Arkansas Tech University
MATH 4033: Elementary Modern Algebra
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1 The Concept of a Mapping

The concept of a mapping (aka function) is important throughout mathematics. We have been dealing with functions for a long time. You recall from calculus that a function is a rule which assigns with each real number in the domain a unique real number in the codomain. So both the domain and codomain are subsets of the real numbers system. In this section, we would like to define functions on domains and codomains other than the set of real numbers.

Definition 1.1
If S and T are nonempty sets, then a **mapping** (or a function) from S into T is a rule which assigns to each member of S a unique member in T. We call S the **domain** and T the **codomain** of the mapping.

In what follows, we will use the terms function and mapping interchangeably. If α is a function from S to T we shall adopt the notation:

$$\alpha : S \longrightarrow T.$$

Note that every member x in the domain S is associated to a unique member y of the codomain T. In function notation, we write $y = \alpha(x)$. However, not every element in the codomain need be associated to an element in the domain.

Example 1.1
If α is a mapping from S to S and A is a subset of S then the rule $\iota_A(x) = x$ defines a mapping from A into A. We call ι_A the **identity mapping** on A.■

Example 1.2
Assume that S and T are finite sets containing m and n elements, respectively. How many mappings are there from S to T?

Solution.
The problem of finding the number of mappings from S to T is the same as that of computing the number of different ways each element of S can be assigned an image in T. For the first element, there are m possibilities, for the second element there are also m possibilitites, etc, for the nth element there are m possibilities. By the Principle of Counting, there are m^n mappings from S to T.■

Remark 1.1
In the notation $y = \alpha(x)$, x is sometimes referred to as the **preimage** of y with respect to α. ■

Sometimes it is necessary to identify the elements of T which can be associated to some elements in the domain S. This subset of the codomain is called the **image** or **range** of α (denoted $\alpha(S)$).

Definition 1.2
If $\alpha : S \longrightarrow T$ is a mapping and A is a subset of S then the set of all images of the members of A will be denoted by $\alpha(A)$.(See Figure 1.1). In set-builder notation

$$\alpha(A) = \{\alpha(x) : x \in A\}.$$

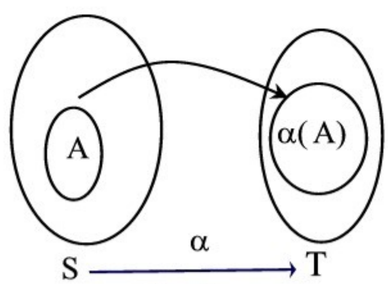

Figure 1.1

Example 1.3
The rules α and β defined from the set $S = \{x, y, z\}$ to $T = \{1, 2, 3\}$ represent mappings with $\alpha(S) = T$ and $\beta(S) = \{1, 3\}$, respectively.(See Figure 1.2) On the other hand, the rule γ does not define a mapping for two reasons: first, the member y has no image, and second, the member x is associated to two members 1 and 3 of T.■

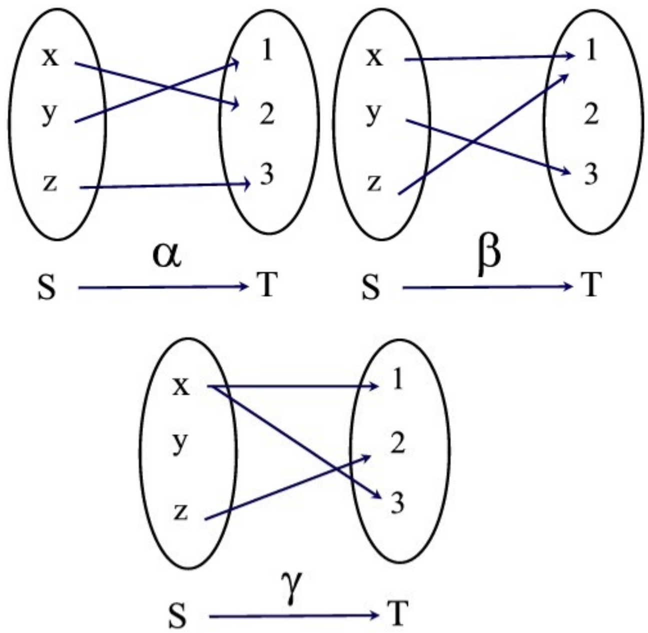

Figure 1.2

Example 1.4
With α and β as in Example 1.3

$$\alpha(\{x,z\}) = \{2,3\} \ \ and \ \ \beta(\{x,z\}) = \{1\}. \blacksquare$$

The first operation of mappings that we consider is the equality of two mappings.

Example 1.5 *(Equality of two mappings)*
We say that two mappings α and β from S into T are *equal* if and only if $\alpha(x) = \beta(x)$ for all $x \in S$, i.e. the range of α is equal to the range of β. We write $\alpha = \beta$. When two functions are not equal we write $\alpha \neq \beta$. This occurs, when there is a member in the common domain such that $\alpha(x) \neq \beta(x)$. For example, the mappings α and β of Example 1.3 are different since $\alpha(y) \neq \beta(y)$. On the other hand, the mappings $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ and $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\alpha(x) = (x+1)^2$ and $\beta(x) = x^2 + 2x + 1$ are equal.$\blacksquare$

Next, we introduce a family of mappings with the property that the range is the whole codomain.

Definition 1.3
A mapping $\alpha : S \longrightarrow T$ is called **onto** (or **surjective**) if and only if $\alpha(S) = T$. That is, if and only if for every member in the codomain there is a member in the domain associated to it. Using quantifiers, α is onto if and only if $\forall y \in T$, $\exists x \in S$ such that $\alpha(x) = y$.

Example 1.6
In terms of a Venn diagram, a mapping is onto if and only if each element in the codomain has at least one arrow pointed to it. Thus, since $\alpha(S) = T$ in Example 1.3 then α is onto whereas β is not since $\beta(S) \neq T$. ■

Example 1.7
The functions $f(x) = x^2$ and $g(x) = \sin x$ are not onto as functions from $\mathbb{R}$ to $\mathbb{R}$. However, if the codomain is restricted to $\mathbb{R}^+$ then f is onto. Also, if the codomain of g is restricted to $[-1, 1]$ then g is onto.■

Example 1.8
Let $\alpha : S \to T$ be a mapping between finite sets such that the number of elements of S is less than that of T. Can α be onto? Explain.

Solution.
If T has more elements than S and since each element of S is associated to exactly one element in T then some elements in T has no preimages in S. Thus, α can not be onto.■

Next, you recall from calculus that a function α from $\mathbb{R}$ to $\mathbb{R}$ is one-to-one if and only if its graph satisfies the horizontal line test, i.e. every horizontal line crosses the graph of α at most once. That is, no two different members of the domain of α share the same member in the range. This concept can be generalized to any sets.

Definition 1.4
A mapping $\alpha : S \longrightarrow T$ is called **one-to-one** (or **injective**) if and only if for any $x_1, x_2 \in S$

$$x_1 \neq x_2 \ \ \textit{implies} \ \ \alpha(x_1) \neq \alpha(x_2)$$

that is, unequal elements in the domain of α have unequal images in the range.

Example 1.9
In terms of a Venn diagram, a mapping is one-to-one if and only if no two arrows point to a same member in the codomain. The function α in Example 1.3 is one-to-one whereas β is not since $x \neq z$ but $\beta(x) = \beta(z)$. ■

Example 1.10
Show that the functions $\alpha(x) = x^2$ and $\beta(x) = \sin x$ defined on the set $\mathbb{R}$ are not one-to-one functions. Modify the domain of each so that they become one-to-one functions.

Solution.
The domain of α is the set of all real numbers. Since $\alpha(-1) = \alpha(1) = 1$ then α is not one-to-one on its domain. Similarly, β is defined for all real numbers. Since $\beta(\frac{\pi}{2}) = \beta(\frac{5\pi}{2}) = 1$ then β is not one-to-one. These functions become one-to-one if α is restricted to either the interval $[0, \infty)$ or the interval $(-\infty, 0]$ whereas β can be restricted to intervals of the form $\left[(2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right]$, where k is an integer.■

Example 1.11
Let $\alpha : S \to T$ be a mapping between two finite sets such that S has more elements than its range. Can α be one-to-one? Explain.

Solution.
If S has more elements than $\alpha(S)$ then there must exist at least two distinct members of S with the same image in $\alpha(S)$. Thus, by Definition 1.4, α can not be one-to-one.■

An equivalent statement to

$$x_1 \neq x_2 \ \ implies \ \ \alpha(x_1) \neq \alpha(x_2) \ \ (x_1, x_2 \in S)$$

is its contrapositive, i.e. the statement

$$\alpha(x_1) = \alpha(x_2) \ \ implies \ \ x_1 = x_2.$$

This latter condition is usually much easier to work with than the one given in the definition of one-to-one as shown in the next example.

Example 1.12
The mapping $\alpha(x) = 2x - 1$ defined on $\mathbb{R}$ is a one-to-one function. To see this, suppose that $x_1 = x_2$. Then multiplying both sides of this equality by 2 to obtain $2x_1 = 2x_2$. Finally, subtract 1 from both sides to obtain $2x_1 - 1 = 2x_2 - 1$. That is, $\alpha(x_1) = \alpha(x_2)$. Hence, α is one-to-one. ■

A mapping can be one-to-one but not onto, onto but not one-one, neither one-to-one nor onto, or both one-to-one and onto. See Example 1.13. We single out the last case in the next definition.

Definition 1.5
A mapping $\alpha : S \longrightarrow T$ is called a **one-to-one correspondence** (or **bijective**) if and only if α is both one-to-one and onto.

Example 1.13
(a) The mapping $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\alpha(x) = x^3$ is a one-to-one correspondence. To see this, for any $y \in \mathbb{R}$ we can find an $x \in \mathbb{R}$ such that $\alpha(x) = y$. Indeed, let $x = \sqrt[3]{y}$. Hence, α is onto. Now, if $x_1^3 = x_2^3$ then taking the cube root of both sides to obtain $x_1 = x_2$. That is, α is one-to-one.
(b) The function $\beta : \mathbb{N} \longrightarrow \mathbb{N}$ defined by $\beta(n) = 2n$ is one-to-one but not onto. Indeed, $\beta(n_1) = \beta(n_2)$ implies $n_1 = n_2$ so that β is one-to-one. The fact that β is not onto follows from the fact that no arrow is pointed to the numbers 1, 3, 5, etc.
(c) The function $\gamma : \mathbb{N} \longrightarrow \mathbb{N}$ defined by

$$\beta(n) = \begin{cases} \frac{n+1}{2} & \text{if n is odd} \\ \frac{n}{2} & \text{if n is even} \end{cases}$$

is onto. To see this, let n be a positive integer. If n is odd then $m = 2n - 1 \in \mathbb{N}$ is also odd. Moreover, $\beta(m) = \frac{m+1}{2} = n$. Now, if n is even then $m = 2n \in \mathbb{N}$ is also even and $\beta m = \frac{m}{2} = n$. β is not one-to-one since $\beta 3 = \beta 4 = 2$.
(d) The Ceiling function $\alpha(x) = \lceil x \rceil$ is the piecewise defined function given by

$$\lceil x \rceil = \text{smallest integer greater than or equal to x.}$$

α is neither one-to-one nor onto as seen in Figure 1.3.■

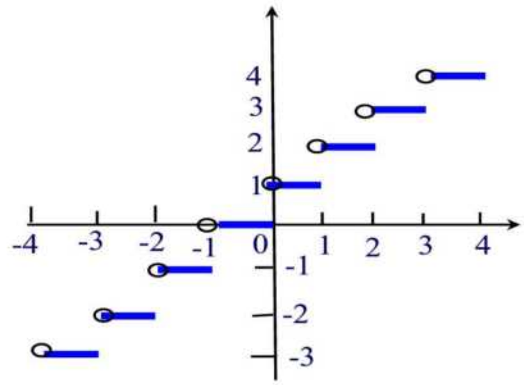

Figure 1.3