theorem( i.e., complex analysis, residue computations, and/or Fourier's integral inversion theorem) are generally beyond the scope of an introductory differential equations course. The interested reader can find a proof in the book "Operational Mathematics" by Ruel Vance Churchill or in D.V. Widder "The Laplace Transform".
With the above theorem, we can now officially define the inverse Laplace transform as follows: For a piecewise continuous function $f$ of exponential order at infinity whose Laplace transform is $F$, we call $f$ the **inverse Laplace transform** of $F$ and write $f = \mathcal{L}^{-1}[F(s)]$. Symbolically

$$f(t) = \mathcal{L}^{-1}[F(s)] \Longleftrightarrow F(s) = \mathcal{L}[f(t)].$$

**Example 43.5**
Find $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right), \quad s > 1.$

**Solution.**
From Example 43.1(a), we have that $\mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad s > a$. In particular, for $a = 1$ we find that $\mathcal{L}[e^t] = \frac{1}{s-1},\ s > 1$. Hence, $\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t,\ t \geq 0$ ■.

The above theorem states that if $f(t)$ is continuous and has a Laplace transform $F(s)$, then there is no other function that has the same Laplace transform. To find $\mathcal{L}^{-1}[F(s)]$, we can inspect tables of Laplace transforms of known functions to find a particular $f(t)$ that yields the given $F(s)$.
When the function $f(t)$ is not continuous, the uniqueness of the inverse

Laplace transform is not assured. The following example addresses the uniqueness issue.

**Example 43.6**
Consider the two functions $f(t) = h(t)h(3-t)$ and $g(t) = h(t) - h(t-3)$.

(a) Are the two functions identical?
(b) Show that $\mathcal{L}[f(t)] = \mathcal{L}[g(t)$.

**Solution.**
(a) We have

$$f(t) = \begin{cases} 1, & 0 \le t \le 3 \\ 0, & t > 3 \end{cases}$$

and

$$g(t) = \begin{cases} 1, & 0 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

So the two functions are equal for all $t \neq 3$ and so they are not identical.
(b) We have

$$\mathcal{L}[f(t)] = \mathcal{L}[g(t)] = \int_0^3 e^{-st}dt = \frac{1-e^{-3s}}{s}, s > 0.$$

Thus, both functions $f(t)$ and $g(t)$ have the same Laplace transform even though they are not identical. However, they are equal on the interval(s) where they are both continuous ■

The inverse Laplace transform possesses a linear property as indicated in the following result.

**Theorem 43.5**
Given two Laplace transforms $F(s)$ and $G(s)$ then

$$\mathcal{L}^{-1}[aF(s) + bG(s)] = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)]$$

for any constants $a$ and $b$.

**Proof.**
Suppose that $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[g(t)] = G(s)$. Since $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s) + bG(s)$ we have $\mathcal{L}^{-1}[aF(s) + bG(s)] = af(t) + bg(t) = a\mathcal{L}^{-1}[F(s)] + b\mathcal{L}^{-1}[G(s)]$ ■

## Practice Problems

**Problem 43.1**
Determine whether the integral $\int_0^\infty \frac{1}{1+t^2}dt$ converges. If the integral converges, give its value.

**Problem 43.2**
Determine whether the integral $\int_0^\infty \frac{t}{1+t^2}dt$ converges. If the integral converges, give its value.

**Problem 43.3**
Determine whether the integral $\int_0^\infty e^{-t}\cos{(e^{-t})}dt$ converges. If the integral converges, give its value.

**Problem 43.4**
Using the definition, find $\mathcal{L}[e^{3t}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

**Problem 43.5**
Using the definition, find $\mathcal{L}[t-5]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

**Problem 43.6**
Using the definition, find $\mathcal{L}[e^{(t-1)^2}]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

**Problem 43.7**
Using the definition, find $\mathcal{L}[(t-2)^2]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

**Problem 43.8**
Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t)=\begin{cases} 0, & 0\le t<1 \\ t-1, & t\ge 1 \end{cases}$$

**Problem 43.9**
Using the definition, find $\mathcal{L}[f(t)]$, if it exists. If the Laplace transform exists then find the domain of $F(s)$.

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ t-1, & 1 \leq t < 2 \\ 0, & t \geq 2. \end{cases}$$

**Problem 43.10**
Let $n$ be a positive integer. Using integration by parts establish the reduction formula

$$\int t^n e^{-st} dt = -\frac{t^n e^{-st}}{s} + \frac{n}{s} \int t^{n-1} e^{-st} dt, \quad s > 0.$$

**Problem 43.11**
For $s > 0$ and $n$ a positive integer evaluate the limits

$\lim_{t\to 0} t^n e^{-st}$ (b) $\lim_{t\to\infty} t^n e^{-st}$

**Problem 43.12**
(a) Use the previous two problems to derive the reduction formula for the Laplace transform of $f(t) = t^n$,

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}], \quad s > 0.$$

(b) Calculate $\mathcal{L}[t^k]$, for $k = 1, 2, 3, 4, 5$.
(c) Formulate a conjecture as to the Laplace transform of $f(t), t^n$ with $n$ a positive integer.

From a table of integrals,

$$\begin{aligned} \int e^{\alpha u} \sin \beta u du &= e^{\alpha u} \frac{\alpha \sin \beta u - \beta \sin \beta u}{\alpha^2 + \beta^2} \\ \int e^{\alpha u} \cos \beta u du &= e^{\alpha u} \frac{\alpha \cos \beta u + \beta \sin \beta u}{\alpha^2 + \beta^2} \end{aligned}$$

**Problem 43.13**
Use the above integrals to find the Laplace transform of $f(t) = \cos \omega t$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

**Problem 43.14**
Use the above integrals to find the Laplace transform of $f(t) = \sin \omega t$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

**Problem 43.15**
Use the above integrals to find the Laplace transform of $f(t) = \cos\omega(t-2)$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

**Problem 43.16**
Use the above integrals to find the Laplace transform of $f(t) = e^{3t}\sin t$, if it exists. If the Laplace transform exists, give the domain of $F(s)$.

**Problem 43.17**
Use the linearity property of Laplace transform to find $\mathcal{L}[5e^{-7t} + t + 2e^{2t}]$. Find the domain of $F(s)$.

**Problem 43.18**
Consider the function $f(t) = \tan t$.

(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers $a$ and $M$ such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

**Problem 43.19**
Consider the function $f(t) = t^2e^{-t}$.

(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers $a$ and $M$ such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

**Problem 43.20**
Consider the function $f(t) = \frac{e^{t^2}}{e^{2t}+1}$.

(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers $a$ and $M$ such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

**Problem 43.21**
Consider the floor function $f(t) = \lfloor t \rfloor$, where for any integer $n$ we have $\lfloor t \rfloor = n$ for all $n \le t < n+1$.

(a) Is $f(t)$ continuous on $0 \le t < \infty$, discontinuous but piecewise continuous on $0 \le t < \infty$, or neither?
(b) Are there fixed numbers $a$ and $M$ such that $|f(t)| \le Me^{at}$ for $0 \le t < \infty$?

**Problem 43.22**
Find $\mathcal{L}^{-1}\left(\frac{3}{s-2}\right).$

**Problem 43.23**
Find $\mathcal{L}^{-1}\left(-\frac{2}{s^2}+\frac{1}{s+1}\right).$

**Problem 43.24**
Find $\mathcal{L}^{-1}\left(\frac{2}{s+2}+\frac{2}{s-2}\right).$

# 44 Further Studies of Laplace Transform

Properties of the Laplace transform enable us to find Laplace transforms without having to compute them directly from the definition. In this section, we establish properties of Laplace transform that will be useful for solving ODEs.

**Laplace Transform of the Heaviside Step Function**
The Heaviside step function is a piecewise continuous function defined by

$$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Figure 44.1 displays the graph of $h(t)$.

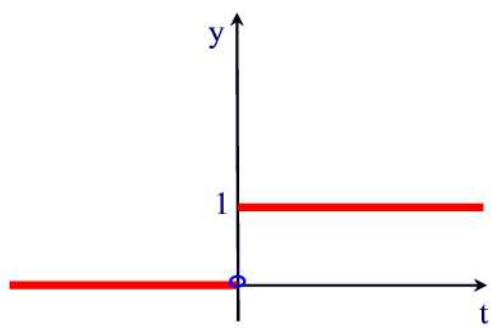



Figure 44.1

Taking the Laplace transform of $h(t)$ we find

$$\mathcal{L}[h(t)] = \int_0^\infty h(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s},\ s > 0.$$

A Heaviside function at $\alpha \geq 0$ is the shifted function $h(t-\alpha)$ ($\alpha$ units to the right). For this function, the Laplace transform is

$$\mathcal{L}[h(t-\alpha)] = \int_0^\infty h(t-\alpha)e^{-st}dt = \int_\alpha^\infty e^{-st}dt = \left[-\frac{e^{-st}}{s}\right]_\alpha^\infty = \frac{e^{-s\alpha}}{s},\ s > 0.$$

**Laplace Tranform of $e^{at}$**
The Laplace transform for the function $f(t) = e^{at}$ is

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-(s-a)t}dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a},\ s > a.$$

**Laplace Tranforms of $\sin at$ and $\cos at$**
Using integration by parts twice we find

$$\begin{aligned}
\mathcal{L}[\sin at] &= \int_0^\infty e^{-st}\sin at dt \\
&= \left[-\frac{e^{-st}\sin at}{s} - \frac{ae^{-st}\cos at}{s^2}\right]_0^\infty - \frac{a^2}{s^2}\int_0^\infty e^{-st}\sin at dt \\
&= -\frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}[\sin at] \\
\left(\frac{s^2+a^2}{s^2}\right)\mathcal{L}[\sin at] &= \frac{a}{s^2} \\
\mathcal{L}[\sin at] &= \frac{a}{s^2+a^2}, \quad s > 0
\end{aligned}$$

A similar argument shows that

$$\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}, \quad s > 0.$$

**Laplace Transforms of $\cosh at$ and $\sinh at$**
Using the linear property of $\mathcal{L}$ we can write

$$\begin{aligned}
\mathcal{L}[\cosh at] &= \frac{1}{2}\left(\mathcal{L}[e^{at}] + \mathcal{L}[e^{-at}]\right) \\
&= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right), \quad s > |a| \\
&= \frac{s}{s^2-a^2}, \quad s > |a|
\end{aligned}$$

A similar argument shows that

$$\mathcal{L}[\sin at] = \frac{a}{s^2-a^2}, \quad s > |a|.$$

**Laplace Transform of a Polynomial**
Let $n$ be a positive integer. Using integration by parts we can write

$$\int_0^\infty t^n e^{-st}dt = -\left[\frac{t^n e^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt.$$

By repeated use of L'Hôpital's rule we find $\lim_{t\to\infty} t^n e^{-st} = \lim_{t\to\infty} \frac{n!}{s^n e^{st}} = 0$ for $s > 0$. Thus,

$$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}], \quad s > 0.$$

Using induction on $n = 0, 1, 2, \cdots$ one can easily eastablish that

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad s > 0.$$

Using the above result together with the linearity property of $\mathcal{L}$ one can find the Laplace transform of any polynomial.
The next two results are referred to as the first and second shift theorems. As with the linearity property, the shift theorems increase the number of functions for which we can easily find Laplace transforms.

**Theorem 44.1** *(First Shifting Theorem)*
If $f(t)$ is a piecewise continuous function for $t \geq 0$ and has exponential order at infinity with $|f(t)| \leq Me^{at}, \quad t \geq C$, then for any real number $\alpha$ we have

$$\mathcal{L}[e^{\alpha t}f(t)] = F(s - \alpha), \quad s > a + \alpha$$

where $\mathcal{L}[f(t)] = F(s)$.

**Proof.**
From the definition of the Laplace transform we have

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt.$$

Using the change of variable $\beta = s - a$ the previous equation reduces to

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-\beta t}f(t)dt = F(\beta) = F(s-a), \ \ s > a+\alpha \ \blacksquare$$

**Theorem 44.2** *(Second Shifting Theorem)*
If $f(t)$ is a piecewise continuous function for $t \geq 0$ and has exponential order at infinity with $|f(t)| \leq Me^{at}, \quad t \geq C$, then for any real number $\alpha \geq 0$ we have

$$\mathcal{L}[f(t - \alpha)h(t - \alpha)] = e^{-\alpha s}F(s), \quad s > a$$

where $\mathcal{L}[f(t)] = F(s)$ and $h(t)$ is the Heaviside step function.

**Proof.**
From the definition of the Laplace transform we have

$$\mathcal{L}[f(t - \alpha)h(t - \alpha)] = \int_0^\infty f(t - \alpha)h(s - \alpha)e^{-st}dt = \int_\alpha^\infty f(t - \alpha)e^{-st}dt.$$

Using the change of variable $\beta = t - \alpha$ the previous equation reduces to

$$\begin{aligned}\mathcal{L}[f(t-\alpha)h(t-\alpha)] &= \int_0^\infty f(\beta)e^{-s(\beta+\alpha)}d\beta \\ &= e^{-s\alpha}\int_0^\infty f(\beta)e^{-s\beta}d\beta = e^{-s\alpha}F(s), \quad s > a \blacksquare\end{aligned}$$

**Example 44.1**
Find

(a) $\mathcal{L}[e^{2t}t^2]$ (b) $\mathcal{L}[e^{3t}\cos 2t]$ (c) $\mathcal{L}^{-1}[e^{-2t}s^2]$

**Solution.**
(a) By Theorem 44.1, we have $\mathcal{L}[e^{2t}t^2] = F(s-2)$ where $\mathcal{L}[t^2] = \frac{2!}{s^3} = F(s),\ s > 0$. Thus, $\mathcal{L}[e^{2t}t^2] = \frac{2}{(s-2)^3},\ s > 2$.
(b) As in part (a), we have $\mathcal{L}[e^{3t}\cos 2t] = F(s-3)$ where $\mathcal{L}[\cos 2t] = F(s-3)$. But $\mathcal{L}[\cos 2t] = \frac{s}{s^2+4},\ s > 0$. Thus,

$$\mathcal{L}[e^{3t}\cos 2t] = \frac{s-3}{(s-3)^2+4}, \quad s > 3$$

(c) Since $\mathcal{L}[t] = \frac{1}{s^2}$, by Theorem 44.2, we have

$$\frac{e^{-2t}}{s^2} = \mathcal{L}[(t-2)h(t-2)].$$

Therefore,

$$\mathcal{L}^{-1}\left[\frac{e^{-2t}}{s^2}\right] = (t-2)h(t-2) = \begin{cases} 0, & 0 \le t < 2 \\ t-2, & t \ge 2 \end{cases} \blacksquare$$

The following result relates the Laplace transform of derivatives and integrals to the Laplace transform of the function itself.

**Theorem 44.3**
Suppose that $f(t)$ is continuous for $t \ge 0$ and $f'(t)$ is piecewise continuous of exponential order at infinity with $|f'(t)| \le Me^{at},\ t \ge C$ Then

(a) $f(t)$ is of exponential order at infinity.
(b) $\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0) = sF(s) - f(0),\ s > \max\{a, 0\} + 1$.
(c) $\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0) = s^2F(s) - sf(0) - f(0),\ s > \max\{a, 0\} + 1$.
(d) $\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{\mathcal{L}[f(t)]}{s} = \frac{F(s)}{s}, \quad s > \max\{a, 0\} + 1$.

**Proof.**
(a) By the Fundamental Theorem of Calculus we have $f(t) = f(0) - \int_0^t f'(u)du$. Also, since $f'$ is piecewise continuous, $|f'(t)| \le T$ for some $T > 0$ and all $0 \le t \le C$. Thus,

$$\begin{array}{lll} |f(t)| & = & \left| f(0) - \int_0^t f'(u)du \right| = |f(0) - \int_0^C f'(u)du - \int_C^t f'(u)du| \\ & \le & |f(0)| + TC + M\int_C^t e^{au}du \end{array}$$

Note that if $a > 0$ then

$$\int_C^t e^{au}du = \frac{1}{a}(e^{at} - e^{aC}) \le \frac{e^{at}}{a}$$

and so

$$|f(t)| \le [|f(0)| + TC + \frac{M}{a}]e^{at}.$$

If $a = 0$ then

$$\int_C^t e^{au}du = t - C$$

and therefore

$$|f(t)| \le |f(0)| + TC + M(t - C) \le (|f(0)| + TC + M)e^t.$$

Now, if $a < 0$ then

$$\int_C^t e^{au}du = \frac{1}{a}(e^{at} - e^{aC}) \le \frac{1}{|a|}$$

so that

$$|f(t)| \le (|f(0)| + TC + \frac{M}{|a|})e^t$$

It follows that

$$|f(t)| \le Ne^{bt}, \quad t \ge 0$$

where $b = \max\{a, 0\} + 1$.

(b) From the definition of Laplace transform we can write

$$\mathcal{L}[f'(t)] = \lim_{A\to\infty} \int_0^A f'(t)e^{-st}dt.$$

Since $f'(t)$ may have jump discontinuities at $t_1, t_2, \cdots, t_N$ in the interval $0 \le t \le A$, we can write

$$\int_0^A f'(t)e^{-st}dt = \int_0^{t_1} f'(t)e^{-st}dt + \int_{t_1}^{t_2} f'(t)e^{-st}dt + \cdots + \int_{t_N}^{A} f'(t)e^{-st}dt.$$

Integrating each term on the RHS by parts and using the continuity of $f(t)$ to obtain

$$\begin{array}{rcl}
\int_0^{t_1} f'(t)e^{-st}dt & = & f(t_1)e^{-st_1} - f(0) + s\int_0^{t_1} f(t)e^{-st}dt \\
\int_{t_1}^{t_2} f'(t)e^{-st}dt & = & f(t_2)e^{-st_2} - f(t_1)e^{-st_1} + s\int_{t_1}^{t_2} f(t)e^{-st}dt \\
\vdots & & \\
\int_{t_{N-1}}^{t_N} f'(t)e^{-st}dt & = & f(t_N)e^{-st_N} - f(t_{N-1})e^{-st_{N-1}} + s\int_{t_{N-1}}^{t_N} f(t)e^{-st}dt \\
\int_{t_N}^{A} f'(t)e^{-st}dt & = & f(A)e^{-sA} - f(t_N)e^{-st_N} + s\int_{t_N}^{A} f(t)e^{-st}dt
\end{array}$$

Also, by the continuity of $f(t)$ we can write

$$\int_0^A f(t)e^{-st}dt = \int_0^{t_1} f(t)e^{-st}dt + \int_{t_1}^{t_2} f(t)e^{-st}dt + \cdots + \int_{t_N}^{A} f(t)e^{-st}dt.$$

Hence,

$$\int_0^A f'(t)e^{-st}dt = f(A)e^{-sA} - f(0) + s\int_0^A f(t)e^{-st}dt.$$

Since $f(t)$ has exponential order at infinity,$\lim_{A\to\infty} f(A)e^{-sA} = 0$. Hence,

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0).$$

(c) Using part (b) we find

$$\begin{array}{rcl}
\mathcal{L}[f''(t)] & = & s\mathcal{L}[f'(t)] - f'(0) \\
 & = & s(sF(s) - f(0)) - f'(0) \\
 & = & s^2F(s) - sf(0) - f'(0), \quad s > \max\{a, 0\} + 1
\end{array}$$

(d) Since $\frac{d}{dt}\left(\int_0^t f(u)du\right) = f(t)$, by part (b) we have

$$F(s) = \mathcal{L}[f(t)] = s\mathcal{L}\left\{\int_0^t f(u)du\right\}$$

and therefore

$$\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{\mathcal{L}[f(t)]}{s} = \frac{F(s)}{s}, \quad s > \max\{a, 0\} + 1 \blacksquare$$

The argument establishing part (b) of the previous theorem can be extended to higher order derivatives.

**Theorem 44.4**
Let $f(t), f'(t), \cdots, f^{(n-1)}(t)$ be continuous and $f^{(n)}(t)$ be piecewise continuous of exponential order at infinity with $|f^{(n)}(t)| \leq Me^{at}, \quad t \geq C$. Then

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0), \;\; s > \max\{a, 0\} + 1.$$

We next illustrate the use of the previous theorem in solving initial value problems.

**Example 44.2**
Solve the initial value problem

$$y'' - 4y' + 9y = t, \quad y(0) = 0, \; y'(0) = 1.$$

**Solution.**
We apply Theorem 44.4 that gives the Laplace transform of a derivative. By the linearity property of the Laplace transform we can write

$$\mathcal{L}[y''] - 4\mathcal{L}[y'] + 9\mathcal{L}[y] = \mathcal{L}[t].$$

Now since

$$\begin{array}{lcc} \mathcal{L}[y''] & = & s^2\mathcal{L}[y] - sy(0) - y'(0) = s^2Y(s) - 1 \\ \mathcal{L}[y'] & = & sY(s) - y(0) = sY(s) \\ \mathcal{L}[t] & = & \frac{1}{s^2} \end{array}$$

where $\mathcal{L}[y] = Y(s)$, we obtain

$$s^2Y(s) - 1 - 4sY(s) + 9Y(s) = \frac{1}{s^2}.$$

Rearranging gives

$$(s^2 - 4s + 9)Y(s) = \frac{s^2 + 1}{s^2}.$$

Thus,

$$Y(s) = \frac{s^2+1}{s^2(s^2-4s+9)}$$

and

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^2+1}{s^2(s^2-4s+9)}\right] \blacksquare$$

In the next section we will discuss a method for finding the inverse Laplace transform of the above expression.

**Example 44.3**
Consider the mass-spring oscillator without friction: $y'' + y = 0$. Suppose we add a force which corresponds to a push (to the left) of the mass as it oscillates. We will suppose the push is described by the function

$$f(t) = -h(t-2\pi) + u(t-(2\pi+a))$$

for some $a > 2\pi$ which we are allowed to vary. (A small $a$ will correspond to a short duration push and a large $a$ to a long duration push.) We are interested in solving the initial value problem

$$y'' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0.$$

**Solution.**
To begin, determine the Laplace transform of both sides of the DE:

$$\mathcal{L}[y'' + y] = \mathcal{L}[f(t)]$$

or

$$s^2Y - sy(0) - y'(0) + Y(s) = -\frac{1}{s}e^{-2\pi s} + \frac{1}{s}e^{-(2\pi+a)s}.$$

Thus,

$$Y(s) = \frac{e^{-(2\pi+a)s}}{s(s^2+1)} - \frac{e^{-2\pi s}}{s(s^2+1)} + \frac{s}{s^2+1}.$$

Now since $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$ we see that

$$Y(s) = e^{-(2\pi+a)s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] - e^{-2\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] + \frac{s}{s^2+1}$$

and therefore

$$\begin{array}{rcl} y(t) & = & h(t-(2\pi+a))\left[\mathcal{L}^{-1}\left(\frac{1}{s}-\frac{s}{s^2+1}\right)\right](t-(2\pi+a)) \\ & - & h(t-2\pi)\left[\mathcal{L}^{-1}\left(\frac{1}{s}-\frac{s}{s^2+1}\right)\right](t-2\pi)+\cos t \\ & = & h(t-(2\pi+a))[1-\cos(t-(2\pi+a))]-u(t-2\pi)[1-\cos(t-2\pi)] \\ & + & \cos t \blacksquare \end{array}$$

We conclude this section with the following table of Laplace transform pairs.

| f(t) | F(s) |
|---|---|
| $h(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$ | $\frac{1}{s},\ s>0$ |
| $t^n,\ n=1,2,\cdots$ | $\frac{n!}{s^{n+1}},\ s>0$ |
| $e^{\alpha t}$ | $\frac{s}{s-\alpha},\ s>\alpha$ |
| $\sin(\omega t)$ | $\frac{\omega}{s^2+\omega^2},\ s>0$ |
| $\cos(\omega t)$ | $\frac{s}{s^2+\omega^2},\ s>0$ |
| $\sinh(\omega t)$ | $\frac{\omega}{s^2-\omega^2},\ s>\lvert\omega\rvert$ |
| $\cosh(\omega t)$ | $\frac{s}{s^2-\omega^2},\ s>\lvert\omega\rvert$ |
| $e^{\alpha t}f(t),\ with\ \lvert f(t)\rvert\leq Me^{at}$ | $F(s-\alpha),\ s>\alpha+a$ |
| $e^{\alpha t}h(t)$ | $\frac{1}{s-\alpha},\ s>\alpha$ |
| $e^{\alpha t}t^n,\ n=1,2,\cdots$ | $\frac{n!}{(s-\alpha)^{n+1}},\ s>\alpha$ |
| $e^{\alpha t}\sin(\omega t)$ | $\frac{\omega}{(s-\alpha)^2+\omega^2},\ s>\alpha$ |
| $e^{\alpha t}\cos(\omega t)$ | $\frac{s-\alpha}{(s-\alpha)^2+\omega^2},\ s>\alpha$ |
| $f(t-\alpha)h(t-\alpha),\ \alpha\geq 0$ $with\ \lvert f(t)\rvert\leq Me^{at}$ | $e^{-\alpha s}F(s),\ s>a$ |

| f(t) | F(s) (continued) |
|---|---|
| $h(t-\alpha),\ \alpha \geq 0$ | $\frac{e^{-\alpha s}}{s},\ s > 0$ |
| $tf(t)$ | -$F'(s)$ |
| $\frac{t}{2\omega}\sin\omega t$ | $\frac{s}{(s^2+\omega^2)^2},\ s > 0$ |
| $\frac{1}{2\omega^3}[\sin\omega t - \omega t\cos\omega t]$ | $\frac{1}{(s^2+\omega^2)^2},\ s > 0$ |
| $f'(t),\ \textit{with}\ f(t)\ \textit{continuous}$ <br> $\textit{and}\ \lvert f'(t)\rvert \leq Me^{at}$ | $sF(s) - f(0)$ <br> $s > \max\{a, 0\} + 1$ |
| $f''(t),\ \textit{with}\ f'(t)\ \textit{continuous}$ <br> $\textit{and}\ \lvert f''(t)\rvert \leq Me^{at}$ | $s^2F(s) - sf(0) - f'(0)$ <br> $s > \max\{a, 0\} + 1$ |
| $f^{(n)}(t),\ \textit{with}\ f^{(n-1)}(t)\ \textit{continuous}$ <br> $\textit{and}\ \lvert f^{(n)}(t)\rvert \leq Me^{at}$ | $s^nF(s) - s^{n-1}f(0) - \cdots$ <br> -$sf^{(n-2)}(0) - f^{(n-1)}(0)$ <br> $s > \max\{a, 0\} + 1$ |
| $\int_0^t f(u)du,\ \textit{with}\ \lvert f(t)\rvert \leq Me^{at}$ | $\frac{F(s)}{s},\ \ s > \max\{a, 0\} + 1$ |

Table $\mathcal{L}$

# Practice Problems

**Problem 44.1**
Use Table $\mathcal{L}$ to find $\mathcal{L}[2e^t + 5]$.

**Problem 44.2**
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{3t-3}h(t-1)]$.

**Problem 44.3**
Use Table $\mathcal{L}$ to find $\mathcal{L}[\sin^2 \omega t]$.

**Problem 44.4**
Use Table $\mathcal{L}$ to find $\mathcal{L}[\sin 3t \cos 3t]$.

**Problem 44.5**
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{2t} \cos 3t]$.

**Problem 44.6**
Use Table $\mathcal{L}$ to find $\mathcal{L}[e^{4t}(t^2 + 3t + 5)]$.

**Problem 44.7**
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{10}{s^2+25} + \frac{4}{s-3}]$.

**Problem 44.8**
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{5}{(s-3)^4}]$.

**Problem 44.9**
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{e^{-2s}}{s-9}]$.

**Problem 44.10**
Use Table $\mathcal{L}$ to find $\mathcal{L}^{-1}[\frac{e^{-3s}(2s+7)}{s^2+16}]$.

**Problem 44.11**
Graph the function $f(t) = h(t-1) + h(t-3)$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

**Problem 44.12**
Graph the function $f(t) = t[h(t-1) - h(t-3)]$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

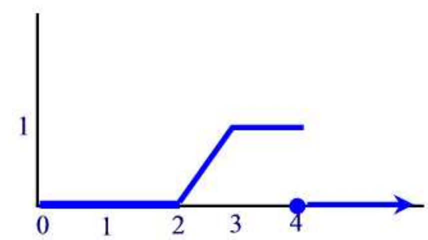
**Problem 44.13**
Graph the function $f(t) = 3[h(t-1) - h(t-4)]$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

**Problem 44.14**
Graph the function $f(t) = |2-t|[h(t-1) - h(t-3)]$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

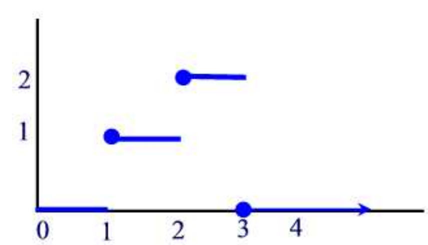
**Problem 44.15**
Graph the function $f(t) = h(2-t)$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

**Problem 44.16**
Graph the function $f(t) = h(t-1) + h(4-t)$ for $t \geq 0$, where $h(t)$ is the Heaviside step function, and use Table $\mathcal{L}$ to find $\mathcal{L}[f(t)]$.

**Problem 44.17**
The graph of $f(t)$ is given below. Represent $f(t)$ as a combination of Heaviside step functions, and use Table $\mathcal{L}$ to calculate the Laplace transform of $f(t)$.




**Problem 44.18**
The graph of $f(t)$ is given below. Represent $f(t)$ as a combination of Heaviside step functions, and use Table $\mathcal{L}$ to calculate the Laplace transform of $f(t)$.



**Problem 44.19**
Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{12}{(s-3)(s+1)}\right]$.

**Problem 44.20**
Using the partial fraction decomposition find $\mathcal{L}^{-1}\left[\frac{24e^{-5s}}{s^2-9}\right]$.

**Problem 44.21**
Use Laplace transform technique to solve the initial value problem

$$y' + 4y = g(t), \quad y(0) = 2$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 1 \\ 12, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

**Problem 44.22**
Use Laplace transform technique to solve the initial value problem

$$y'' - 4y = e^{3t}, \quad y(0) = 0, \quad y'(0) = 0.$$

**Problem 44.23**
Obtain the Laplace transform of the function $\int_2 t f(\lambda) d\lambda$ in terms of $\mathcal{L}[f(t)] = F(s)$ given that $\int_0^2 f(\lambda) d\lambda = 3$.

# 45 The Laplace Transform and the Method of Partial Fractions

In the last example of the previous section we encountered the equation

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^2+1}{s^2(s^2-4s+9)}\right].$$

We would like to find an explicit expression for $y(t)$. This can be done using the method of partial fractions which is the topic of this section. According to this method, finding $\mathcal{L}^{-1}\left(\frac{N(s)}{D(s)}\right)$ , where $N(s)$ and $D(s)$ are polynomials, require decomposing the rational function into a sum of simpler expressions whose inverse Laplace transform can be recognized from a table of Laplace transform pairs.
The method of integration by partial fractions is a technique for integrating rational functions, i.e. functions of the form

$$R(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials.
The idea consists of writing the rational function as a sum of simpler fractions called **partial fractions**. This can be done in the following way:

*Step 1.* Use long division to find two polynomials $r(s)$ and $q(s)$ such that

$$\frac{N(s)}{D(s)} = q(s) + \frac{r(s)}{D(s)}.$$

Note that if the degree of $N(s)$ is smaller than that of $D(s)$ then $q(s) = 0$ and $r(s) = N(s)$.

*Step 2.* Write $D(s)$ as a product of factors of the form $(as+b)^n$ or $(as^2+bs+c)^n$ where $as^2+bs+c$ is irreducible, i.e. $as^2+bs+c=0$ has no real zeros.

*Step 3.* Decompose $\frac{r(s)}{D(s)}$ into a sum of partial fractions in the following way:
(1) For each factor of the form $(s-\alpha)^k$ write

$$\frac{A_1}{s-\alpha} + \frac{A_2}{(s-\alpha)^2} + \cdots + \frac{A_k}{(s-\alpha)^k},$$

where the numbers $A_1, A_2, \cdots, A_k$ are to be determined.
(2) For each factor of the form $(as^2+bs+c)^k$ write

$$\frac{B_1s+C_1}{as^2+bs+c}+\frac{B_2s+C_2}{(as^2+bs+c)^2}+\cdots+\frac{B_ks+C_k}{(as^2+bs+c)^k},$$

where the numbers $B_1, B_2, \cdots, B_k$ and $C_1, C_2, \cdots, C_k$ are to be determined.

*Step 4.* Multiply both sides by $D(s)$ and simplify. This leads to an expression of the form

$$r(s) = \text{a polynomial whose coefficients are combinations of } \mathrm{A_i}, \mathrm{B_i}, \text{ and } \mathrm{C_i}.$$

Finally, we find the constants, $A_i, B_i$, and $C_i$ by equating the coefficients of like powers of $s$ on both sides of the last equation.

**Example 45.1**
Decompose into partial fractions $R(s) = \frac{s^3+s^2+2}{s^2-1}$.

**Solution.**
Step 1. $\frac{s^3+s^2+2}{s^2-1} = s+1+\frac{s+3}{s^2-1}$.
Step 2. $s^2-1=(s-1)(s+1)$.
Step 3. $\frac{s+3}{(s+1)(s-1)} = \frac{A}{s+1}+\frac{B}{s-1}$.
Step 4. Multiply both sides of the last equation by $(s-1)(s+1)$ to obtain

$$s+3 = A(s-1)+B(s+1).$$

Expand the right hand side, collect terms with the same power of $s$, and identify coefficients of the polynomials obtained on both sides:

$$s+3=(A+B)s+(B-A).$$

Hence, $A+B=1$ and $B-A=3$. Adding these two equations gives $B=2$. Thus, $A=-1$ and so

$$\frac{s^3+s^2+2}{s^2-1} = s+1-\frac{1}{s+1}+\frac{2}{s-1}.\blacksquare$$

Now, after decomposing the rational function into a sum of partial fractions all we need to do is to find the Laplace transform of expressions of the form $\frac{A}{(s-\alpha)^n}$ or $\frac{Bs+C}{(as^2+bs+c)^n}$.

**Example 45.2**
Find $\mathcal{L}^{-1}\left[\frac{1}{s(s-3)}\right]$.

**Solution.**
We write
$$\frac{1}{s(s-3)} = \frac{A}{s} + \frac{B}{s-3}.$$
Multiply both sides by $s(s-3)$ and simplify to obtain
$$1 = A(s-3) + Bs$$
or
$$1 = (A+B)s - 3A.$$
Now equating the coefficients of like powers of $s$ to obtain $-3A = 1$ and $A + B = 0$. Solving for $A$ and $B$ we find $A = -\frac{1}{3}$ and $B = \frac{1}{3}$. Thus,
$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s(s-3)}\right] &= -\frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] \\ &= -\frac{1}{3}h(t) + \frac{1}{3}e^{3t},\ t \geq 0\end{aligned}$$
where $h(t)$ is the Heaviside unit step function ■

**Example 45.3**
Find $\mathcal{L}^{-1}\left[\frac{3s+6}{s^2+3s}\right]$.

**Solution.**
We factor the denominator and split the integrand into partial fractions:
$$\frac{3s+6}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}.$$
Multiplying both sides by $s(s+3)$ to obtain
$$\begin{aligned}3s+6 &= A(s+3) + Bs \\ &= (A+B)s + 3A\end{aligned}$$
Equating the coefficients of like powers of $x$ to obtain $3A = 6$ and $A + B = 3$. Thus, $A = 2$ and $B = 1$. Finally,
$$\mathcal{L}^{-1}\left[\frac{3s+6}{s^2+3s}\right] = 2\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = 2h(t) + e^{-3t},\ t \geq 0.\ ■$$

**Example 45.4**
Find $\mathcal{L}^{-1}\left[\frac{s^2+1}{s(s+1)^2}\right]$.

**Solution.**
We factor the denominator and split the rational function into partial fractions:

$$\frac{s^2+1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}.$$

Multiplying both sides by $s(s+1)^2$ and simplifying to obtain

$$\begin{array}{rcl} s^2+1 & = & A(s+1)^2 + Bs(s+1) + Cs \\ & = & (A+B)s^2 + (2A+B+C)s + A. \end{array}$$

Equating coefficients of like powers of $s$ we find $A = 1, 2A + B + C = 0$ and $A + B = 1$. Thus, $B = 0$ and $C = -2$. Now finding the inverse Laplace transform to obtain

$$\mathcal{L}^{-1}\left[\frac{s^2+1}{s(s+1)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right] = h(t) - 2te^{-t},\ t \geq 0. \blacksquare$$

**Example 45.5**
Use Laplace transform to solve the initial value problem

$$y'' + 3y' + 2y = e^{-t}, \quad y(0) = y'(0) = 0.$$

**Solution.**
By the linearity property of the Laplace transform we can write

$$\mathcal{L}[y''] + 3\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[e^{-t}].$$

Now since

$$\begin{array}{rcl} \mathcal{L}[y''] & = & s^2\mathcal{L}[y] - sy(0) - y'(0) = s^2Y(s) \\ \mathcal{L}[y'] & = & sY(s) - y(0) = sY(s) \\ \mathcal{L}[e^{-t}] & = & \frac{1}{s+1} \end{array}$$

where $\mathcal{L}[y] = Y(s)$, we obtain

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s+1}.$$

Rearranging gives

$$(s^2+3s+2)Y(s)=\frac{1}{s+1}.$$

Thus,

$$Y(s)=\frac{1}{(s+1)(s^2+3s+2)}.$$

and

$$y(t)=\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+3s+2)}\right].$$

Using the method of partial fractions we can write

$$\frac{1}{(s+1)(s^2+3s+2)}=\frac{1}{s+2}-\frac{1}{s+1}+\frac{1}{(s+1)^2}.$$

Thus,

$$y(t)=\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]-\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]+\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right]=e^{-2t}-e^{-t}+te^{-t},\ \ t\geq 0 \blacksquare$$

## Practice Problems

In Problems 45.1 - 45.4, give the form of the partial fraction expansion for $F(s)$. You need not evaluate the constants in the expansion. However, if the denominator has an irreducible quadratic expression then use the completing the square process to write it as the sum/difference of two squares.

**Problem 45.1**

$$F(s) = \frac{s^3+3s+1}{(s-1)^3(s-2)^2}.$$

**Problem 45.2**

$$F(s) = \frac{s^2+5s-3}{(s^2+16)(s-2)}.$$

**Problem 45.3**

$$F(s) = \frac{s^3-1}{(s^2+1)^2(s+4)^2}.$$

**Problem 45.4**

$$F(s) = \frac{s^4+5s^2+2s-9}{(s^2+8s+17)(s-2)^2}.$$

**Problem 45.5**
Find $\mathcal{L}^{-1}\left[\frac{1}{(s+1)^3}\right]$.

**Problem 45.6**
Find $\mathcal{L}^{-1}\left[\frac{2s-3}{s^2-3s+2}\right]$.

**Problem 45.7**
Find $\mathcal{L}^{-1}\left[\frac{4s^2+s+1}{s^3+s}\right]$.

**Problem 45.8**
Find $\mathcal{L}^{-1}\left[\frac{s^2+6s+8}{s^4+8s^2+16}\right]$.

**Problem 45.9**
Use Laplace transform to solve the initial value problem

$$y' + 2y = 26\sin 3t,\ \ y(0) = 3.$$

**Problem 45.10**
Use Laplace transform to solve the initial value problem

$$y' + 2y = 4t,\ \ y(0) = 3.$$

**Problem 45.11**
Use Laplace transform to solve the initial value problem

$$y'' + 3y' + 2y = 6e^{-t},\ \ y(0) = 1,\ \ y'(0) = 2.$$

**Problem 45.12**
Use Laplace transform to solve the initial value problem

$$y'' + 4y = \cos 2t,\ \ y(0) = 1,\ \ y'(0) = 1.$$

**Problem 45.13**
Use Laplace transform to solve the initial value problem

$$y'' - 2y' + y = e^{2t},\ \ y(0) = 0,\ \ y'(0) = 0.$$

**Problem 45.14**
Use Laplace transform to solve the initial value problem

$$y'' + 9y = g(t),\ \ y(0) = 1,\ \ y'(0) = 0$$

where

$$g(t) = \begin{cases} 6, & 0 \le t < \pi \\ 0, & \pi \le t < \infty \end{cases}$$

**Problem 45.15**
Determine the constants $\alpha, \beta, y_0$, and $y_0'$ so that $Y(s) = \frac{2s-1}{s^2+s+2}$ is the Laplace transform of the solution to the initial value problem

$$y'' + \alpha y' + \beta y = 0,\ \ y(0) = y_0,\ \ y'(0) = y_0'.$$

**Problem 45.16**
Determine the constants $\alpha, \beta, y_0$, and $y_0'$ so that $Y(s) = \frac{s}{(s+1)^2}$ is the Laplace transform of the solution to the initial value problem

$$y'' + \alpha y' + \beta y = 0,\ \ y(0) = y_0,\ \ y'(0) = y_0'.$$

# 46 Laplace Transforms of Periodic Functions

In many applications, the nonhomogeneous term in a linear differential equation is a periodic function. In this section, we derive a formula for the Laplace transform of such periodic functions.

Recall that a function $f(t)$ is said to be $T-$**periodic** if we have $f(t+T) = f(t)$ whenever $t$ and $t+T$ are in the domain of $f(t)$. For example, the sine and cosine functions are $2\pi-$periodic whereas the tangent and cotangent functions are $\pi-$periodic.

If $f(t)$ is $T-$periodic for $t \geq 0$ then we define the function

$$f_T(t) = \begin{cases} f(t), & 0 \leq t \leq T \\ 0, & t > T \end{cases}$$

The Laplace transform of this function is then

$$\mathcal{L}[f_T(t)] = \int_0^\infty f_T(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt.$$

The Laplace transform of a $T-$periodic function is given next.

**Theorem 46.1**

If $f(t)$ is a $T-$periodic, piecewise continuous fucntion for $t \geq 0$ then

$$\mathcal{L}[f(t)] = \frac{\mathcal{L}[f_T(t)]}{1 - e^{-sT}}, \quad s > 0.$$

**Proof.**

Since $f(t)$ is piecewise continuous, it is bounded on the interval $0 \leq t \leq T$. By periodicity, $f(t)$ is bounded for $t \geq 0$. Hence, it has an exponential order at infinity. By Theorem 43.2, $\mathcal{L}[f(t)]$ exists for $s > 0$. Thus,

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt = \sum_{n=0}^\infty \int_0^T f_T(t - nT)h(t - nT)e^{-st}dt,$$

where the last sum is the result of decomposing the improper integral into a sum of integrals over the constituent periods.

By the Second Shifting Theorem (i.e. Theorem 44.2) we have

$$\mathcal{L}[f_T(t - nT)h(t - nT)] = e^{-nTs}\mathcal{L}[f_T(t)], \; s > 0$$

Hence,

$$\mathcal{L}[f(t)] = \sum_{n=0}^{\infty} e^{-nTs}\mathcal{L}[f_T(t)] = \mathcal{L}[f_T(t)]\left(\sum_{n=0}^{\infty} e^{-nTs}\right).$$

Since $s > 0$, it follows that $0 < e^{-nTs} < 1$ so that the series $\sum_{n=0}^{\infty} e^{-nTs}$ is a convergent geometric series with limit $\frac{1}{1-e^{-sT}}$. Therefore,

$$\mathcal{L}[f(t)] = \frac{\mathcal{L}[f_T(t)]}{1-e^{-sT}}, \quad s > 0 \blacksquare$$

**Example 46.1**
Determine the Laplace transform of the function

$$f(t) = \begin{cases} 1, & 0 \le t \le \frac{T}{2} \\ 0, & \frac{T}{2} < t < T \end{cases} \qquad f(t+T) = f(t), \ t \ge 0.$$

**Solution.**
The graph of $f(t)$ is shown in Figure 46.1.

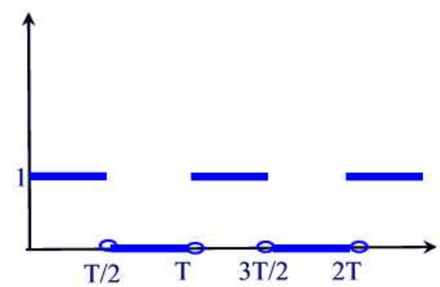



Figure 46.1

By Theorem 46.1,

$$\mathcal{L}[f(t)] = \frac{\int_0^{\frac{T}{2}} e^{-st}dt}{1-e^{-sT}}, \quad s > 0.$$

Evaluating this last integral, we find

$$\mathcal{L}[f(t)] = \frac{\frac{1-e^{-\frac{sT}{2}}}{s}}{1-e^{-sT}} = \frac{1}{s(1+e^{-\frac{sT}{2}})}, \quad s > 0 \blacksquare$$

**Example 46.2**
Find the Laplace transform of the sawtooth curve shown in Figure 46.2

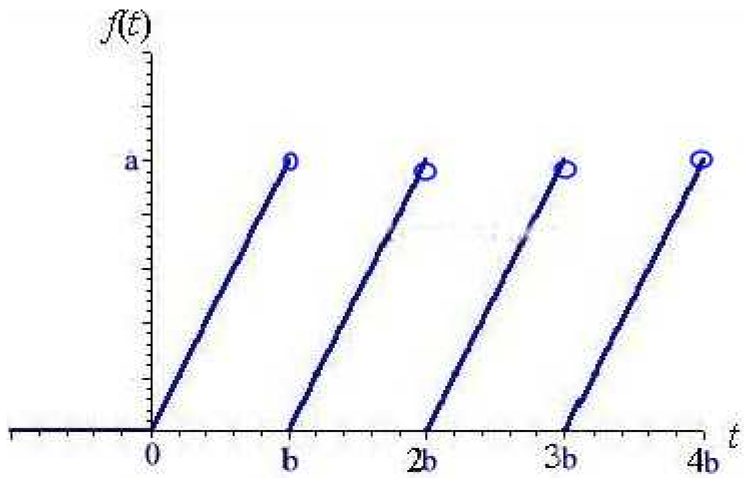



Figure 46.2

**Solution.**
The given function is periodic of period $b$. For the first period the function is defined by

$$f_b(t) = \frac{a}{b}t[h(t) - h(t-b)].$$

So we have

$$\begin{array}{rcl}\mathcal{L}[f_b(t)] &=& \mathcal{L}[\frac{a}{b}t(h(t) - h(t-b))]\\ &=& -\frac{a}{b}\frac{d}{ds}\mathcal{L}[h(t) - h(t-b)]\end{array}$$

But

$$\begin{array}{rcl}\mathcal{L}[h(t) - h(t-b)] &=& \mathcal{L}[h(t)] - \mathcal{L}[h(t-b)]\\ &=& \frac{1}{s} - \frac{e^{-bs}}{s}, \quad s > 0\end{array}$$

Hence,

$$\mathcal{L}[f_b(t)] = \frac{a}{b}\left(\frac{1}{s^2} - \frac{bse^{-bs} + e^{-bs}}{s^2}\right).$$

Finally,

$$\mathcal{L}[f(t)] = \frac{\mathcal{L}[f_b(t)]}{1 - e^{-bs}} = \frac{a}{b}\left[\frac{1 - e^{-bs} - bse^{-bs}}{s^2(1 - e^{-bs})}\right] \blacksquare$$

**Example 46.3**
Find $\mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}\right]$.

**Solution.**
Note first that

$$\frac{1}{s^2} - \frac{e^{-s}}{s(1 - e^{-s})} = \frac{1 - e^{-s} - se^{-s}}{s^2(1 - e^{-s})}.$$

According to the previous example with $a = 1$ and $b = 1$ we find that $\mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}\right]$ is the sawtooth function shown in Figure 46.2 ■

**Linear Time Invariant Systems and the Transfer Function**

The Laplace transform is a powerful technique for analyzing linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems, to name just a few. A mathematical model described by a linear differential equation with constant coefficients of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \cdots + b_1 u' + b_0 u$$

is called a **linear time invariant system.** The function $y(t)$ denotes the system output and the function $u(t)$ denotes the system input. The system is called time-invariant because the parameters of the system are not changing over time and an input now will give the same result as the same input later. Applying the Laplace transform on the linear differential equation with null initial conditions we obtain

$$a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \cdots + a_0 Y(s) = b_m s^m U(s) + b_{m-1} s^{m-1} U(s) + \cdots + b_0 U(s).$$

The function

$$\Phi(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

is called the **system transfer function.** That is, the transfer function of a linear time-invariant system is the ratio of the Laplace transform of its output to the Laplace transform of its input.

**Example 46.4**

Consider the mathematical model described by the initial value problem

$$my'' + \gamma y' + ky = f(t), \quad y(0) = 0, \; y'(0) = 0.$$

The coefficients $m, \gamma$, and $k$ describe the properties of some physical system, and $f(t)$ is the input to the system. The solution $y$ is the output at time $t$. Find the system transfer function.

**Solution.**
By taking the Laplace transform and using the initial conditions we obtain

$$(ms^2 + \gamma s + k)Y(s) = F(s).$$

Thus,

$$\mathbf{\Phi}(s) = \frac{Y(s)}{F(s)} = \frac{1}{ms^2 + \gamma s + k} \blacksquare \tag{29}$$

**Parameter Identification**
One of the most useful applications of system transfer functions is for system or parameter identification.

**Example 46.5**
Consider a spring-mass system governed by

$$my'' + \gamma y' + ky = f(t), \quad y(0) = 0, \ y'(0) = 0. \tag{30}$$

Suppose we apply a unit step force $f(t) = h(t)$ to the mass, initially at equilibrium, and you observe the system respond as

$$y(t) = -\frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t + \frac{1}{2}.$$

What are the physical parameters $m, \gamma$, and $k$?

**Solution.**
Start with the model (30)) with $f(t) = h(t)$ and take the Laplace transform of both sides, then solve to find $Y(s) = \frac{1}{s(ms^2+\gamma s+k)}$. Since $f(t) = h(t)$, $F(s) = \frac{1}{s}$. Hence

$$\Phi(s) = \frac{Y(s)}{F(s)} = \frac{1}{ms^2 + \gamma s + k}.$$

On the other hand, for the input $f(t) = h(t)$ the corresponding observed output is

$$y(t) = -\frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t + \frac{1}{2}.$$

Hence,

$$\begin{aligned} Y(s) &= \mathcal{L}[-\tfrac{1}{2}e^{-t}\cos t - \tfrac{1}{2}e^{-t}\sin t + \tfrac{1}{2}] \\ &= -\tfrac{1}{2}\tfrac{s+1}{(s+1)^2+1} - \tfrac{1}{2}\tfrac{1}{(s+1)^2+1} + \tfrac{1}{2s} \\ &= \tfrac{1}{s(s^2+2s+2)} \end{aligned}$$

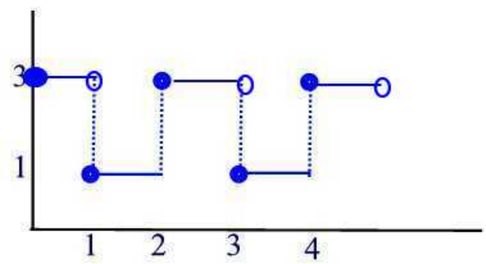
Thus,

$$\Phi(s) = \frac{Y(s)}{F(s)} = \frac{1}{s^2 + 2s + 2}.$$

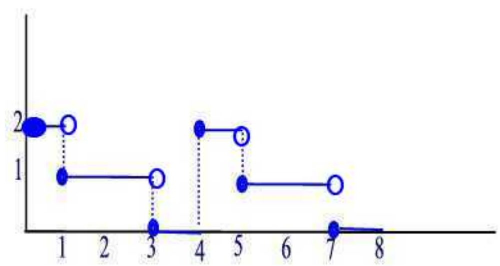
By comparison we conclude that $m = 1,\ \gamma = 2$, and $k = 2$ ■

## Practice Problems

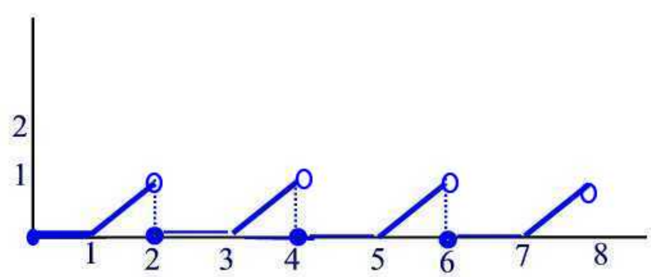
**Problem 46.1**
Find the Laplace transform of the periodic function whose graph is shown.



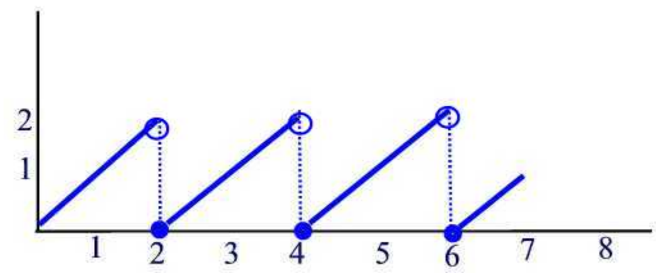

**Problem 46.2**
Find the Laplace transform of the periodic function whose graph is shown.




**Problem 46.3**
Find the Laplace transform of the periodic function whose graph is shown.



**Problem 46.4**
Find the Laplace transform of the periodic function whose graph is shown.




**Problem 46.5**
State the period of the function $f(t)$ and find its Laplace transform where

$$f(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & \pi \leq t < 2\pi \end{cases} \qquad f(t + 2\pi) = f(t), \ t \geq 0.$$

**Problem 46.6**
State the period of the function $f(t) = 1 - e^{-t}, \ 0 \leq t < 2, \quad f(t+2) = f(t)$, and find its Laplace transform.

**Problem 46.7**
Using Example 44.3 find

$$\mathcal{L}^{-1}\left[\frac{s^2 - s}{s^3} + \frac{e^{-s}}{s(1 - e^{-s})}\right].$$

**Problem 46.8**
An object having mass $m$ is initially at rest on a frictionless horizontal surface. At time $t = 0$, a periodic force is applied horizontally to the object, causing it to move in the positive x-direction. The force, in newtons, is given by

$$f(t) = \begin{cases} f_0, & 0 \leq t \leq \frac{T}{2} \\ 0, & \frac{T}{2} < t < T \end{cases} \qquad f(t + T) = f(t), \ t \geq 0.$$

The initial value problem for the horizontal position, $x(t)$, of the object is

$$mx''(t) = f(t), \quad x(0) = x'(0) = 0.$$

(a) Use Laplace transforms to determine the velocity, $v(t) = x'(t)$, and the position, $x(t)$, of the object.
(b) Let $m = 1\ kg,\ f_0 = 1\ N$, and $T = 1\ sec$. What is the velocity, $v$, and position, $x$, of the object at $t = 1.25$ sec?

**Problem 46.9**
Consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = y'(0) = 0, \quad t > 0$$

Suppose that the transfer function of this system is given by $\Phi(s) = \frac{1}{2s^2+5s+2}$.
(a) What are the constants $a,\ b$, and $c$?
(b) If $f(t) = e^{-t}$, determine $F(s),\ Y(s)$, and $y(t)$.

**Problem 46.10**
Consider the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = y'(0) = 0, \quad t > 0$$

Suppose that an input $f(t) = t$, when applied to the above system produces the output $y(t) = 2(e^{-t} - 1) + t(e^{-t} + 1), \quad t \geq 0$.
(a) What is the system transfer function?
(b) What will be the output if the Heaviside unit step function $f(t) = h(t)$ is applied to the system?

**Problem 46.11**
Consider the initial value problem

$$y'' + y' + y = f(t), \quad y(0) = y'(0) = 0,$$

where

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ -1, & 1 < t < 2 \end{cases} \qquad f(t+2) = f(t)$$

(a) Determine the system transfer function $\Phi(s)$.
(b) Determine $Y(s)$.

**Problem 46.12**
Consider the initial value problem

$$y''' - 4y = e^t + t, \quad y(0) = y'(0) = y''(0) = 0.$$

(a) Determine the system transfer function $\Phi(s)$.
(b) Determine $Y(s)$.

**Problem 46.13**
Consider the initial value problem

$$y'' + by' + cy = h(t), \quad y(0) = y_0, \ y'(0) = y_0', \ t > 0.$$

Suppose that $\mathcal{L}[y(t)] = Y(s) = \frac{s^2+2s+1}{s^3+3s^2+2s}$. Determine the constants $b, \ c, \ y_0,$ and $y_0'$.

# 47 Convolution Integrals

We start this section with the following problem.

**Example 47.1**
A spring-mass system with a forcing function $f(t)$ is modeled by the following initial-value problem

$$mx'' + kx = f(t), \quad x(0) = x_0, \ x'(0) = x'_0.$$

Find solution to this initial value problem using the Laplace transform method.

**Solution.**
Apply Laplace transform to both sides of the equation to obtain

$$ms^2X(s) - msx_0 - mx'_0 + kX(s) = F(s).$$

Solving the above algebraic equation for $X(s)$ we find

$$\begin{array}{rcl} X(s) & = & \frac{F(s)}{ms^2+k} + \frac{msx_0}{ms^2+k} + \frac{mx'_0}{ms^2+k} \\ & = & \frac{1}{m}\frac{F(s)}{s^2+\frac{k}{m}} + \frac{sx_0}{s^2+\frac{k}{m}} + \frac{x'_0}{s^2+\frac{k}{m}} \end{array}$$

Apply the inverse Laplace transform to obtain

$$\begin{array}{rcl} x(t) & = & \mathcal{L}^{-1}[X(s)] \\ & = & \frac{1}{m}\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2+\frac{k}{m}}\right\} + x_0\mathcal{L}^{-1}\left\{\frac{s}{s^2+\frac{k}{m}}\right\} + x'_0\mathcal{L}^{-1}\left\{\frac{1}{s^2+\frac{k}{m}}\right\} \\ & = & \frac{1}{m}\mathcal{L}^{-1}\left\{F(s)\cdot\frac{1}{s^2+\frac{k}{m}}\right\} + x_0\cos\left(\sqrt{\frac{k}{m}}\right)t + x'_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}\right)t \end{array}$$

Finding $\mathcal{L}^{-1}\left\{F(s)\cdot\frac{1}{s^2+\frac{k}{m}}\right\}$,i.e., the inverse Laplace transform of a product, requires the use of the concept of convolution, a topic we discuss in this section ■

Convolution integrals are useful when finding the inverse Laplace transform of products $H(s) = F(s)G(s)$. They are defined as follows: The **convolution** of two scalar piecewise continuous functions $f(t)$ and $g(t)$ defined for $t \geq 0$ is the integral

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds.$$

**Example 47.2**
Find $f * g$ where $f(t) = e^{-t}$ and $g(t) = \sin t$.

**Solution.**
Using integration by parts twice we arrive at

$$\begin{aligned}(f * g)(t) &= \int_0^t e^{-(t-s)} \sin s ds \\ &= \frac{1}{2}\left[e^{-(t-s)}(\sin s - \cos s)\right]_0^t \\ &= \frac{e^{-t}}{2} + \frac{1}{2}(\sin t - \cos t) \blacksquare\end{aligned}$$

**Graphical Interpretation of Convolution Operation**
For the convolution

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds$$

we perform the following:
Step 1. Given the graphs of $f(s)$ and $g(s)$.(Figure 47.1(a) and (b))
Step 2. Time reverse $f(-s)$. (See Figure 47.1(c))
Step 3. Shift $f(-s)$ right by an amount $t$ to get $f(t-s)$. (See Figure 47.1(d))
Step 4. Determine the product $f(t-s)g(s)$. (See Figure 47.1(e))
Step 5. Determine the area under the graph of $f(t-s)g(s)$ between 0 and $t$. (See Figure 47.1(e))

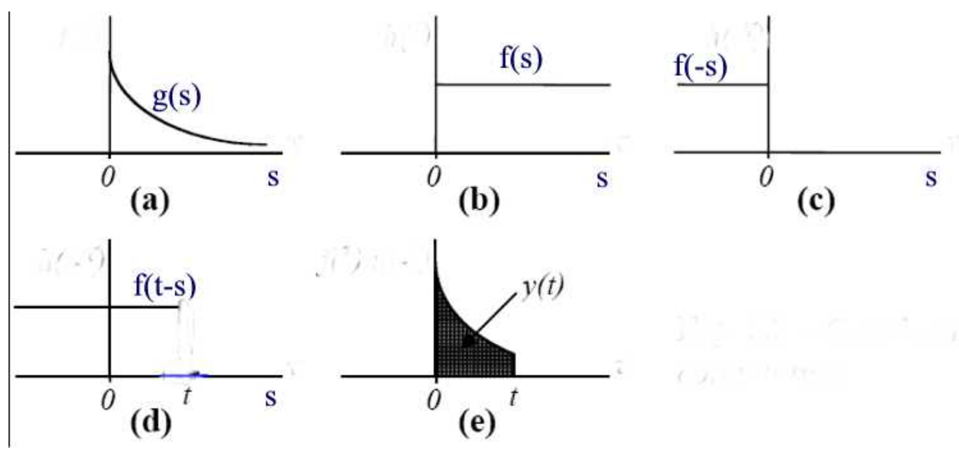



Figure 47.1

Next, we state several properties of convolution product, which resemble those of ordinary product.

**Theorem 47.1**
Let $f(t), g(t)$, and $k(t)$ be three piecewise continuous scalar functions defined for $t \geq 0$ and $c_1$ and $c_2$ are arbitrary constants. Then
(i) $f * g = g * f$ (Commutative Law)
(ii) $(f * g) * k = f * (g * k)$ (Associative Law)
(iii) $f * (c_1 g + c_2 k) = c_1 f * g + c_2 f * k$ (Distributive Law)

**Proof.**
(i) Using the change of variables $\tau = t - s$ we find

$$\begin{aligned}(f * g)(t) &= \int_0^t f(t-s)g(s)ds \\ &= -\int_t^0 f(\tau)g(t-\tau)d\tau \\ &= \int_0^t g(t-\tau)f(\tau)d\tau = (g * f)(t)\end{aligned}$$

(ii) By definition, we have

$$\begin{aligned}[(f * g) * k](t) &= \int_0^t (f * g)(t-u)k(u)du \\ &= \int_0^t \left[\int_0^{t-u} f(t-u-w)g(w)k(u)dw\right] du\end{aligned}$$

For the integral in the bracket, make change of variable $w = s - u$. We have

$$[(f * g) * k](t) = \int_0^t \left[\int_u^t f(t-s)g(s-u)k(u)ds\right] du.$$

This multiple integral is carried over the region

$$\{(s, u) : 0 \leq u \leq s \leq t\}$$

as depicted by shaded region in the following graph.

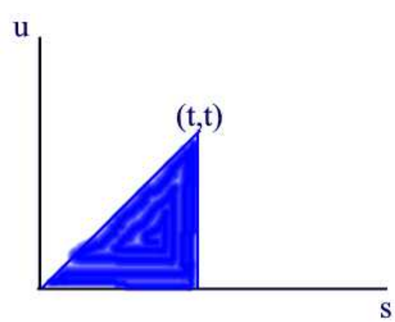



Figure 47.2

Changing the order of integration, we have

$$\begin{array}{rcl}[(f*g)*k)](t) & = & \int_0^t \left[\int_0^s f(t-s)g(s-u)k(u)du\right] ds \\ & = & \int_0^t f(t-s)(g*k)(s)ds \\ & = & [f*(g*k)](t)\end{array}$$

(iii) We have

$$\begin{array}{rcl}(f*(c_1g+c_2k))(t) & = & \int_0^t f(t-s)(c_1g(s)+c_2k(s))ds \\ \\ & = & c_1\int_0^t f(t-s)g(s)ds + c_2\int_0^t f(t-s)k(s)ds \\ \\ & = & c_1(f*g)(t)+c_2(f*k)(t) \blacksquare\end{array}$$

**Example 47.3**
Express the solution to the initial value problem $y' + \alpha y = g(t),\ y(0) = y_0$ in terms of a convolution integral.

**Solution.**
Solving this initial value problem by the method of integrating factor we find

$$y(t) = e^{-\alpha t}y_0 + \int_0^t e^{-\alpha(t-s)}g(s)ds = e^{-\alpha t}y_0 + e^{-\alpha t} * g(t) \blacksquare$$

**Example 47.4**
If $\mathbf{f}(t)$ is an $m \times n$ matrix function and $\mathbf{g}$(t) is an $n \times p$ matrix function then we define

$$(\mathbf{f} * \mathbf{g})(t) = \int_0^t \mathbf{f}(t-s)\mathbf{g}(s)ds, \quad t \geq 0.$$

Express the solution to the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(t),\ \mathbf{y}(0) = \mathbf{y}_0$ in terms of a convolution integral.

**Solution.**
The unique solution is given by

$$\mathbf{y}(t) = e^{t\mathbf{A}}\mathbf{y}_0 + \int_0^t e^{\mathbf{A}(t-s)}\mathbf{g}(s)ds = e^{t\mathbf{A}}\mathbf{y}_0 + e^{t\mathbf{A}} * \mathbf{g}(t) \blacksquare$$

The following theorem, known as the Convolution Theorem, provides a way for finding the Laplace transform of a convolution integral and also finding the inverse Laplace transform of a product.

**Theorem 47.2**
If $f(t)$ and $g(t)$ are piecewise continuous for $t \geq 0$, and of exponential order at infinity then

$$\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)] = F(s)G(s).$$

Thus, $(f*g)(t) = \mathcal{L}^{-1}[F(s)G(s)]$.

**Proof.**
First we show that $f*g$ has a Laplace transform. From the hypotheses we have that $|f(t)| \leq M_1 e^{a_1 t}$ for $t \geq C_1$ and $|g(t)| \leq M_2 e^{a_2 t}$ for $t \geq C_2$. Let $M = M_1 M_2$ and $C = C_1 + C_2$. Then for $t \geq C$ we have

$$\begin{array}{lll} |(f*g)(t)| & = & \left|\int_0^t f(t-s)g(s)ds\right| \leq \int_0^t |f(t-s)||g(s)|ds \\ & \leq & M_1 M_2 \int_0^t e^{a_1(t-s)} e^{a_2 s} ds \\ & = & \left\{\begin{array}{cc} Mte^{a_1 t}, & a_1 = a_2 \\ M\frac{e^{a_2 t} - e^{a_1 t}}{a_2 - a_1}, & a_1 \neq a_2 \end{array}\right. \end{array}$$

This shows that $f*g$ is of exponential order at infinity. Since $f$ and $g$ are piecewise continuous, the first fundamental theorem of calculus implies that $f*g$ is also piecewise continuous. Hence, $f*g$ has a Laplace transform.
Next, we have

$$\begin{array}{lll} \mathcal{L}[(f*g)(t)] & = & \int_0^\infty e^{-st}\left(\int_0^t f(t-\tau)g(\tau)d\tau\right)dt \\ & = & \int_{t=0}^\infty \int_{\tau=0}^t e^{-st} f(t-\tau)g(\tau)d\tau dt \end{array}$$

Note that the region of integration is an infinite triangular region and the integration is done vertically in that region. Integration horizontally we find

$$\mathcal{L}[(f*g)(t)] = \int_{\tau=0}^\infty \int_{t=\tau}^\infty e^{-st} f(t-\tau)g(\tau)dtd\tau.$$

We next introduce the change of variables $\beta = t - \tau$. The region of integration becomes $\tau \geq 0, t \geq 0$. In this case, we have

$$\begin{array}{lll} \mathcal{L}[(f*g)(t)] & = & \int_{\tau=0}^\infty \int_{\beta=0}^\infty e^{-s(\beta+\tau)} f(\beta)g(\tau)d\tau d\beta \\ & = & \left(\int_{\tau=0}^\infty e^{-s\tau} g(\tau)d\tau\right)\left(\int_{\beta=0}^\infty e^{-s\beta} f(\beta)d\beta\right) \\ & = & G(s)F(s) = F(s)G(s) \blacksquare \end{array}$$

**Example 47.5**
Use the convolution theorem to find the inverse Laplace transform of

$$H(s) = \frac{1}{(s^2+a^2)^2}.$$

**Solution.**
Note that

$$H(s) = \left(\frac{1}{s^2+a^2}\right)\left(\frac{1}{s^2+a^2}\right).$$

So, in this case we have, $F(s) = G(s) = \frac{1}{s^2+a^2}$ so that $f(t) = g(t) = \frac{1}{a}\sin(at)$. Thus,

$$(f*g)(t) = \frac{1}{a^2}\int_0^t \sin(at-as)\sin(as)ds = \frac{1}{2a^3}(\sin(at) - at\cos(at)) \blacksquare$$

Convolution integrals are useful in solving initial value problems with forcing functions.

**Example 47.6**
Solve the initial value problem

$$4y'' + y = g(t), \quad y(0) = 3, \; y'(0) = -7$$

**Solution.**
Take the Laplace transform of all the terms and plug in the initial conditions to obtain

$$4(s^2Y(s) - 3s + 7) + Y(s) = G(s)$$

or

$$(4s^2+1)Y(s) - 12s + 28 = G(s).$$

Solving for $Y(s)$ we find

$$\begin{aligned} Y(s) &= \frac{12s-28}{4\left(s^2+\frac{1}{4}\right)} + \frac{G(s)}{4\left(s^2+\frac{1}{4}\right)} \\ &= \frac{3s}{s^2+(\left(\frac{1}{2}\right)^2} - 7\frac{\left(\frac{1}{2}\right)^2}{s^2+\left(\frac{1}{2}\right)^2} + \frac{1}{4}G(s)\frac{\left(\frac{1}{2}\right)^2}{s^2+\left(\frac{1}{2}\right)^2} \end{aligned}$$

Hence,

$$y(t) = 3\cos\left(\frac{t}{2}\right) - 7\sin\left(\frac{t}{2}\right) + \frac{1}{2}\int_0^t \sin\left(\frac{s}{2}\right)g(t-s)ds.$$

So, once we decide on a $g(t)$ all we need to do is to evaluate the integral and we'll have the solution ■

# Practice Problems

**Problem 47.1**
Consider the functions $f(t) = g(t) = h(t),\ t \geq 0$ where $h(t)$ is the Heaviside unit step function. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

**Problem 47.2**
Consider the functions $f(t) = e^t$ and $g(t) = e^{-2t},\ t \geq 0$. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

**Problem 47.3**
Consider the functions $f(t) = \sin t$ and $g(t) = \cos t,\ t \geq 0$. Compute $f * g$ in two different ways.
(a) By directly evaluating the integral.
(b) By computing $\mathcal{L}^{-1}[F(s)G(s)]$ where $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

**Problem 47.4**
Use Laplace transform to comput the convolution $\mathbf{P} * \mathbf{y}$, where $|bfP(t) = \left[\begin{array}{cc} h(t) & e^t \\ 0 & t \end{array}\right]$ and $\mathbf{y}(t) = \left[\begin{array}{c} h(t) \\ e^{-t} \end{array}\right]$.

**Problem 47.5**
Compute and graph $f * g$ where $f(t) = h(t)$ and $g(t) = t[h(t) - h(t-2)]$.

**Problem 47.6**
Compute and graph $f * g$ where $f(t) = h(t) - h(t-1)$ and $g(t) = h(t-1) - 2h(t-2)]$.

**Problem 47.7**
Compute $t * t * t$.

**Problem 47.8**
Compute $h(t) * e^{-t} * e^{-2t}$.

**Problem 47.9**
Compute $t * e^{-t} * e^t$.

**Problem 47.10**
Suppose it is known that $\overbrace{h(t) * h(t) * \cdots * h(t)}^{n\ functions} = Ct^8$. Determine the constants $C$ and the poisitive integer $n$.

**Problem 47.11**
Use Laplace transform to solve for $y(t)$ :

$$\int_0^t \sin{(t-\lambda)} y(\lambda) d\lambda = t^2.$$

**Problem 47.12**
Use Laplace transform to solve for $y(t)$ :

$$y(t) - \int_0^t e^{(t-\lambda)} y(\lambda) d\lambda = t.$$

**Problem 47.13**
Use Laplace transform to solve for $y(t)$ :

$$t * y(t) = t^2(1 - e^{-t}).$$

**Problem 47.14**
Use Laplace transform to solve for $y(t)$ :

$$\mathbf{y}' = h(t) * \mathbf{y}, \quad \mathbf{y}(0) = \left[ \begin{array}{c} 1 \\ 2 \end{array} \right].$$

**Problem 47.15**
Solve the following initial value problem.

$$y' - y = \int_0^t (t-\lambda) e^{\lambda} d\lambda, \quad y(0) = -1.$$

# 48 The Dirac Delta Function and Impulse Response

In applications, we are often encountered with linear systems, originally at rest, excited by a sudden large force (such as a large applied voltage to an electrical network) over a very short time frame. In this case, the output corresponding to this sudden force is referred to as the "impulse response". Mathematically, an impulse can be modeled by an initial value problem with a special type of function known as the **Dirac delta function** as the external force, i.e., the nonhomogeneous term. To solve such IVP requires finding the Laplace transform of the delta function which is the main topic of this section.

**An Example of Impulse Response**
Consider a spring-mass system with a time-dependent force $f(t)$ applied to the mass. The situation is modeled by the second-order differential equation

$$my'' + \gamma y' + ky = f(t) \tag{31}$$

where $t$ is time and $y(t)$ is the displacement of the mass from equilibrium. Now suppose that for $t \le 0$ the mass is at rest in its equilibrium position, so $y(0) = y'(0) = 0$. Hence, the situation is modeled by the initial value problem

$$my'' + \gamma y' + ky = f(t), \quad y(0) = 0, \quad y'(0) = 0. \tag{32}$$

Solving this equation by the method of variation of parameters one finds the unique solution

$$y(t) = \int_0^t \phi(t-s)f(s)ds \tag{33}$$

where

$$\phi(t) = \frac{e^{(-\gamma/2m)t}\sin\left(t\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}\right)}{m\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}}.$$

Next, we consider the problem of strucking the mass by an "instantaneous" hammer blow at $t = 0$. This situation actually occurs frequently in practice-a system sustains a forceful, almost-instantaneous input. Our goal is to model the situation mathematically and determine how the system will respond.

In the above situation we might describe f(t) as a large constant force applied on a very small time interval. Such a model leads to the forcing function

$$f_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & 0 \le t \le \epsilon \\ 0, & \text{otherwise} \end{cases}$$

where $\epsilon$ is a small positive real number. When $\epsilon$ is close to zero the applied force is very large during the time interval $0 \le t \le \epsilon$ and zero afterwards. A possible graph of $f_\epsilon(t)$ is given in Figure 48.1

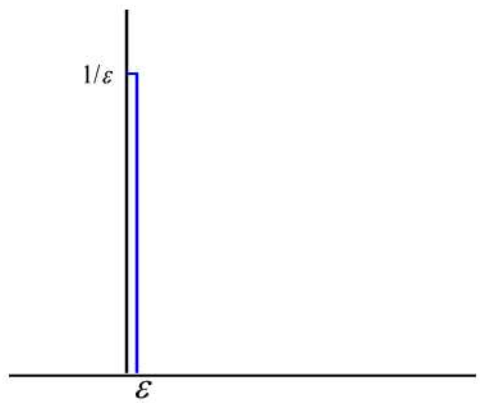



Figure 48.1

In this case it's easy to see that for any choice of $\epsilon$ we have

$$\int_{-\infty}^{\infty} f_\epsilon dt = 1$$

and

$$\lim_{\epsilon \to 0^+} f_\epsilon(t) = 0, \ t \neq 0, \quad \lim_{\epsilon \to 0^+} f_\epsilon(0) = \infty. \tag{34}$$

Our ultimate interest is the behavior of the solution to equation (31) with forcing function $f_\epsilon(t)$ in the limit $\epsilon \to 0^+$. That is, what happens to the system output as we make the applied force progressively "sharper" and "stronger?".
Let $y_\epsilon(t)$ be the solution to equation (31) with $f(t) = f_\epsilon(t)$. Then the unique solution is given by

$$y_\epsilon(t) = \int_0^t \phi(t-s) f_\epsilon(s) ds.$$

For $t \geq \epsilon$ the last equation becomes

$$y_\epsilon(t) = \frac{1}{\epsilon}\int_0^\epsilon \phi(t-s)ds.$$

Since $\phi(t)$ is continuous for all $t \geq 0$ we can apply the mean value theorem for integrals and write

$$y_\epsilon(t) = \phi(t-\psi)$$

for some $0 \leq \psi \leq \epsilon$. Letting $\epsilon \to 0^+$ and using the continuity of $\phi$ we find

$$y(t) = \lim_{\epsilon \to 0^+} y_\epsilon(t) = \phi(t).$$

We call $y(t)$ the **impulse response** of the linear system.

**The Dirac Delta Function**

The problem with the integral

$$\int_0^t \phi(t-s)f_\epsilon(s)ds$$

is that $\lim_{\epsilon\to 0^+} f_\epsilon(0)$ is undefined. So it makes sense to ask the question of whether we can find a function $\delta(t)$ such that

$$\begin{array}{rcl}\lim_{\epsilon\to 0^+} y_\epsilon(t) & = & \lim_{\epsilon\to 0^+}\int_0^t \phi(t-s)f_\epsilon(s)ds\\ & = & \int_0^t \phi(t-s)\delta(s)ds\\ & = & \phi(t)\end{array}$$

where the role of $\delta(t)$ would be to evaluate the integrand at $s = 0$. Note that because of Fig 48.1 and (34), we cannot interchange the opeartions of limit and integration in the above limit process. Such a function $\delta$ exist in the theory of distributions and can be defined as follows:
If $f(t)$ is continuous in $a \leq t \leq b$ then we define the function $\delta(t)$ by the integral equation

$$\int_a^b f(t)\delta(t-t_0)dt = \lim_{\epsilon\to 0^+}\int_a^b f(t)f_\epsilon(t-t_0)dt.$$

The object $\delta(t)$ on the left is called the **Dirac Delta function,** or just the **delta function** for short.

**Finding the Impulse Function Using Laplace Transform**
For $\epsilon > 0$ we can solve the initial value problem (32) using Laplace transforms. To do this we need to compute the Laplace transform of $f_\epsilon(t)$, given by the integral

$$\mathcal{L}[f_\epsilon(t)] = \int_0^\infty f_\epsilon(t)e^{-st}dt = \frac{1}{\epsilon}\int_0^\epsilon e^{-st}dt = \frac{1-e^{-\epsilon s}}{\epsilon s}.$$

Note that by using L'Hôpital's rule we can write

$$\lim_{\epsilon\to 0^+} \mathcal{L}[f_\epsilon(t)] = \lim_{\epsilon\to 0^+} \frac{1-e^{-\epsilon s}}{\epsilon s} = 1, \quad s > 0.$$

Now, to find $y_\epsilon(t)$, we apply the Laplace transform to both sides of equation (31) and using the initial conditions we obtain

$$ms^2 Y_\epsilon(s) + \gamma s Y_\epsilon(s) + kY_\epsilon(s) = \frac{1-e^{-\epsilon s}}{\epsilon s}.$$

Solving for $Y_\epsilon(s)$ we find

$$Y_\epsilon(s) = \frac{1}{ms^2+\gamma s+k}\frac{1-e^{-\epsilon s}}{\epsilon s}.$$

Letting $\epsilon \to 0^+$ we find

$$Y(s) = \frac{1}{ms^2+\gamma s+k}$$

which is the transfer function of the system. Now inverse transform $Y(s)$ to find the solution to the initial value problem. That is,

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{ms^2+\gamma s+k}\right) = \phi(t).$$

Now, impulse inputs are usually modeled in terms of delta functions. Thus, knowing the Laplace transform of such functions is important when solving differential equations. The next theorem finds the Laplace transform of the delta function.

**Theorem 48.1**
With $\delta(t)$ defined as above, if $a \leq t_0 < b$

$$\int_a^b f(t)\delta(t-t_0)dt = f(t_0).$$

**Proof.**
We have

$$\begin{array}{rcl}
\int_a^b f(t)\delta(t-t_0) & = & \lim_{\epsilon\to 0^+}\int_a^b f(t)f_\epsilon(t-t_0)dt \\
& = & \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} f(t)dt \\
& = & \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}f(t_0+\beta\epsilon)\epsilon = f(t_0)
\end{array}$$

where $0 < \beta < 1$ and the mean-value theorem for integrals has been used ■

**Remark 48.1**
Since $p_\epsilon(t-t_0) = \frac{1}{\epsilon}$ for $t_0 \leq t \leq t_0+\epsilon$ and 0 otherwise we see that $\int_a^b f(t)\delta(t-a)dt = f(a)$ and $\int_a^b f(t)\delta(t-t_0)dt = 0$ for $t_0 \geq b$.

It follows immediately from the above theorem that

$$\mathcal{L}[\delta(t-t_0)] = \int_0^\infty e^{-st}\delta(t-t_0)dt = e^{-st_0}, \quad t_0 \geq 0.$$

In particular, if $t_0 = 0$ we find

$$\mathcal{L}[\delta(t)] = 1.$$

The following example illustrates the formal use of the delta function.

**Example 48.1**
A spring-mass system with mass 2, damping 4, and spring constant 10 is subject to a hammer blow at time $t = 0$. The blow imparts a total impulse of 1 to the system, which was initially at rest. Find the response of the system.

**Solution.**
The situation is modeled by the initial value problem

$$2y'' + 4y' + 10y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Taking Laplace transform of both sides we find

$$2s^2Y(s) + 4sY(s) + 10Y(s) = 1.$$

Solving for $Y(s)$ we find

$$Y(s) = \frac{1}{2s^2 + 4s + 10}.$$

The impulsive response is

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{2}\frac{1}{(s+1)^2 + 2^2}\right) = \frac{1}{4}e^{-2t}\sin 2t \blacksquare$$

**Example 48.2**
A 16 lb weight is attached to a spring with a spring constant equal to 2 lb/ft. Neglect damping. The weight is released from rest at 3 ft below the equilibrium position. At $t = 2\pi$ sec, it is struck with a hammer, providing an impulse of 4 lb-sec. Determine the displacement function $y(t)$ of the weight.

**Solution.**
This situation is modeled by the initial value problem

$$\frac{16}{32}y'' + 2y = 4\delta(t - 2\pi), \quad y(0) = 3, \ y'(0) = 0.$$

Apply Laplace transform to both sides to obtain

$$s^2Y(s) - 3s + 4Y(s) = 8e^{-2\pi s}.$$

Solving for $Y(s)$ we find

$$Y(s) = \frac{3s}{s^2 + 4} + \frac{e^{-2\pi s}}{s^2 + 4}.$$

Now take the inverse Laplace transform to get

$$y(t) = \mathcal{L}^{-1}[Y(s)] = 3\cos 2t + 8h(t - 2\pi)f(t - 2\pi)$$

where

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\} = \frac{1}{2}\sin 2t.$$

Hence,

$$y(t) = 3\cos 2t + 4h(t - 2\pi)\sin 2(t - 2\pi) = 3\cos 2t + 4h(t - 2\pi)\sin 2t$$

or more explicitly

$$y(t) = \begin{cases} 3\cos 2t, & t < 2\pi \\ 3\cos 2t + 4\sin 2t, & t \geq 2\pi \end{cases} \blacksquare$$

# Practice Problems

**Problem 48.1**
Evaluate

(a) $\int_0^3 (1+e^{-t})\delta(t-2)dt$.
(b) $\int_{-2}^1 (1+e^{-t})\delta(t-2)dt$.
(c) $\int_{-1}^2 \begin{bmatrix} \cos 2t \\ te^{-t} \end{bmatrix} \delta(t)dt$.
(d) $\int_{-1}^2 (e^{2t}+t) \begin{bmatrix} \delta(t+2) \\ \delta(t-1) \\ \delta(t-3) \end{bmatrix} dt$.

**Problem 48.2**
Let $f(t)$ be a function defined and continuous on $0 \le t < \infty$. Determine

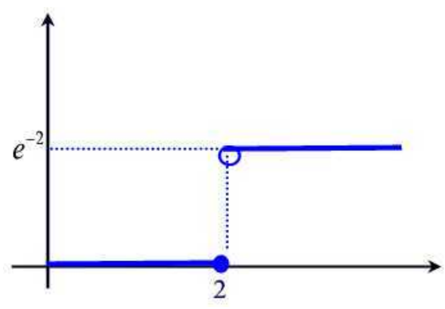
$$(f * \delta)(t) = \int_0^t f(t-s)\delta(s)ds.$$

**Problem 48.3**
Determine a value of the constant $t_0$ such that $\int_0^1 \sin^2[\pi(t-t_0)]\delta(t-\frac{1}{2})dt = \frac{3}{4}$.

**Problem 48.4**
If $\int_1^5 t^n \delta(t-2)dt = 8$, what is the exponent $n$?

**Problem 48.5**
Sketch the graph of the function $g(t)$ which is defined by $g(t) = \int_0^t \int_0^s \delta(u-1)duds,\ 0 \le t < \infty$.

**Problem 48.6**
The graph of the function $g(t) = \int_0^t e^{\alpha t}\delta(t-t_0)dt,\ 0 \le t < \infty$ is shown. Determine the constants $\alpha$ and $t_0$.



**Problem 48.7**
(a) Use the method of integarting factor to solve the initial value problem $y' - y = h(t),\ y(0) = 0$.
(b) Use the Laplace transform to solve the initial value problem $\phi' - \phi = \delta(t),\ \phi(0) = 0$.
(c) Evaluate the convolution $\phi * h(t)$ and compare the resulting function with the solution obtained in part(a).

**Problem 48.8**
Solve the initial value problem

$$y' + y = 2 + \delta(t-1), \quad y(0) = 0,\ 0 \le t \le 6.$$

Graph the solution on the indicated interval.

**Problem 48.9**
Solve the initial value problem

$$y'' = \delta(t-1) - \delta(t-3), \quad y(0) = 0,\ y'(0) = 0,\ 0 \le t \le 6.$$

Graph the solution on the indicated interval.

**Problem 48.10**
Solve the initial value problem

$$y'' - 2y' = \delta(t-1), \quad y(0) = 1,\ y'(0) = 0,\ 0 \le t \le 2.$$

Graph the solution on the indicated interval.

**Problem 48.11**
Solve the initial value problem

$$y'' + 2y' + y = \delta(t-2), \quad y(0) = 0,\ y'(0) = 1,\ 0 \le t \le 6.$$

Graph the solution on the indicated interval.

# 49 Solving Systems of Differential Equations Using Laplace Transform

In this section we extend the definition of Laplace transform to matrix-valued functions and apply this extension to solving systems of differential equations. Let $y_1(t), y_2(t), \cdots, y_n(t)$ be members of $\mathcal{PE}$. Consider the vector-valued function

$$\mathbf{y}(t) = \left[\begin{array}{c} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{array}\right]$$

The Laplace transform of $\mathbf{y}(t)$ is

$$\begin{array}{rcl} \mathcal{L}[\mathbf{y}(t)] & = & \int_0^\infty \mathbf{y}(t)e^{-st}dt \\ & = & \left[\begin{array}{c} \int_0^\infty y_1(t)e^{-st}dt \\ \int_0^\infty y_2(t)e^{-st}dt \\ \vdots \\ \int_0^\infty y_n(t)e^{-st}dt \end{array}\right] \\ & = & \left[\begin{array}{c} \mathcal{L}[y_1(t)] \\ \mathcal{L}[y_2(t)] \\ \vdots \\ \mathcal{L}[y_n(t)] \end{array}\right] \end{array}$$

In a similar way, we define the Laplace transform of an $m \times n$ matrix to be the $m \times n$ matrix consisting of the Laplace transforms of the component functions. If the Laplace transform of each component exists then we say $\mathbf{y}$(t) is **Laplace transformable.**

**Example 49.1**
Find the Laplace transform of the vector-valued function

$$\mathbf{y}(t) = \left[\begin{array}{c} t^2 \\ 1 \\ e^t \end{array}\right]$$

**Solution.**
The Laplace transform is

$$\mathcal{L}[\mathbf{y}(t)] = \begin{bmatrix} \frac{6}{s^3} \\ \\ \frac{1}{s} \\ \\ \frac{1}{s-1} \end{bmatrix}, \quad s > 1 \blacksquare$$

The linearity property of the Laplace transform can be used to establish the following result.

**Theorem 49.1**
If $\mathbf{A}$ is a constant $n \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix-valued function then

$$\mathcal{L}[\mathbf{AB}(t)] = \mathbf{A}\mathcal{L}[\mathbf{B}(t)].$$

**Proof.**
Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B}(t) = (b_{ij}(t))$. Then $\mathbf{AB}(t) = (\sum_{k=1}^n a_{ik}b_{kp})$. Hence,

$$\mathcal{L}[\mathbf{AB}(t)] = [\mathcal{L}(\sum_{k=1}^n a_{ik}b_{kp})] = [\sum_{k=1}^n a_{ik}\mathcal{L}(b_{kp})] = \mathbf{A}\mathcal{L}[\mathbf{B}(t)] \blacksquare$$

Theorem 42.3 can be extended to vector-valued functions.

**Theorem 49.2**
(a) Suppose that $\mathbf{y}(t)$ is continuous for $t \geq 0$ and let the components of the derivative vector $\mathbf{y}'$ be members of $\mathcal{PE}$. Then

$$\mathcal{L}[\mathbf{y}'(t)] = s\mathcal{L}[\mathbf{y}(t)] - \mathbf{y}(0).$$

(b) Let $\mathbf{y}'(t)$ be continuous for $t \geq 0$, and let the entries of $\mathbf{y}''(t)$ be members of $\mathcal{PE}$. Then

$$\mathcal{L}[\mathbf{y}''(t)] = s^2\mathcal{L}[\mathbf{y}(t)] - s\mathbf{y}(0) - \mathbf{y}'(0).$$

(c) Let the entries of $\mathbf{y}(t)$ be members of $\mathcal{PE}$. Then

$$\mathcal{L}\left\{\int_0^t \mathbf{y}(s)ds\right\} = \frac{\mathcal{L}[\mathbf{y}(t)]}{s}.$$

**Proof.**
(a) We have

$$\begin{aligned}
\mathcal{L}[\mathbf{y}'(t)] &= \begin{bmatrix} \mathcal{L}[y_1'(t)] \\ \mathcal{L}[y_2'(t)] \\ \vdots \\ \mathcal{L}[y_n'(t)] \end{bmatrix} \\
&= \begin{bmatrix} s\mathcal{L}[y_1(t)] - y_1(0) \\ s\mathcal{L}[y_2(t)] - y_2(0) \\ \vdots \\ s\mathcal{L}[y_n(t)] - y_n(0) \end{bmatrix} \\
&= s\mathcal{L}[\mathbf{y}(t)] - \mathbf{y}(0)
\end{aligned}$$

(b) We have

$$\begin{aligned}
\mathcal{L}[\mathbf{y}''(t)] &= s\mathcal{L}[\mathbf{y}'(t)] - \mathbf{y}'(0) \\
&= s(s\mathcal{L}[\mathbf{y}(t)] - \mathbf{y}(0)) - \mathbf{y}'(0) \\
&= s^2\mathcal{L}[\mathbf{y}(t)] - s\mathbf{y}(0) - \mathbf{y}'(0)
\end{aligned}$$

(c) We have

$$\mathcal{L}[\mathbf{y}(t)] = s\mathcal{L}\left\{\int_0^t \mathbf{y}(s)ds\right\}$$

so that

$$\mathcal{L}\left\{\int_0^t \mathbf{y}(s)ds\right\} = \frac{\mathcal{L}[\mathbf{y}(t)]}{s} \blacksquare$$

The above two theorems can be used for solving the following initial value problem

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y} + \mathbf{g}(t), \quad \mathbf{y}(0) = \mathbf{y}_0, \quad t > 0 \tag{35}$$

where $\mathbf{A}$ is a constant matrix and the components of $\mathbf{g}(t)$ are members of $\mathcal{PE}$.
Using the above theorems we can write

$$s\mathbf{Y}(s) - \mathbf{y}_0 = \mathbf{A}\mathbf{Y}(s) + \mathbf{G}(s)$$

or

$$(s\mathbf{I} - \mathbf{A})\mathbf{Y}(s) = \mathbf{y}_0 + \mathbf{G}(s)$$

where $\mathcal{L}[\mathbf{g}(t)] = \mathbf{G}(s)$. If $s$ is not an eigenvalue of $\mathbf{A}$ then the matrix $s\mathbf{I} - \mathbf{A}$ is invertible and in this case we have

$$\mathbf{Y}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{y}_0 + \mathbf{G}(s)]. \tag{36}$$

To compute $\mathbf{y}(t) = \mathcal{L}^{-1}[\mathbf{Y}(s)]$ we compute the inverse Laplace transform of each component of $\mathbf{Y}(s)$. We illustrate the above discussion in the next example.

**Example 49.2**
Solve the initial value problem

$$\mathbf{y}' = \left[\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}\right]\mathbf{y} + \left[\begin{array}{c} e^{2t} \\ -2t \end{array}\right], \quad \mathbf{y}(0) = \left[\begin{array}{c} 1 \\ -2 \end{array}\right]$$

**Solution.**
We have

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s-3)}\left[\begin{array}{cc} s-1 & 2 \\ 2 & s-1 \end{array}\right]$$

and

$$\mathbf{G}(s) = \left[\begin{array}{c} \frac{1}{s-2} \\ -\frac{2}{s^2} \end{array}\right].$$

Thus,

$$\begin{aligned} \mathbf{Y}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{y}_0 + \mathbf{G}(s)] \\ &= \frac{1}{(s+1)(s-3)}\left[\begin{array}{cc} s-1 & 2 \\ 2 & s-1 \end{array}\right]\left[\begin{array}{c} 1 + \frac{1}{s-2} \\ -2 - \frac{2}{s^2} \end{array}\right] \\ &= \left[\begin{array}{c} \frac{s^4-6s^3+9s^2-4s+8}{s^2(s+1)(s-2)(s-3)} \\ \frac{-2s^4+8s^3-8s^2+6s-4}{s^2(s+1)(s-2)(s-3)} \end{array}\right] \end{aligned}$$

Using the method of partial fractions we can write

$$\begin{aligned} Y_1(s) &= \frac{4}{3}\frac{1}{s^2} - \frac{8}{9}\frac{1}{s} + \frac{7}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s-2} - \frac{1}{9}\frac{1}{s-3} \\ Y_2(s) &= -\frac{2}{3}\frac{1}{s^2} + \frac{10}{9}\frac{1}{s} - \frac{7}{3}\frac{1}{s+1} - \frac{2}{3}\frac{1}{s-2} - \frac{1}{9}\frac{1}{s-3} \end{aligned}$$

Therefore

$$\begin{aligned} y_1(t) &= \mathcal{L}^{-1}[Y_1(s)] = \frac{4}{3}t - \frac{8}{9} + \frac{7}{3}e^{-t} - \frac{1}{3}e^{2t} - \frac{1}{9}e^{3t} \\ y_2(t) &= \mathcal{L}^{-1}[Y_2(s)] = -\frac{2}{3}t + \frac{10}{9} - \frac{7}{3}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{9}e^{3t}, \quad t \geq 0 \end{aligned}$$

Hence, for $t \geq 0$

$$\mathbf{y}(t) = t\begin{bmatrix} \frac{4}{3} \\ -\frac{2}{3} \end{bmatrix} + \begin{bmatrix} -\frac{8}{9} \\ \frac{10}{9} \end{bmatrix} + e^{-t}\begin{bmatrix} \frac{7}{3} \\ \frac{7}{3} \end{bmatrix} + e^{2t}\begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} + e^{3t}\begin{bmatrix} -\frac{1}{9} \\ -\frac{1}{9} \end{bmatrix} \blacksquare$$

**System Transfer Matrix and the Laplace Transform of $e^{t\mathbf{A}}$**

The vector equation (35) is a linear time invariant system whose Laplace input is given by $\mathbf{y}_0 + G(s)$ and the Laplace output $\mathbf{Y}(s)$. According to (36) the system tranform matrix is given by $(s\mathbf{I} - \mathbf{A})^{-1}$. We will show that this matrix is the Laplace transform of the exponential matrix function $e^{t\mathbf{A}}$. Indeed, $e^{t\mathbf{A}}$ is the solution to the initial value problem

$$\mathbf{\Phi}'(t) = \mathbf{A}\mathbf{\Phi}(t), \quad \mathbf{\Phi}(0) = \mathbf{I},$$

where $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{A}$ is a constant $n \times n$ matrix. Taking Laplace of both sides yields

$$s\mathcal{L}[\Phi(t)] - \mathbf{I} = \mathbf{A}\mathcal{L}[\Phi(t)].$$

Solving for $\mathcal{L}[\Phi(t)]$ we find

$$\mathcal{L}[\Phi(t)] = (s\mathbf{I} - \mathbf{A})^{-1} = \mathcal{L}[e^{t\mathbf{A}}].$$

## Practice Problems

**Problem 49.1**
Find $\mathcal{L}[\mathbf{y}(t)]$ where

$$\mathbf{y}(t) = \frac{d}{dt}\left[\begin{array}{c} e^{-t}\cos 2t \\ 0 \\ t + e^{t} \end{array}\right]$$

**Problem 49.2**
Find $\mathcal{L}[\mathbf{y}(t)]$ where

$$\mathbf{y}(t) = \int_0^t \left[\begin{array}{c} 1 \\ u \\ e^{-u} \end{array}\right] du$$

**Problem 49.3**
Find $\mathcal{L}^{-1}[\mathbf{Y}(s)]$ where

$$\mathbf{Y}(s) = \left[\begin{array}{c} \frac{1}{s} \\ \frac{2}{s^2+2s+2} \\ \frac{1}{s^2+s} \end{array}\right]$$

**Problem 49.4**
Find $\mathcal{L}^{-1}[\mathbf{Y}(s)]$ where

$$\mathbf{Y}(s) = \left[\begin{array}{ccc} 1 & -1 & 2 \\ 2 & 0 & 3 \\ 1 & -2 & 1 \end{array}\right]\left[\begin{array}{c} \mathcal{L}[t^3] \\ \mathcal{L}[e^{2t}] \\ \mathcal{L}[\sin t] \end{array}\right]$$

**Problem 49.5**
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}' = \left[\begin{array}{cc} 5 & -4 \\ 5 & -4 \end{array}\right]\mathbf{y} + \left[\begin{array}{c} 0 \\ 1 \end{array}\right], \quad \mathbf{y}(0) = \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

**Problem 49.6**
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}' = \left[\begin{array}{cc} 5 & -4 \\ 3 & -2 \end{array}\right]\mathbf{y}, \quad \mathbf{y}(0) = \left[\begin{array}{c} 3 \\ 2 \end{array}\right]$$

**Problem 49.7**
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ 3e^t \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

**Problem 49.8**
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}'' = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{y}'(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**Problem 49.9**
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}'' = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{y}'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

**Problem 49.10**
Use the Laplace transform to solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^t \\ 1 \\ -2t \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

**Problem 49.11**
The Laplace transform was applied to the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y}, \ \mathbf{y}(0) = \mathbf{y}_0$, where $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, $\mathbf{A}$ is a $2 \times 2$ constant matrix, and $\mathbf{y}_0 = \begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix}$.
The following transform domain solution was obtained

$$\mathcal{L}[\mathbf{y}(t)] = \mathbf{Y}(s) = \frac{1}{s^2 - 9s + 18} \begin{bmatrix} s-2 & -1 \\ 4 & s-7 \end{bmatrix} \begin{bmatrix} y_{1,0} \\ y_{2,0} \end{bmatrix}.$$

(a) what are the eigenvalues of $\mathbf{A}$?
(b) Find $\mathbf{A}$.

# 50 Numerical Methods for Solving First Order Linear Systems: Euler's Method

Whenever a mathematical problem is encountered in science or engineering, which cannot readily or rapidly be solved by a traditional mathematical method, then a numerical method is usually sought and carried out. In this section, we study Euler's method for approximating the solution to the initial value problem

$$\mathbf{y}'(t) = P(t)\mathbf{y}(t) + \mathbf{g}(t), \quad \mathbf{y}(a) = \mathbf{y}_0, \quad a \le t \le b$$

where $P(t)$ is an $n \times n$ matrix.

**Euler's Method for First Order Scalar Differential Equation**
We first develop Euler's method for the scalar equation

$$y'(t) = f(t, y), \quad y(t_0) = y_0, \quad a \le t \le b. \tag{37}$$

Divide the interval $a \le t \le b$ to $N$ equal subintervals each of length

$$h = \frac{b-a}{N}$$

using the gride points

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b.$$

Note that for $0 \le i \le N$ we have

$$t_i = a + ih \text{ and } t_{i+1} = t_i + h, \quad 0 \le i \le N-1.$$

The (unique) exact solution $y(t)$ to Equation(37) is differentiable so that we can write

$$y'(t) = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}.$$

This says that for small $h$ we can estimate $y'(t)$ by the difference quotient

$$\frac{y(t+h) - y(t)}{h} \approx y'(t) = f(t, y).$$

Evaluating the above approximation at the gride points $t_0, t_1, \cdots, t_{N-1}$ we can write

$$y(t_k + h) \approx y(t_k) + hf(t_k, y(t_k)), \quad 0 \le k \le N-1.$$

If we let $y_k$ denote the approximation of $y(t_k)$ then the previous equation becomes

$$y_{k+1} = y_k + hf(t_k, y_k), \quad y(t_0) = y_0. \tag{38}$$

Equation (38) is known as **Euler's method.** We illustrate Euler's method in the next example.

**Example 50.1**
Suppose that $y(0) = 1$ and $\frac{dy}{dt} = y$. Estimate $y(0.5)$ in 5 steps using Euler's method.

**Solution.**
The step size is $h = \frac{0.5-0}{5} = 0.1$. The following chart lists the steps needed:

| $k$ | $t_k$ | $y_k$ | $f(t_k, y_k)h$ |
|---|---|---|---|
| 0 | 0 | 1 | 0.1 |
| 1 | 0.1 | 1.1 | 0.11 |
| 2 | 0.2 | 1.21 | 0.121 |
| 3 | 0.3 | 1.331 | 0.1331 |
| 4 | 0.4 | 1.4641 | 0.14641 |
| 5 | 0.5 | 1.61051 | |

Thus, $y(0.5) \approx 1.61051$. Note that the exact value is $y(0.5) = e^{0.5} \approx 1.6487213$ ■

**Remark 50.1**
1. Euler's method approximates the value of the solution at a given point; it does not give an explicit formula of the solution.

2. It can be shown that the error in Euler's method is proportional to $\frac{1}{N}$. Thus, doubling the number of mesh points will decrease the error by $\frac{1}{2}$.

**Euler's Method for First Order Linear Systems** Next, we want to extend Euler's method to the intial value problem

$$\mathbf{y}'(t) = P(t)\mathbf{y}(t) + \mathbf{g}(t), \quad \mathbf{y}(a) = \mathbf{y}_0, \quad a \le t \le b. \tag{39}$$

Let the exact solution be

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

Then $\mathbf{y}(t)$ is differentiable with derivative

$$\mathbf{y}'(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} \lim_{h\to 0} \frac{y_1(t+h)-y_1(t)}{h} \\ \lim_{h\to 0} \frac{y_2(t+h)-y_2(t)}{h} \\ \vdots \\ \lim_{h\to 0} \frac{y_n(t+h)-y_n(t)}{h} \end{bmatrix}$$

$$= \lim_{h\to 0} \frac{1}{h} \begin{bmatrix} y_1(t+h) - y_1(t) \\ y_2(t+h) - y_2(t) \\ \vdots \\ y_n(t+h) - y_n(t) \end{bmatrix}$$

$$= \lim_{h\to 0} \frac{1}{h}[\mathbf{y}(t+h) - \mathbf{y}(t)]$$

Thus, for small $h$ we can estimate $\mathbf{y}'$ with the difference quotient

$$\lim_{h\to 0} \frac{1}{h}[\mathbf{y}(t+h) - \mathbf{y}(t)] \approx \mathbf{y}'(t) = P(t)\mathbf{y}(t) + \mathbf{g}(t).$$

Evaluating the above approximation at the gride points $t_0, t_1, \cdots, t_{N-1}$ we can write

$$\mathbf{y}(t_k + h) \approx \mathbf{y}(t_k) + h[P(t_k)\mathbf{y}(t_k) + \mathbf{g}(t_k)],\ 0 \le k \le N-1.$$

Letting $\mathbf{y}_k$ be an approximation of $\mathbf{y}(t_k)$, we define

$$\mathbf{y}_{k+1} = \mathbf{y}_k + h[P(t_k)\mathbf{y}_k + \mathbf{g}(t_k)],\ 0 \le k \le N-1. \tag{40}$$

Iteration (40) is the **Euler's method** for the initial value problem (39).

**Example 50.2**
Consider the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{y}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\quad 0 \le t \le 1.$$

(a) Write the Euler's method algorithm in explicit form. Specify the starting values $t_0$ and $\mathbf{y}_0$.
(b) Give a formula for the $k$th $t-$value, $t_k$. What is the range of the index $k$ if we choose $h = 0.01$?

**Solution.**
(a) The Euler's iterations are given by the formula

$$\mathbf{y}_{k+1} = \mathbf{y}_k + h[\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\mathbf{y}_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We have $t_0 = 0$ and

$$\mathbf{y}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(b) Since $a = t_0 = 0$, we have $t_k = kh,\ 0 \le k \le N-1$. In the case $h = 0.01$ and $b = 1$ we find $0.01 = \frac{1-0}{N}$ which implies that $N = 100$. So the range of the index $k$ is $k = 0, 1, 2, \cdots, 100$ ■

**Solving Variable-Coefficient Scalar Equations**
We conclude this section by using the Euler's method developped for first order differential equations to scalar differential equations of any order. We will illustrate the process by considering the following second order initial value problem

$$y'' + p(t)y' + q(t)y = g(t),\ y(t_0) = y_0, +y'(t_0) = y'_0,\ a < t < b$$

where $p(t), q(t)$, and $g(t)$ are continuous in the interval $a < t < b$ and $a < t_0 < b$.
The above equation can be recast as a first order linear system by using the substitution

$$z_1(t) = y(t),\quad z_2(t) = y'(t) \text{ and } \mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

Indeed, since

$$\begin{aligned} z'_1 =& y' = z_2 \\ z'_2 =& y'' = -p(t)y' - q(t)y + g(t) = -q(t)z_1 - p(t)z_2 + g(t) \end{aligned}$$

we can write this as the system

$$\mathbf{z}' = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix},\ \mathbf{z}(t_0) = \begin{bmatrix} y_0 \\ y'_0 \end{bmatrix}.$$

This is a first order linear system that can be solved using Euler's method.

**Example 50.3**
Consider the scalar initial value problem

$$y'' + y = t^{\frac{3}{2}}, \quad y(0) = 1, y'(0) = 0.$$

(a) Rewrite the given initial value problem as an equivalent initial value problem for a first order system.
(b) Write the Euler's method algorithm $\mathbf{z}_{k+1} = \mathbf{z}_k + h[P(t_k)\mathbf{z}_k + \mathbf{g}(t_k)]$, in explicit form. Specify the starting values $t_0$ and $\mathbf{z}_0$.
(c) Using a calculator with step size $h = 0.01$, carry out two steps of Euler's method, finding $\mathbf{z}_1$ and $\mathbf{z}_2$ What are the corresponding numerical approximations to the solution $\mathbf{y}(t)$ at times $t = 0.01$ and $t = 0.02$?

**Solution.**
(a) Let $z_1 = y$ and $z_2 = y'$. Then $z_1' = y' = z_2$ and $z_2' = y'' = -y + t^{\frac{3}{2}} = -z_1 + t^{\frac{3}{2}}$. This leads to the following intial value problem of a first order system

$$\mathbf{z}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ t^{\frac{3}{2}} \end{bmatrix}, \quad \mathbf{z}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(b) The Euler's method algorithm is

$$\mathbf{z}_{k+1} = \mathbf{z}_k + h\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{z}_k + \begin{bmatrix} 0 \\ t_k^{\frac{3}{2}} \end{bmatrix} \right\}, \quad \mathbf{z}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus, $t_0 = 0$ and

$$\mathbf{z}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(c) We have

$$\mathbf{z}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.01 \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} 1 \\ -0.01 \end{bmatrix}$$

and

$$\mathbf{z}_2 = \begin{bmatrix} 1 \\ -0.01 \end{bmatrix} + 0.01 \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -0.01 \end{bmatrix} + \begin{bmatrix} 0 \\ (0.01)^{\frac{3}{2}} \end{bmatrix} \right\} = \begin{bmatrix} 0.\overline{9} \\ -0.01\overline{9} \end{bmatrix}$$

Finally, $y(0.01) = z_1(0.01) = 1$ and $y(0.02) = z_1(0.02) = 0.\overline{9}$ ■

## Practice Problems

In Problems 50.1 - 50.3 answer the following questions:
(a) Solve the differential equation analytically using the appropriate method of solution.
(b) Write the Euler's iterates: $y_{k+1} = y_k + hf(t_k, y_k)$.
(c) Using step size $h = 0.1$, compute the Euler approximations $y_k,\ k = 1, 2, 3$ at times $t_k = a + kh$.
(d) For $k = 1, 2, 3$ compute the error $y(t_k) - y_k$ where $y(t_k)$ is the exact value of $y$ at $t_k$.

**Problem 50.1**

$$y' = 2t - 1, \quad y(1) = 0.$$

**Problem 50.2**

$$y' = -ty, \quad y(0) = 1.$$

**Problem 50.3**

$$y' = y^2, \quad y(0) = 1.$$

In Problems 50.4 - 50.6 answer the following questions:
(a) Write the Euler's method algorithm in explicit form. Specify the starting values $t_0$ and $\mathbf{y}_0$.
(b) Give a formula for the $k$th $t-$value, $t_k$. What is the range of the index $k$ if we choose $h = 0.01$?
(c) Use a calculator to carry out two steps of Euler's method, finding $\mathbf{y}_1$ and $\mathbf{y}_2$.

**Problem 50.4**

$$\mathbf{y}' = \left[\begin{array}{cc} -t^2 & t \\ 2-t & 0 \end{array}\right]\mathbf{y} + \left[\begin{array}{c} 1 \\ t \end{array}\right], \ \mathbf{y}(1) = \left[\begin{array}{c} 2 \\ 0 \end{array}\right], \quad 1 \le t \le 4.$$

**Problem 50.5**

$$\mathbf{y}' = \left[\begin{array}{ccc} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 0 \end{array}\right]\mathbf{y} + \left[\begin{array}{c} 0 \\ 2 \\ t \end{array}\right], \ \mathbf{y}(-1) = \left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right], \quad -1 \le t \le 0.$$

**Problem 50.6**

$$\mathbf{y}' = \begin{bmatrix} \frac{1}{t} & \sin t \\ 1-t & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ t^2 \end{bmatrix}, \ \mathbf{y}(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 1 \le t \le 6.$$

In Problems 50.7 - 50.8 answer the following questions.
(a) Rewrite the given initial value problem as an equivalent initial value problem for a first order system, using the substitution $z_1 = y, z_2 = y', z_3 = y'', \cdots$ .
(b) Write the Euler's method algorithm $\mathbf{z}_{k+1} = \mathbf{z}_k + h[P(t_k)\mathbf{z}_k + \mathbf{g}(t_k)]$, in explicit form. Specify the starting values $t_0$ and $\mathbf{z}_0$.
(c) Using a calculator with step size $h = 0.01$, carry out two steps of Euler's method, finding $\mathbf{z}_1$ and $\mathbf{z}_2$ What are the corresponding numerical approximations to the solution $\mathbf{y}(t)$ at times $t = 0.01$ and $t = 0.02$?

**Problem 50.7**

$$y'' + y' + t^2 y = 2, \quad y(1) = 1, \quad y'(1) = 1.$$

**Problem 50.8**

$$y''' + 2y' + ty = t + 1, \quad y(0) = 1, \ y'(0) = -1, \ y''(0) = 0.$$