Solutions to Section 1

Exercise 1.1
Show that $|a| \geq a$ and $|a| \geq -a$.

Solution.
This follows from the fact that $\max\{-a, a\} \geq a$ and $\max\{-a, a\} \geq -a$ ■

Exercise 1.2
Show that

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

That is, the absolute value function is a piecewise defined function. Graph this function in the rectangular coordinate system.

Solution.
If $a \geq 0$ then $-a \leq 0$ so that $|a| = \max\{-a, a\} = a$. If $a < 0$ then $-a > 0$ so that $|a| = \max\{-a, a\} = -a$. The graph is shown in Figure 1. ■

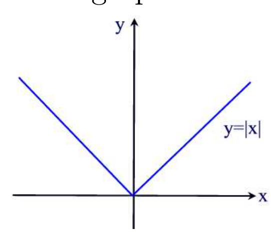


Figure 1

Exercise 1.3
Show that $|a| \geq 0$ with $|a| = 0$ if and only if $a = 0$.

Solution.
We see from the graph of $|a|$ that $|a| \geq 0$ and $|a| = 0$ if and only if $a = 0$ ■

Exercise 1.4
Show that if $|a| = |b|$ then $a = \pm b$.

Solution.
Suppose first that $a \geq 0$. Then $|a| = a$.
• If $b \geq 0$ then $|b| = b$. In this case, $|a| = |b|$ implies that $a = b$.

• If $b < 0$ then $|b| = -b$. In this case, $|a| = |b|$ implies that $a = -b$.
Suppose now that $a < 0$. Then $|a| = -a$.
• If $b \geq 0$ then $|b| = b$. In this case, $|a| = |b|$ implies that $-a = b$ which is the same as $a = -b$.
• If $b < 0$ then $|b| = -b$. In this case, $|a| = |b|$ implies that $-a = -b$ which is equivalent to $a = b$ ■

Exercise 1.5
Solve the equation $|3x - 2| = |5x + 4|$.

Solution.
We have either $3x - 2 = 5x + 4$ or $3x - 2 = -(5x + 4)$. Solving the first equation we find $x = -3$. Solving the second equation, we find $x = -\frac{1}{4}$ ■

Exercise 1.6
Show that $|-a| = |a|$.

Solution.
If $a \geq 0$ then $-a < 0$. Thus, $|a| = a$ and $|-a| = -(-a) = a$. Hence, $|-a| = |a|$. If $a < 0$ then $-a > 0$. In this case, $|a| = -a$ and $|-a| = -a$. That is, $|-a| = |a|$ ■

Exercise 1.7
Show that $|ab| = |a| \cdot |b|$.

Solution.
Suppose first that $a \geq 0$. Then $|a| = a$.
• If $b \geq 0$ then $|b| = b$. Moreover, $ab \geq 0$. In this case, $|ab| = ab = |a| \cdot |b|$.
• If $b < 0$ then $|b| = -b$. Moreover, $ab \leq 0$. In this case, $|ab| = -ab = a(-b) = |a| \cdot |b|$.
Suppose now that $a < 0$. Then $|a| = -a$.
• If $b \geq 0$ then $|b| = b$. Moreover, $ab \leq 0$. In this case, $|ab| = -ab = (-a)b = |a| \cdot |b|$.
• If $b < 0$ then $|b| = -b$. Moreover, $ab > 0$. In this case, $|ab| = ab = |a| \cdot |b|$ ■

Exercise 1.8
Show that $\left|\frac{1}{a}\right| = \frac{1}{|a|}$, where $a \neq 0$.

Solution.
If $a > 0$ then $\frac{1}{a} > 0$. Thus, $\left|\frac{1}{a}\right| = \frac{1}{a} = \frac{1}{|a|}$. If $a < 0$ then $\frac{1}{a} < 0$. Thus, $\left|\frac{1}{a}\right| = -\frac{1}{a} = \frac{1}{-a} = \frac{1}{|a|}$ ■

Exercise 1.9
Show that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ where $b \neq 0$.

Solution.
Using both Exercise 1.7 and Exercise 1.8 we can write the following $\left|\frac{a}{b}\right| = \left|a \cdot \frac{1}{b}\right| = |a| \cdot \left|\frac{1}{b}\right| = |a| \cdot \frac{1}{|b|} = \frac{|a|}{|b|}$ ■

Exercise 1.10
Show that for any two real numbers a and b we have $ab \leq |a| \cdot |b|$.

Solution.
From Exercise 1.1, we have $ab \leq |ab| = |a| \cdot |b|$, where we used Exercise 1.7 ■

Exercise 1.11
Recall that a number $b \geq 0$ is the **square root** of a number a, written $\sqrt{a} = b$, if and only if $a = b^2$. Show that

$$\sqrt{a^2} = |a|.$$

Solution.
Since $(\pm a)^2 = a^2$ we can write $\sqrt{a^2} = -a$ if $a < 0$ or $\sqrt{a^2} = a$ if $a \geq 0$. But this is equivalent to writing $\sqrt{a^2} = |a|$ by Exercise 1.2 ■

Exercise 1.12
suppose that A and B are points on a coordinate line that have coordinates a and b, respectively. Show that $|a - b|$ is the distance between the points A and B. Thus, if $b = 0$, $|a|$ measures the distance from the number a to the origin.

Solution.
Let d be the distance between A and B. If $a < b$ (See Figure 2(a)) then $d = b - a$ and $|a - b| = -(a - b) = b - a = d$. If $a > b$ (See Figure 2(b)) then $d = a - b$ and $|a - b| = a - b = d$ ■

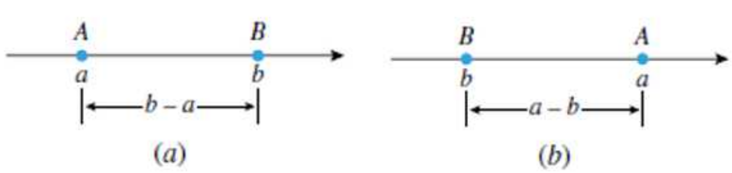


Figure 2

Exercise 1.13
Graph the portion of the real line given by the inequality
(a) $|x - a| < \delta$
(b) $0 < |x - a| < \delta$
where $\delta > 0$. Represent each graph in interval notation.

Solution.
(a) The graph is given in Figure 3(a). In interval notation, we have $(a - \delta, a + \delta)$.
(b) The graph is given in Figure 3(b). In interval notation, we have $(a - \delta, a) \cup (\delta, a + \delta)$ ■

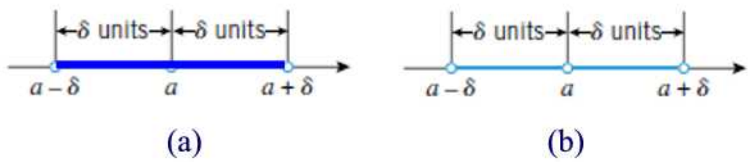


Figure 3

Exercise 1.14
Show that $|x - a| < k$ if and only if $a - k < x < a + k$, where $k > 0$.

Solution.
From the previous exercise, we can write $|x - a| < k \Leftrightarrow a - k < x < a + k$ ■

Exercise 1.15
Show that $|x - a| > k$ if and only if $x < a - k$ or $x > a + k$, where $k \geq 0$.

Solution.
Using Figure 4, we see that $|x - a| > k \Leftrightarrow x - a < -k$ or $x - a > k$. This is equivalent to $x < a - k$ or $x > a + k$ ■

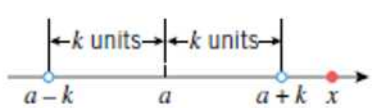


Figure 4

Exercise 1.16
Solve each of the following inequalities: (a) $|2x-3|<5$ and (b) $|x+4|>2$.

Solution.
(a) Using Exercise 1.14 we can write $|2x-3|<5 \Leftrightarrow -5<2x-3<5 \Leftrightarrow -5+3<2x<5+3 \Leftrightarrow -1<x<4$.
(b) Using Exercise 1.15 we can write $|x+4|>2 \Leftrightarrow x+4<-2$ or $x+4>2$ which is equivalent to $x<-6$ or $x>-2$ ■

Exercise 1.17 *(Triangle inequality)*
Use Exercise 1.1, Exercise 1.7, and the expansion of $(|a+b|)^2$ to establish the inequality

$$|a+b| \leq |a|+|b|,$$

where a and b are arbitrary real numbers.

Solution.
We have

$$(|a+b|)^2=(a+b)(a+b)=a^2+2ab+b^2 \leq a^2+2|a|\cdot|b|+b^2=(|a|+|b|)^2.$$

Now, the result follows by taking the square root of both sides ■

Exercise 1.18
Show that for any real numbers a and b we have $|a|-|b| \leq |a-b|$. Hint: Notice that $a=(a-b)+b$.

Solution.
Using Exercise 1.17 we can write $|a|=|(a-b)+b| \leq |a-b|+|b|$. Subtracting $|b|$ from both sides to obtain the desired inequaltiy ■

Exercise 1.19
Let $a \in \mathbb{R}$. Show that $\max\{a,0\}=\frac{1}{2}(a+|a|)$ and $\min\{a,0\}=\frac{1}{2}(a-|a|)$.

Solution.
The results are clear if $a=0$. If $a>0$ then $|a|=a$ and $\max\{a,0\}=a=\frac{1}{2}(a+|a|)$. If $a<0$ then $|a|=-a$ $\max\{a,0\}=0=\frac{1}{2}(a+|a|)$. Likewise for the minimum ■

Exercise 1.20
Show that $|a+b|=|a|+|b|$ if and only if $ab \geq 0$.

Solution.
Suppose first that $|a+b| = |a| + |b|$. Squaring both sides we find $a^2 + 2ab + b^2 = a^2 + 2|a||b| + b^2$ or equivalently $ab = |a||b| = |ab|$. But this is true only when $ab \geq 0$. Conversely, suppose that $ab \geq 0$. If $a = 0$ we have $|0+b| = |b| = |0| + |b|$. Likewise when $b = 0$. So assume that $ab > 0$. Suppose that $a > 0$ and $b > 0$. Then $a+b > 0$ and in this case $|a+b| = a+b = |a|+|b|$. Similar argument when $a < 0$ and $b < 0$ ■

Exercise 1.21
Suppose $0 < x < \frac{1}{2}$. Simplify $\frac{x+3}{|2x^2+5x-3|}$.

Solution.
We have

$$\frac{x+3}{|2x^2+5x-3|} = \frac{x+3}{|(2x-1)(x+3)|} = \frac{1}{|2x-1|} = \frac{1}{1-2x} \blacksquare$$

Exercise 1.22
Write the function $f(x) = |x+2| + |x-4|$ as a piecewise defined function. Sketch its graph.

Solution.
We have

$$|x+2| = \begin{cases} x+2 & \text{if } x \geq -2 \\ -(x+2) & \text{if } x < -2 \end{cases}$$

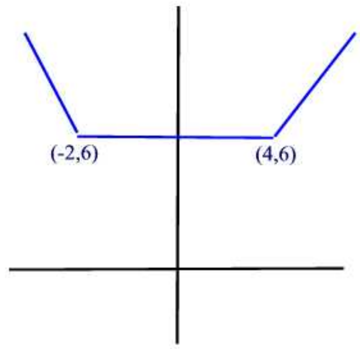
Likewise,

$$|x-4| = \begin{cases} x-4 & \text{if } x \geq 4 \\ -(x-4) & \text{if } x < 4 \end{cases}$$

Combining we find

$$f(x) = \begin{cases} -2x+2 & \text{if } x < -2 \\ 6 & \text{if } -2 \leq x < 4 \\ 2x- & \text{if } x \geq 4 \end{cases}$$

The graph is given below ■



Exercise 1.23
Prove that $||a| - |b|| \leq |a - b|$ for any real numbers a and b.

Solution.
We have $|b| - |a| \leq |b - a| = |a - b|$ so that $-|a - b| \leq |a| - |b| \leq |a - b$ by Exercise 1.18. Now using Exercise 1.14, the result follows ■

Exercise 1.24
Solve the equation $4|x - 3|^2 - 3|x - 3| = 1$.

Solution.
Let $u = |x - 3|$. Then $4u^2 - 3u - 1 = 0$ implies $u = 1$ or $u = \frac{-1}{4}$. Since $u \geq 0$ we must have $|x - 3| = u = 1$. Hence, $x - 3 = -1$ or $x - 3 = 1$. Thus, $x = 2$ or $x = 4$ ■

Exercise 1.25
What is the range of the function $f(x) = \frac{|x|}{x}$ for all $x \neq 0$?

Solution.
If $x > 0$ then $\frac{|x|}{x} = \frac{x}{x} = 1$. If $x < 0$ then $\frac{|x||}{x} = \frac{-x}{x} = -1$. Thus, the range is $\{-1, 1\}$ ■

Exercise 1.26
Solve $3 \leq |x - 2| \leq 7$. Write your answer in interval notation.

Solution.
Solving the inequality $|x - 2| \leq 7$ we find $-5 \leq x \leq 9$. Solving the inequality $|x - 2| \geq 3$ we find $x \leq -1$ or $x \geq 5$. Thus, the common intervals are $[-5, -1] \cup [5, 9]$ ■

Exercise 1.27
Simplify $\frac{\sqrt{x^2}}{|x|}$.

Solution.
The answer is $\frac{\sqrt{x^2}}{|x|} = \frac{|x|}{|x|} = 1$ ■

Exercise 1.28
Solve the inequality $\left|\frac{x+1}{x-2}\right| < 3$. Write your answer in interval notation.

Solution.
We have $-3 < \frac{x+1}{x-2} < 3$. The inequality $\frac{x+1}{x-2} < 3$ implies $\frac{-2x+7}{x-2} < 0$. Solving this inequality we find $x < 2$ or $x > \frac{7}{2}$. Likewise, The inequality $\frac{x+1}{x-2} > -3$ implies $\frac{4x-5}{x-2} > 0$. Solving this inequality we find $x < \frac{5}{4}$ or $x > 2$. Hence, the common interval is $(-\infty, \frac{5}{4}) \cup (\frac{7}{2}, \infty)$ ■

Exercise 1.29
Suppose x and y are real numbers such that $|x - y| < |x|$. Show that $xy > 0$.

Solution.
Since $|x - y| < |x|$ we have $-|x| < x - y < |x|$. Multiplying through by -1 and adding x we obtain $x - |x| < y < x + |x|$. If $x = 0$ then $0 < y < 0$ which is impossible. Therefore either $x > 0$ or $x < 0$. If $x > 0$ then $0 < y < 2x$. Hence $xy > 0$. If $x < 0$ then $2x < y < 0$ and so $xy > 0$ ■

Solutions to Section 2

Exercise 2.1
Prove that A is bounded if and only if there is a positive constant C such that $|x| \leq C$ for all $x \in A$.

Solution.
Suppose that A is bounded. Then there exist real numbers m and M such that $m \leq x \leq M$ for all $x \in A$. Let $C = \{|m|, |M| + 1\} > 0$. Then $-C \leq -|m| \leq m \leq x \leq M \leq |M| + 1 \leq C$. That is, $|x| \leq C$ for all $x \in A$.
Conversely, suppose that $|x| \leq C$ for all $x \in A$ and for some $C > 0$. Then , by Exercise 1.4, we have $-C \leq x \leq C$ for all $x \in A$. Let $m = -C$ and $M = C$ ■

Exercise 2.2
Let $A = [0, 1]$.
(a) Find an upper bound of A. How many upper bounds are there?
(b) Find a lower bound of A. How many lower bounds are there?

Solution.
(a) Any number greater than 1 is an upper bound.
(b) Any number less than 0 is a lower bound ■

Exercise 2.3
Consider the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$.
(a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?
(b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

Solution.
(a) The supremum is 1 which is also the maximum of A.
(b) The infimum is 0 which is not a minimum of A ■

Exercise 2.4
Consider the set $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$.
(a) Show that 1 is an upper bound of A.
(b) Suppose $L < 1$ is another upper bound of A. Let n be a positive integer such that $n > \frac{1}{1-L}$. Such a number n exist by the Archimedian property which we will discuss below. Show that this leads to a contradiction. Thus, $L \geq 1$. This shows that 1 is the least upper bound of A and hence $\sup A = 1$.

Solution.
(a) Since $\frac{1}{n} > 0$ we have $1 - \frac{1}{n} < 1$ for all $n \in \mathbb{N}$. Thus, 1 is an upper bound of A.
(b) Since $n > \frac{1}{1-L}$, we find $L < 1 - \frac{1}{n}$. But this contradicts the fact that L is an upper bound of A. Hence, we must have $1 < L$. This shows, that 1 is the smallest upper bound and so is the supremum of A ■

Exercise 2.5
Let $a, b \in \mathbb{R}$ with $a > 0$.
(a) Suppose that $na \leq b$ for all $n \in \mathbb{N}$. Show that the set $A = \{na : n \in \mathbb{N}\}$ has a supremum. Call it c.
(b) Show that $na \leq c - a$ for all $n \in \mathbb{N}$. That is, $c - a$ is an upper bound of A. Hint: $n + 1 \in \mathbb{N}$ for all $n \in \mathbb{N}$.
(c) Conclude from (b) that there must be a positive integer n such that $na > b$.

Solution.
(a) Since $na \leq b$ for all $n \in \mathbb{N}$, the set A is bounded from above. By the completeness axiom of $\mathbb{R}$, A has a supremum, denote it by $c = \sup\{A\}$.
(b) Let $n \in \mathbb{N}$. Then $n + 1 \in \mathbb{N}$ so that $(n + 1)a \leq c$ or $na \leq c - a$.
(c) From (b), we have that $c - a$ is an upper bound of A. By the definition of supremum, we must have $c \leq c - a$ which is impossible. This shows that A cannot have an upper bound. That is, there must be a positive integer n such that $na > b$ ■

Exercise 2.6
Let a and b be two real numbers such that $a < b$.
(a) Let $[a]$ denote the greatest integer less than or equal to a. Show that $[a] - 1 < a < [a] + 1$.
(b) Let n be a positive integer such that $n > \frac{1}{b-a}$. Show that $na + 1 < nb$.
(c) Let $m = [na] + 1$. Show that $na < m < nb$. Thus, $a < \frac{m}{n} < b$. We see that between any two distinct real numbers there is a rational number.

Solution.
(a) Since $|[a] - a| < 1$, we have $-1 < [a] - a < 1$ which is equivalent to $[a] - 1 < a < [a] + 1$.
(b) Since $n > \frac{1}{b-a}$ and $b - a > 0$ we can have $n(b - a) > 1$ or $na + 1 < nb$.
(c) We have $m - 1 = [na] \leq na < [na] + 1 = m < na + 1 < nb$. Thus, $na < m < nb$. Dividing through by $n > 0$ we obtain $a < \frac{m}{n} < b$ ■

Exercise 2.7
Consider the set $A = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}$.
(a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?
(b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

Solution.
(a) The supremum is $\frac{1}{2}$ which is also a maximum.
(b) The infimum is -1 which is also a minimum $\blacksquare$

Exercise 2.8
Consider the set $A = \{x \in \mathbb{R} : 1 < x < 2\}$.
(a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?
(b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

Solution.
(a) 2 is a supremum that is not a maximum.
(b) 1 is an infimum that is not a minimum $\blacksquare$

Exercise 2.9
Consider the set $A = \{x \in \mathbb{R} : x^2 > 4\}$.
(a) Show $x \in A$ and $x < 2$ leads to a contradiction. Hence, we must have that $x \geq 2$ for all $x \in A$. That is, 2 is a lower bound of A.
(b) Let L be a lower bound of A such that $L > 2$. Let $y = \frac{L+2}{2}$. Show that $2 < y < L$.
(c) Use (a) to show that $y \in A$ and $L \leq y$. Show that this leads to a contradiction. Hence, we must have $L \leq 2$ which means that 2 is the infimum of A.

Solution.
(a) If $x \in A$ and $x < 2$ then $x^2 < 4$ which contradicts the fact that $x \in A$. Thus, for all $x \in A$ we have $x \geq 2$. This shows that 2 is a lower bound of A.
(b) Since $L > 2$ we have $L + 2 > 4$ and this implies $y = \frac{L+2}{2} > 2$. Also, $y = \frac{L+2}{2} < \frac{L+L}{2} = L$.
(c) Since $y > 2$ we have $y^2 > 4$ so that $y \in A$. But L is a lower bound of A so we must have $L \leq y$. But this contradicts $y < L$ from (b). It follows that 2 is the least lower bound of A $\blacksquare$

Exercise 2.10
Show that for any real number x there is a positive integer n such that $n > x$.

Solution.
Let $a = 1$ and $b = x$ in the Archimedean property ■

Exercise 2.11
Let a and b be any two real numbers such that $a < b$.
(a) Let w be a fixed positive irrational number. Show that there is a rational number r such that $a < wr < b$.
(e) Show that wr is irrational. Hence, between any two distinct real numbers there is an irrational number.

Solution.
(a) Since $a < b$, we have $\frac{a}{w} < \frac{b}{w}$. By Exercise **??**, there is a rational number r such that $\frac{a}{w} < r < \frac{b}{w}$ or $a < rw < b$.
(e) If $rw = s$ with s rational then $w = \frac{s}{r}$ which is a rational, a contradiction. Hence, rw is irrational ■

Exercise 2.12
Suppose that $\alpha = \sup A < \infty$. Let $\epsilon > 0$ be given. Prove that there is an $x \in A$ such that $\alpha - \epsilon < x$.

Solution.
Suppose the contrary. That is, $\alpha - \epsilon \geq x$ for all $x \in A$. In this case, $\alpha - \epsilon$ is an upper bound of A. Thus, we must have $\alpha \leq \alpha - \epsilon$ which is impossible ■

Exercise 2.13
Suppose that $\beta = \inf A < \infty$. Let $\epsilon > 0$ be given. Prove that there is an $x \in A$ such that $\beta + \epsilon > x$.

Solution.
Suppose the contrary. That is, $\beta + \epsilon \leq x$ for all $x \in A$. In this case, $\beta + \epsilon$ is a lower bound of A. Thus, we must have $\beta + \epsilon \leq \beta$ which is impossible ■

Exercise 2.14
For each of the following sets S find $\sup\{S\}$ and $\inf\{S\}$ if they exist.
(a) $S = \{x \in \mathbb{R} : x^2 < 5\}$.
(b) $S = \{x \in \mathbb{R} : x^2 > 7\}$.
(c) $S = \{-\frac{1}{n} : n \in \mathbb{N}\}$.

Solution.
(a) $\sup\{S\} = \sqrt{5}$ and $\inf\{S\} = -\sqrt{5}$.
(b) $\sup\{S\} = \infty$ and $\inf\{S\} = -\infty$.
(c) $\sup\{S\} = 0$ and $\inf\{S\} = -1$ ■

Solutions to Section 3

Exercise 3.1
Find a simple expression for the general term of each sequence.
(a) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots$
(b) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \cdots$
(c) $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \cdots$
(d) $-1, 1, -1, 1, -1, 1, \cdots$

Solution.
(a) $a_n = (-1)^{n-1}\frac{1}{n}$.
(b) $a_n = \frac{n+1}{n}$.
(c) $a_n = \frac{1}{2n-1}$
(d) $a_n = (-1)^n$ ■

Exercise 3.2
Show that the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ converges to 0.

Solution.
Let $\epsilon > 0$ be given. We want to find a positive integer N_ϵ such that if $n \ge N_\epsilon$ then $\left|\frac{1}{n} - 0\right| < \epsilon$. But this last inequality implies that $n > \frac{1}{\epsilon}$. Let N_ϵ be a positive integer greater than $\frac{1}{\epsilon}$. If $n \ge N_\epsilon > \frac{1}{\epsilon}$ then $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon$. This shows that

$$\lim_{n\to\infty} \frac{1}{n} = 0 \ ■$$

Exercise 3.3
Show that the sequence $\left\{1 + \frac{C}{n}\right\}_{n=1}^{\infty}$ converges to 1, where $C \neq 0$ is a constant.

Solution.
Let $\epsilon > 0$. We want to find a positive integer N_ϵ such that $\left|1 + \frac{C}{n} - 1\right| < \epsilon$ for all $n \ge N_\epsilon$. But $\left|1 + \frac{C}{n} - 1\right| < \epsilon$ implies $n > \frac{|C|}{\epsilon}$. By choosing N_ϵ to be a positive integer greater than $\frac{|C|}{\epsilon}$ the result follows ■

Exercise 3.4
Is there a number L with the property that $|(-1)^n - L| < 1$ for all $n \ge N_1$, where N_1 is some positive integer? Hint: Consider the inequality with an even integer greater than N_1 and an odd integer greater than N_1.

Solution.
If $n_e \geq N_1$ is an even integer than $|(-1)^{n_e} - L| = |1 - L| < 1$. If $n_o \geq N_1$ is an odd integer than $|(-1)^{n_o} - L| = |-1 - L| < 1$. This shows that L is within one unit of both -1 and 1 which is impossible. Thus, L does not exist ■

Exercise 3.5
Use the previous exercise to show that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent.

Solution.
Assume the contrary. That is, suppose there is an L such that $\lim_{n\to\infty}(-1)^n = L$. Let $\epsilon = 1$. Then, there is a positive integer N_1 such that $n \geq N_1$ implies $|(-1)^n - L| < 1$. But by the previous exercise, this is impossible. Hence, the given sequence is divergent ■

Exercise 3.6
Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$ with $a < b$. Show that by choosing $\epsilon = \frac{b-a}{2} > 0$ we end up with the impossible inequality $b - a < b - a$. A similar result holds if $b < a$. Thus, we must have $a = b$. Hint: Exercise 1.6 and Exercise 1.17.

Solution.
Let $\epsilon = \frac{b-a}{2} > 0$. Since the sequence converges to a, we can find a positive integer N_1 such that

$$n \geq N_1 \Longrightarrow |a_n - a| < \frac{b-a}{2}.$$

Similarly, since the sequence converges to b we can find a positive integer N_2 such that

$$n \geq N_2 \Longrightarrow |a_n - b| < \frac{b-a}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$ we have $n \geq N_1$ and $n \geq N_2$. Moreover, by using Exercise 1.6 and Exercise 1.17 we have

$$\begin{aligned} b - a =&|b - a| = |(b - a_n) + (a_n - a)| \\ \leq&|b - a_n| + |a_n - a| \\ =&|a_n - b| + |a_n - a| < \frac{b-a}{2} + \frac{b-a}{2} = b - a \end{aligned}$$

Thus, we conclude that $b - a < b - a$ which is impossible. Likewise, if $b < a$ we end up with $a - b < a - b$ which is impossible. That is, either $a < b$ or $b < a$ leads to a contradiction. Hence, $a = b$ ■

Exercise 3.7
Show that each of the following sequences is bounded. Identify M in each case.
(a) $a_n = (-1)^n$.
(b) $a_n = \frac{1}{\sqrt{n}\ln(n+1)}$.

Solution.
(a) We have $|a_n| = |(-1)^n| = 1 \leq 1$ so that $M = 1$.
(b) $n \geq 1 \Rightarrow \sqrt{n} \geq 1 \Rightarrow \frac{1}{\sqrt{n}} \leq 1$. Also, $\ln(n+1) \geq \ln 2 \Rightarrow \frac{1}{\ln(n+1)} \leq \frac{1}{\ln 2}$. Hence, $|a_n| \leq \frac{1}{\ln 2} = M$ ■

Exercise 3.8
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $|a_n| \leq K$ for all $n \geq N$. Show that this sequence is bounded. Identify your M.

Solution.
Let $M = |a_1| + |a_2| + \cdots + |a_{N-1}| + K$. Then $|a_n| \leq M$ for all $n \geq 1$. That is, the sequence is bounded ■

Exercise 3.9
Show that a convergent sequence is bounded. Hint: use the definition of convergence with $\epsilon = 1$.

Solution.
Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence with limit L. Let $\epsilon = 1$. There is a positive integer N_1 such that $|a_n - L| < 1$ for all $n \geq N_1$. By Exercise 1.18, we obtain $|a_n| - |L| < 1$ or $|a_n| < 1 + |L|$ for all $n \geq N_1$. By the previous problem with $K = 1 + |L|$, the sequence is bounded ■

Exercise 3.10
Give an example of a bounded sequence that is divergent.

Solution.
Let $a_n = (-1)^n$. We know that this sequence is bounded (Exercise 3.7(a)). We also know that this sequence is divergent (Exercise) ■

Exercise 3.11
Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}$ be three sequences with the following conditions:

(1) $b_n \leq a_n \leq c_n$ for all $n \geq K$, where K is some positive integer.
(2) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$.
Show that $\lim_{n\to\infty} a_n = L$. Hint: Use the definition of convergence along with Exercise 1.14.

Solution.
Let $\epsilon > 0$. By hypothesis, there exist positive integers N_1 and N_2 such that

$|b_n - L| < \epsilon$ for all $n \geq N_1$ or equivalently $L - \epsilon < b_n < L + \epsilon$ for all $n \geq N_1$

and

$|c_n - L| < \epsilon$ for all $n \geq N_2$ or equivalently $L - \epsilon < c_n < L + \epsilon$ for all $n \geq N_2$.

Let $N = N_1 + N_2 + K$. Suppose $n \geq N$. Then $L - \epsilon < b_n \leq a_n \leq c_n < L + \epsilon$. That is $L - \epsilon < a_n < L + \epsilon$ for all $n \geq N$. This implies that $\lim_{n\to\infty} a_n = L$ ■

Exercise 3.12
An expansion of $(a+b)^n$, where n is a positive integer is given by the Binomial formula

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)a^k b^{n-k}$$

where $C(n,k) = \frac{n!}{k!(n-k)!}$.
(a) Use the Binomial formula to establish the inequality

$$(1+x)^{\frac{1}{n}} \leq 1 + \frac{x}{n}, \ x \geq 0$$

(b) Show that if $a \geq 1$ then $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$. Hint: Use Exercise 3.3.

Solution.
(a) Using the Binomial formula with $a = 1$ and $b = \frac{x}{n}$ we find

$$\left(1 + \tfrac{x}{n}\right)^n = 1 + n\tfrac{x}{n} + (\text{other positive terms}) \geq 1 + x.$$

or

$$1 + x \leq \left(1 + \frac{x}{n}\right)^n.$$

Taking the n^{th} root of both sides we find

$$(1+x)^{\frac{1}{n}} \leq 1 + \frac{x}{n}.$$

(b) If $a \geq 1$ then $a^{\frac{1}{n}} \geq 1$. By letting $x = a - 1 \geq 0$ in (a) we find

$$1 \leq a^{\frac{1}{n}} \leq 1 + \frac{a-1}{n}.$$

By Exercise 3.3 we know that $\lim_{n\to\infty} 1 + \frac{a-1}{n} = 1$. Thus, by the squeeze rule we obtain

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1, \quad a \geq 1 \blacksquare$$

Exercise 3.13
Prove that the sequence $\{\cos(n\pi)\}_{n=1}^{\infty}$ is divergent.

Solution.
Note that $\{\cos(n\pi)\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$ and by Exercise , this sequence is divergent $\blacksquare$

Exercise 3.14
Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined by $a_n = n$ for all $n \in \mathbb{N}$. Explain why the sequence $\{a_n\}_{n=1}^{\infty}$ does not converge to any limit.

Solution.
The sequence is unbounded $\blacksquare$

Exercise 3.15
(a) Show that for all $n \in \mathbb{N}$ we have

$$\frac{n!}{n^n} \leq \frac{1}{n}.$$

(b) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = \frac{n!}{n^n}$ is convergent and find its limit.

Solution.
(a) We know that $\frac{n-i}{n} \leq 1$ for all $0 \leq i \leq n-1$. Thus, $\frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots 2\cdot 1}{n\cdot n\cdots n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{2}{n} \cdot \frac{1}{n} \leq \frac{1}{n}$.
(b) By the Squeeze rule we find that $\lim_{n\to\infty} \frac{n!}{n^n} = 0 \blacksquare$

Exercise 3.16
Using only the definition of convergence show that

$$\lim_{n\to\infty} \frac{\sqrt[3]{n} - 5001}{\sqrt[3]{n} - 1001} = 1.$$

Solution.
Let $\epsilon > 0$. We want to find a positive integer N such that if $n \geq N$ then

$$\left|\frac{\sqrt[3]{n} - 5001}{\sqrt[3]{n} - 1001} - 1\right| < \epsilon$$

or

$$\left|\frac{-4001}{\sqrt[3]{n} - 1001}\right| < \epsilon.$$

Let $n > 1001^3$. Then $\sqrt[3]{n} - 1001 > 0$ so that the previous inequality becomes

$$\frac{4001}{\sqrt[3]{n} - 1001} < \epsilon.$$

Solving this for n we find

$$n > \left(\frac{4001}{\epsilon} + 1001\right)^3.$$

Let N be a positive integer greater than $\left(\frac{4001}{\epsilon} + 1001\right)^3$. Then for $n \geq N$ we have

$$\left|\frac{\sqrt[3]{n} - 5001}{\sqrt[3]{n} - 1001} - 1\right| < \epsilon \blacksquare$$

Exercise 3.17
Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \sqrt{2 + a_n}$ for all $n \in \mathbb{N}$. Show that $a_n \leq 2$ for all $n \in \mathbb{N}$.

Solution.
The proof is by induction on n. For $n = 1$ we have $a_1 = 1 \leq 2$. Suppose that $a_n \leq 2$. Then $a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{2 + 2} = 2 \blacksquare$

Exercise 3.18
Calculate $\lim_{n\to\infty} \frac{(n^2+1)\cos n}{n^3}$.

Solution.
We have

$$-\frac{n^2 + 1}{n^3} \leq \frac{(n^2 + 1)\cos n}{n^3} \leq \frac{n^2 + 1}{n^3}.$$

By the Squeeze rule we conclude that the limit is 0 $\blacksquare$

Exercise 3.19
Calculate $\lim_{n\to\infty}\frac{2(-1)^{n+3}}{\sqrt{n}}$.

Solution.
We have
$$-\frac{2}{\sqrt{n}} \le \frac{2(-1)^{n+3}}{\sqrt{n}} \le \frac{2}{\sqrt{n}}.$$
By the Squeeze rule the limit is 0 ■

Exercise 3.20
Suppose that $\lim_{n\to\infty} a_n = L$ with $L > 0$. Show that there is a positive integer N such that $2x_N > x$.

Solution.
Let $\epsilon = \frac{L}{2}$. Then there is a positive integer N such that if $n \ge N$ we have $|a_n - L| < \frac{L}{2}$. Thus, $|a_N - L| < \frac{L}{2}$ or $-\frac{L}{2} < a_N - L$. Hence, $a_N > \frac{L}{2}$ or $2a_N > L$ ■

Exercise 3.21
Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Clearly, $a < a + \frac{1}{n}$.
(a) Show that there is $a_1 \in \mathbb{Q}$ such that $a < a_1 < a + \frac{1}{n}$. Hint: Exercise 2.6(c).
(b) Show that there is $a_2 \in \mathbb{Q}$ such that $a < a_2 < a_1$.
(c) Continuiung the above process we can find a sequence $\{a_n\}_{n=1}^{\infty}$ such that $a < a_n < a + \frac{1}{n}$ for all $n \in \mathbb{N}$. Show that this sequence converges to a.
We have proved that if a is a real number then there is a sequence of rational numbers converging a. We say that the set $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

Solution.
(a) This follows from Exercise 2.6(c).
(b) Similar to (a).
(c) Applying the Squeeze rule, we obtain $\lim_{n\to\infty} a_n = a$ ■

Solutions to Section 4

Exercise 4.1
Suppose that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Show that

$$\lim_{n\to\infty} a_n \pm b_n = A \pm B.$$

Solution.
We will prove the result for addition. The difference case is similar. Let $\epsilon > 0$. Since the two sequences are convergent, there exist positive integers N_1 and N_2 such that

$$|a_n - A| < \frac{\epsilon}{2} \text{ for all } n \geq N_1$$

and

$$|b_n - B| < \frac{\epsilon}{2} \text{ for all } n \geq N_2.$$

Let $N = N_1 + N_2$. Then for all $n \geq N$ we have $n \geq N_1$ and $n \geq N_2$. Hence,

$$|(a_n+b_n)-(A+B)| = |(a_n-A)+(b_n-B)| \leq |a_n-A|+|b_n-B| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

This establishes the desired result ■

Exercise 4.2
Suppose that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.
(a) Show that $|b_n| \leq M$ for all $n \in \mathbb{N}$, where M is a positive constant.
(b) Show that $a_nb_n - AB = (a_n - A)b_n + A(b_n - B)$.
(c) Let $\epsilon > 0$ be arbitrary and $K = M + |A|$. Show that there exists a positive integer N_1 such that $|a_n - A| < \frac{\epsilon}{2K}$ for all $n \geq N_1$.
(d) Let $\epsilon > 0$ and K be as in (c). Show that there exists a positive integer N_2 such that $|b_n - B| < \frac{\epsilon}{2K}$ for all $n \geq N_2$.
(e) Show that $\lim_{n\to\infty} a_nb_n = AB$.

Solution.
(a) Since $\{b_n\}_{n=1}^{\infty}$ is convergent, the sequence is bounded. Thus, there is a positive constant M such that $|b_n| \leq M$ for all $n \geq 1$.
(b) We have $(a_n - A)b_n + A(b_n - B) = a_nb_n - Ab_n + Ab_n - AB = a_nb_n - AB$.
(c) Let $\epsilon_1 = \frac{\epsilon}{2K}$. Since $\lim_{n\to\infty} a_n = A$, we can find a positive integer N_1 such that $|a_n - A| < \epsilon_1 = \frac{\epsilon}{2K}$ for all $n \geq N_1$.
(d) Let $\epsilon_2 = \frac{\epsilon}{2K}$. Since $\lim_{n\to\infty} b_n = B$, we can find a positive integer N_2 such

that $|b_n - B| < \epsilon_2 = \frac{\epsilon}{2K}$ for all $n \ge N_2$.
(e) Let $N = N_1 + N_2$. Then $n \ge N$ implies $n \ge N_1$ and $n \ge N_2$. In this case,

$$\begin{aligned}|a_nb_n - AB| =& |(a_n - A)b_n + A(b_n - B)| \le |a_n - A||b_n| + |A||b_n - B| \\ <& \frac{\epsilon}{2K} \cdot M + \frac{\epsilon}{2K} \cdot |A| \\ <& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

where we have used the fact that $\frac{M}{K} = \frac{M}{M+|A|} < 1$ and $\frac{|A|}{K} = \frac{|A|}{M+|A|} < 1$ ■

Exercise 4.3
Give an example of two divergent sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $\{a_nb_n\}$ and $\{a_n + b_n\}$ are convergent.

Solution.
Let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Both sequences are divergent. Moreover, $a_nb_n = (-1)^{2n+1} = -1$ for all $n \ge 1$. Hence, $\lim_{n\to\infty} a_nb_n = -1$. Finally, $a_n + b_n = (-1)^n - (-1)^n = 0$ for all $n \ge 1$. Therefore, $\lim_{n\to\infty}(a_n + b_n) = 0$ ■

Exercise 4.4
Let $k \ne 0$ be an arbitrary constant and $\lim_{n\to\infty} a_n = A$. Show that $\lim_{n\to\infty} ka_n = kA$.

Solution.
Let $\epsilon > 0$ be arbitrary. There is a positive integer N such that $|a_n - A| < \frac{\epsilon}{|k|}$ for all $n \ge N$. Moreover, for $n \ge N$ we have

$$|ka_n - kA| = |k(a_n - A)| = |k||a_n - A| < |k| \cdot \frac{\epsilon}{|k|} = \epsilon.$$

This establishes the desired result ■

Exercise 4.5
Suppose that $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}_{n=1}^{\infty}$ is bounded. Show that $\lim_{n\to\infty} a_nb_n = 0$.

Solution.
Since $\{b_n\}_{n=1}^{\infty}$ is bounded, we can find a positive constant M such that $|b_n| \le M$ for all $n \ge 1$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} a_n = 0$, there is a

positive integer N such that $|a_n - 0| = |a_n| < \frac{\epsilon}{M}$ for all $n \geq N$. Thus, for $n \geq N$, we have

$$|a_nb_n - 0| = |a_nb_n| = |a_n||b_n| < \frac{\epsilon}{M} \cdot M = \epsilon.$$

This shows that $\lim_{n\to\infty} a_nb_n = 0$ ■

Exercise 4.6
(a) Use the previous exercise to show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.
(b) Show that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ using the squeeze rule.

Solution.
(a) Let $a_n = \frac{1}{n}$ and $b_n = \sin n$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ and $|b_n| = |\sin n| \leq 1$. Now the result follows from the previous exercise.
(b) Since $-1 \leq \sin n \leq 1$, we obtain $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$. But $\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} -\frac{1}{n} = 0$ so that by the squeeze rule

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0 \blacksquare$$

Exercise 4.7
Suppose that $\lim_{n\to\infty} a_n = A$, with $A \neq 0$. Show that there is a positive integer N such that $|a_n| > \frac{|A|}{2}$ for all $n \geq N$. Hint: Use Exercise 1.18.

Solution.
Let $\epsilon = \frac{|A|}{2} > 0$. Then there is a positive integer N such that $|a_n - A| < \frac{|A|}{2}$ for all $n \geq N$. By Exercise 1.18, we have $|A| - |a_n| < \frac{|A|}{2}$ or $|a_n| > \frac{|A|}{2}$ for $n \geq N$ ■

Exercise 4.8
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with the following conditions:
(1) $a_n \neq 0$ for all $n \geq 1$.
(2) $\lim_{n\to\infty} a_n = A$, with $A \neq 0$.
(a) Show that there is a positive integer N_1 such that for all $n \geq N_1$ we have

$$\left|\frac{1}{a_n} - \frac{1}{A}\right| < \frac{2}{|A|^2}|a_n - A|.$$

(b) Let $\epsilon > 0$ be arbitrary. Show that there is a positive integer N_2 such that for all $n \geq N_2$ we have

$$|a_n - A| < \frac{|A|^2}{2}\epsilon.$$

(c) Using (a) and (b), show that

$$\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}.$$

Solution.
(a) From the previous exercise, we can find a positive integer N_1 such that $|a_n| > \frac{|A|}{2}$ for $n \geq N_1$. Thus, for $n \geq N_1$ we obtain

$$\left|\frac{1}{a_n} - \frac{1}{A}\right| = \frac{|a_n - A|}{|a_n||A|} < \frac{2}{|A|^2}|a_n - A|.$$

(b) Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} a_n = A$, we can find a positive integer N_2 such that for all $n \geq N_2$ we have

$$|a_n - A| < \frac{|A|^2}{2}\epsilon.$$

(c) Let $\epsilon > 0$ be arbitrary. Let $N = N_1 + N_2$. Then $n \geq N$ implies that $n \geq N_1$ and $n \geq N_2$. Moreover,

$$\left|\frac{1}{a_n} - \frac{1}{A}\right| = \frac{|a_n - A|}{|a_n||A|} < \frac{2}{|A|^2}|a_n - A| < \frac{2}{|A|^2}\frac{|A|^2}{2}\epsilon = \epsilon.$$

This establishes the required result ■

Exercise 4.9
Let $0 < a < 1$. Show that $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$. Hint: Use Exercise 3.12 (b).

Solution.
Since $0 < a < 1$, we have $\frac{1}{a} > 1$. By Exercise 3.12(b), we find

$$\lim_{n\to\infty} \left(\frac{1}{a}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{a^{\frac{1}{n}}} = 1.$$

By the previous exercise, we can write

$$\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\frac{1}{a^{\frac{1}{n}}}} = \frac{1}{1} = 1 \blacksquare$$

Exercise 4.10
Show that if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ with $b_n \neq 0$ for all $n \geq 1$ and $B \neq 0$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}.$$

Solution.
Using Exercise **??** and Exercise **??** we can write

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = A \cdot \frac{1}{B} = \frac{A}{B} \blacksquare$$

Exercise 4.11
Given that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$ with $a_n \leq b_n$ for all $n \geq 1$.
(a) Suppose that $B < A$. Let $\epsilon = \frac{A-B}{2} > 0$. Show that there exist positive integers N_1 and N_2 such that $A - \epsilon < a_n < A + \epsilon$ for $n \geq N_1$ and $B - \epsilon < b_n < B + \epsilon$ for $n \geq N_2$.
(b) Let $N = N_1 + N_2$. Show that for $n \geq N$ we obtain the contradiction $b_n < a_n$. Thus, we must have $A \leq B$.

Solution.
(a) Since $\lim_{n\to\infty} a_n = A$, there exists a positive integer N_1 such that $|a_n - A| < \epsilon$ for all $n \geq N_1$. By Exercise 1.14, this is equivalent to $A-\epsilon < a_n < A+\epsilon$ for $n \geq N_1$. Similarly, since $\lim_{n\to\infty} b_n = B$, there exists a positive integer N_2 such that $|b_n - B| < \epsilon$ for all $n \geq N_2$. By Exercise 1.14, this is equivalent to $B - \epsilon < b_n < B + \epsilon$ for $n \geq N_2$.
(b) If $n \geq N$ then $n \geq N_1$ and $n \geq N_2$. Thus,

$$\begin{aligned} b_n <& B + \epsilon = B + \frac{A-B}{2} = \frac{A+B}{2} \\ =& A - \frac{A-B}{2} = A - \epsilon < a_n \end{aligned}$$

This contradicts the fact that $a_n \leq b_n$ for all $n \geq 1$. Hence, we conclude that $A \leq B$ $\blacksquare$

Exercise 4.12
Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $\lim_{n\to\infty} b_n = 0$ where $b_n \neq 0$ for all $n \in \mathbb{N}$. Find $\lim_{n\to\infty} a_n$.

Solution.
We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_n}{b_n} \cdot b_n = \left(\lim_{n\to\infty} \frac{a_n}{b_n}\right)\left(\lim_{n\to\infty} b_n\right) = L \cdot 0 = 0 \blacksquare$$

Exercise 4.13
The Fibonacci numbers are defined recursively as follows:

$$a_1 = a_2 = 1 \text{ and } a_{n+2} = a_{n+1} + a_n \text{ for all } n \in \mathbb{N}.$$

Suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. Find the value of L.

Solution.
We have

$$a_{n+2} = a_{n+1} + a_n$$

Divide through by a_{n+1} to obtain

$$\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_n}{a_{n+1}}.$$

Take the limit as $n \to \infty$ to obtain

$$L = 1 + \frac{1}{L}$$

or

$$L^2 - L - 1 = 0.$$

Solving this quadratic equation for $L > 0$ we find $L = \frac{1+\sqrt{5}}{2}$ $\blacksquare$

Exercise 4.14
Show that the sequence defined by

$$a_n = \frac{n}{n+1} + (-1)^n \frac{n^2+3}{n^2+7}$$

have two limits by finding $\lim_{n\to\infty} a_{2n}$ and $\lim_{n\to\infty} a_{2n+1}$.

Solution. We have

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{2n}{2n+1} + \frac{4n^2+3}{4n^2+7} = 2$$

and

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \frac{2n+1}{2n+2} - \frac{(2n+1)^2+3}{(2n+1)^2+7} = 0 \blacksquare$$

Exercise 4.15
Use the properties of this section to find

$$\lim_{n\to\infty} \frac{\sqrt{2n^2+5n}}{n+4}.$$

Solution.
We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{2n^2+5n}}{n+4} &= \lim_{n\to\infty} \frac{\sqrt{2+\frac{5}{n}}}{1+\frac{4}{n}} \\
&= \lim_{n\to\infty} \frac{\lim_{n\to\infty}\sqrt{2+\frac{5}{n}}}{\lim_{n\to\infty}(1+\frac{4}{n})} \\
&= \lim_{n\to\infty} \frac{\sqrt{\lim_{n\to\infty}(2+\frac{5}{n})}}{1+\lim_{n\to\infty}\frac{4}{n}} \\
&= \sqrt{2} \ \blacksquare
\end{aligned}$$

Exercise 4.16
Find the limit of the sequence defined by

$$a_n = n^{\frac{1}{2\ln n}}.$$

Solution.
We have $\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{1}{2\ln n} \cdot \ln n = \frac{1}{2}$. Thus, $\lim_{n\to\infty} a_n = e^{\frac{1}{2}}$ $\blacksquare$

Exercise 4.17
Consider the sequence defined by

$$a_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}.$$

(a) Show that $a_n \geq \sqrt{n}$ for all $n \in \mathbb{N}$.
(b) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is divergent. Hint: Exercise 4.11.

Solution.
(a) By induction on n. If $n = 1$ we have $a_1 = \frac{1}{\sqrt{1}} = \sqrt{1}$. Suppose that $a_n \geq \sqrt{n}$. Then $a_{n+1} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}} \geq \frac{\sqrt{n}\sqrt{n}+1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$.
(b) Since $a_n \geq \sqrt{n}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \sqrt{n} = \infty$, we conclude that $\lim_{n\to\infty} a_n = \infty$ $\blacksquare$

Exercise 4.18
Find the limit of the sequence defined by

$$a_n = \ln(2n + \sqrt{n}) - \ln n.$$

Solution.
We have $\lim_{n\to\infty}[\ln(2n+\sqrt{n}) - \ln n] = \lim_{n\to\infty} \ln\left(\frac{2n+\sqrt{n}}{n}\right) = \lim_{n\to\infty} \ln\left(2 + \frac{1}{\sqrt{n}}\right) = \ln 2$ ■

Exercise 4.19
Consider the sequence defined by $a_n = \sqrt[n]{3^n + 1}$.
(a) Show that $3 < a_n < 3\sqrt[n]{2}$ for all $n \geq 2$.
(b) Find the limit of a_n as $n \to \infty$.

Solution.
(a) We have

$$3 = \sqrt[n]{3^n} < \sqrt[n]{3^n + 1} < \sqrt[n]{3^n + 3^n} < 3\sqrt[n]{2}.$$

(b) Since $\lim_{n\to\infty} \sqrt[n]{2} = 1$ we conclude by the Squeeze rule that $\lim_{n\to\infty} a_n = 3$ ■

Exercise 4.20
Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of nonnegative terms with limit L. Suppose that the terms of sequence satisfy the recursive relation $a_n a_{n+1} = a_n + 2$ for all $N \in \mathbb{N}$. Find L.

Solution.
We have $\lim_{n\to\infty} a_n a_{n+1} = \lim_{n\to\infty}(a_n + 2) \to \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n + \lim_{n\to\infty} 2 \to L \cdot L = L + 2 \to L^2 - L - 2 = 0$. Solving this equation we find $L = -1$ and $L = 2$. But $a_n \geq 0$ for all $n \in \mathbb{N}$ so that $L \geq 0$. Thus, $L = 2$ ■

Exercise 4.21
Find the limit of the sequence defined by

$$a_n = \cos\frac{1}{n} + \frac{\sin n}{n}.$$

Solution.
We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos 1n + \lim_{n\to\infty} \frac{\sin n}{n} = 1 + 0 = 1 \blacksquare$$

Exercise 4.22
Suppose that $a_{n+1} = \frac{a_n^2+1}{a_n}$. Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is divergent.

Solution.
Suppose the contrary and let $L = \lim_{n\to\infty} a_n$. Then $L = \frac{L^2+1}{L}$ or $L^2 = L^2 + 1$ which leads to the contradiction $0 = 1$. Hence, the given sequence must be divergent ■

Solutions to Section 5

Exercise 5.1
Show that the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is decreasing.

Solution.
We have $n+1 \geq n \Rightarrow \frac{1}{n+1} \leq \frac{1}{n} \Rightarrow a_n \geq a_{n+1}$ for all $n \geq 1$. This shows that the sequence is decreasing ■

Exercise 5.2
Show that the sequence $\{\frac{1}{1+e^{-n}}\}_{n=1}^{\infty}$ is increasing.

Solution.
We have $n \leq n+1 \Rightarrow e^n \leq e^{n+1} \Rightarrow e^{-(n+1)} \leq e^{-n} \Rightarrow 1+e^{-(n+1)} \leq 1+e^{-n} \Rightarrow \frac{1}{1+e^{-n}} \leq \frac{1}{1+e^{-(n+1)}} \Rightarrow a_n \leq a_{n+1}$ for all $n \geq 1$. This shows that the sequence is increasing ■

Exercise 5.3
Show that the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is bounded from below. What is a lower bound? Are there more than one lower bound?

Solution.
Since $\frac{1}{n} \geq 0$ for all $n \geq 1$, the given sequence is bounded from below with a lower bound 0. Any negative number is a lower bound ■

Exercise 5.4
Show that the sequence $\{\frac{1}{1+e^{-n}}\}_{n=1}^{\infty}$ is bounded from above. What is an upper bound? Are there more than one upper bound?

Solution.
We have $1+e^{-n} \geq 1 \Rightarrow \frac{1}{1+e^{-n}} \leq 1$. Thus, the given sequence is bounded from above with an upper bound equals to 1. Any number larger than 1 is an upper bound ■

Exercise 5.5
Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence that is bounded from above.
(a) Show that there is a finite number M such that $M = \sup\{a_n : n \geq 1\}$.
(b) Let $\epsilon > 0$ be arbitrary. Show that $M - \epsilon$ cannot be an upper bound of the sequence.

(c) Show that there is a positive integer N such that $M - \epsilon < a_N$.
(d) Show that $M - \epsilon < a_n$ for all $n \geq N$.
(e) Show that $M - \epsilon < a_n < M + \epsilon$ for all $n \geq N$.
(f) Show that $\lim_{n\to\infty} a_n = M$. That is, the given sequence is convergent.

Solution.
(a) Since the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded from above, by the Completeness Axiom there is a finite number M such that $M = \sup\{a_n : n \geq 1\}$. We will show that $\{a_n\}_{n=1}^{\infty}$ converges to M.
(b) Let $\epsilon > 0$ be arbitrary. Consider the number $M - \epsilon$. This number is either an upper bound of the sequence or not. If it is an upper bound then we must have $M \leq M - \epsilon$ which is an impossible inequality.
(c) If such an N does not exist, then we will have $a_n \leq M - \epsilon$ for all $n \geq 1$. This means that $M - \epsilon$ is an upper bound which is impossible by (b). Thus, there is a positive integer N such that

$$M - \epsilon < a_N \leq M.$$

(d) Since the sequence is increasing, for all $n \geq N$ we can write $a_N \leq a_n$. Hence, for all $n \geq N$ we have $M - \epsilon < a_n$.
(e) For all $n \geq N$ we can write

$$M - \epsilon < a_n \leq M < M + \epsilon$$

or equivalently $|a_n - M| < \epsilon$ for all $n \geq N$.
(f) This follows from the definition of convergence and (e) ■

Exercise 5.6
Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined recursively by $a_1 = \frac{3}{2}$ and $a_{n+1} = \frac{1}{2}a_n + 1$ for $n \geq 2$.
(a) Show by induction on $n \geq 2$, that $a_{n+1} = a_n + \frac{1}{2^{n+1}}$.
(b) Show that this sequence is increasing.
(c) Show that $\{a_n\}_{n=1}^{\infty}$ is bounded from above. What is an upper bound?
(d) Show that $\{a_n\}_{n=1}^{\infty}$ is convergent. What is its limit? Hint: In finding the limit, use the arithmetic operations of sequences.

Solution.
(a) We have $a_2 = \frac{1}{2}a_1 + 1 = \frac{3}{4} + 1 = \frac{7}{4} = \frac{3}{2} + \frac{1}{4} = a_1 + \frac{1}{2^2}$. Suppose that $a_n = a_{n-1} + \frac{1}{2^n}$. Then $a_{n+1} = \frac{1}{2}a_n + 1 = \frac{1}{2}a_{n-1} + \frac{1}{2^{n+1}} + 1 = \left(\frac{1}{2}a_{n-1} + 1\right) + \frac{1}{2^{n+1}} =$

$a_n + \frac{1}{2^{n+1}}$.
(b) Since $a_{n+1} - a_n = \frac{1}{2^{n+1}} > 0$, the given sequence is increasing.
(c) By (b), we have $a_{n+1} = \frac{1}{2}a_n + 1 > a_n$. Solving this equality for a_n we find $a_n < 2$ so that the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded from above with an upper bound equals to 2.
(d) By the previous exercise, the sequence is convergent say to A. Using the arithmetic operations of sequences, we can write

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(\frac{1}{2}a_n + 1\right) = \frac{1}{2}\lim_{n\to\infty} a_n + \lim_{n\to\infty} 1.$$

Thus,

$$A = \frac{1}{2}A + 1.$$

Solving this equation for A we find $A = 2$ ■

Exercise 5.7
Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence such that $M \leq a_n$ for all $n \geq 1$. Show that $\{a_n\}_{n=1}^{\infty}$ is convergent. Hint: Let $b_n = -a_n$ and use Exercise 5.5 and Exercise 4.4.

Solution.
Since $\{a_n\}_{n=1}^{\infty}$ is decreasing, the sequence $\{b_n\}_{n=1}^{\infty}$ is increasing. Since the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded from below, the sequence $\{b_n\}_{n=1}^{\infty}$ is bounded from above. By Exercise 5.5, the sequence $\{b_n\}_{n=1}^{\infty}$ is convergent. By Exercise 4.4, the sequence $\{a_n\}_{n=1}^{\infty}$ is also convergent ■

Exercise 5.8
Show that a monotone sequence is convergent if and only if it is bounded.

Solution.
Suppose first that a sequence $\{a_n\}_{n=1}^{\infty}$ is a monotone convergent sequence. By Exercise 3.9, the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded.
Conversely, suppose that $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence. Then there is a positive constant M such that $|a_n| \leq M$ for all $n \geq 1$. By Exercise 1.14, we have $-M \leq a_n \leq M$ for all $n \geq 1$. This shows that the sequence is bounded from below as well from above. If the sequence is either increasing or decreasing, then it is convergent by Exercise 5.5 and Exercise 5.7 ■

Exercise 5.9
Let a_n be defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \in \mathbb{N}$.
(a) Show that $a_n \leq 2$ for all $n \in \mathbb{N}$. That is, $\{a_n\}_{n=1}^{\infty}$ is bounded from above.
(b) Show that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$. That is, $\{a_n\}_{n=1}^{\infty}$ is increasing.
(c) Conclude that $\{a_n\}_{n=1}^{\infty}$ is convergent. Find its limit.

Solution.
(a) By induction on n. For $n = 1$, we have $a_1 = \sqrt{2} \leq 2$. Suppose that $a_n \leq 2$. Then $a_{n+1} = \sqrt{2 + a_n} \leq \sqrt{2 + 2} = 2$. Thus, $a_n \leq 2$ for all $n \in \mathbb{N}$.
(b) By induction on n. For $n = 1$ we have $a_2 - a_1 = \sqrt{2 + \sqrt{2}} - \sqrt{2} \geq 0$. Suppose that $a_n - a_{n-1} \geq 0$. Then

$$a_{n+1} - a_n = \sqrt{2 + a_n} - \sqrt{2 + a_{n-1}} = \frac{a_n - a_{n-1}}{\sqrt{2 + a_n} + \sqrt{2 + a_{n-1}}} \geq 0.$$

(c) Since the sequence is increasing and bounded from above, it is convergent, say with limit a. Thus, we have $a = \sqrt{2 + a}$. Solving this equation we obtain $a = -1$ or $a = 2$. Since $a_1 = \sqrt{2}$ and the sequence is increasing we conclude that $a = 2$ ■

Exercise 5.10
Let $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$.
(a) Show that $a_n < 2$ for all $n \in \mathbb{N}$. Hint: Recall that $\sum_{k=1}^{n} \frac{1}{(n+1)n} = 1 - \frac{1}{n+1}$.
(b) Show that $\{a_n\}_{n=1}^{\infty}$ is increasing.
(c) Conclude that $\{a_n\}_{n=1}^{\infty}$ is convergent.

Solution.
(a) We have $a_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n} \frac{1}{k^2} \leq 1 + 1 - \frac{1}{n} < 2$.
(b) Since $a_{n+1} = a_n + \frac{1}{(n+1)^2} > a_n$, the given sequence is increasing.
(c) This follows from the fact that an increasing sequence that is bounded from above is convergent ■

Exercise 5.11
Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined recursively as follows

$$a_1 = 2 \text{ and } 7a_{n+1} = 2a_n^2 + 3 \text{ for all } n \in \mathbb{N}.$$

(a) show that $\frac{1}{2} < a_n < 3$ for all $n \in \mathbb{N}$.
(b) Show that $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$.
(c) Deduce that $\{a_n\}_{n=1}^{\infty}$ is convergent and find its limit.

Solution.
(a) We prove this by induction on n. If $n = 1$ then $\frac{1}{2} < 2 = a_1 < 3$. Suppose that $\frac{1}{2} < a_n < 3$. Then $\frac{7}{2} < 7a_{n+1} < 21 \rightarrow \frac{1}{2} < a_{n+1} < 3$.
(b) We have $a_{n+1} - a_n = \frac{1}{7}(2a_n^2 - 7a_n + 3) = \frac{1}{7}(2a_n - 1)(a_n - 3) < 0$. Thus, $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$.
(c) We deduce that the given sequence is convergent, say with limit L. Thus, $7L = 2L^2 + 3$. Solving this equation for L we find $L = \frac{1}{2}$ and $L = 3$. Since the sequence is decreasing and $\frac{1}{2} < a_n < 3$ we must have $L = \frac{1}{2}$ ■

Exercise 5.12
Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence. Define $b_n = \frac{a_1+a_2+\cdots+a_n}{n}$. Show that the sequence $\{b_n\}_{n=1}^{\infty}$ is increasing.

Solution.
note that $s_1 \leq s_2 \leq \cdots \leq s_n \leq s_{n+1}$, hence $s_1+s_2+\cdots+s_n \leq ns_{n+1}$. Adding $n(s_1 + \cdots + s_n)$ to both sides, we obtain

$$n(s_1 + \cdots + s_n) + (s_1 + \cdots + s_n) \leq n(s_1 + \cdots + s_n) + ns_{n+1},$$

or, in other words,

$$(n + 1)(s_1 + \cdots + s_n) \leq n(s_1 + \cdots + s_n + s_{n+1}),$$

Dividing both sides by $n(n + 1)$, we obtain

$$\frac{s_1 + \cdots + s_n}{n} \leq \frac{s_1 + \cdots + s_n + s_{n+1}}{n + 1},$$

or, in other words, $b_n \leq b_{n+1}$ for all n. This proves our claim ■

Exercise 5.13
Give an example of a monotone sequence that is divergent.

Solution.
One example is the sequence defined by $a_n = n$ ■

Exercise 5.14
Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = 3 + \frac{a_n}{2}$ for all $n \in \mathbb{N}$.
(a) Show that $a_n \leq 6$ for all $n \in \mathbb{N}$.
(b) Show that $\{a_n\}_{n=1}^{\infty}$ is increasing.
(c) Conclude that the sequence is convergent. Find its limit.

Solution.
(a) If $n = 1$ we have $a_1 = 1 < 6$. Suppose that $a_n \leq 6$. Then $a_{n+1} = 3 + \frac{a_n}{2} \leq 3 + \frac{6}{2} = 6$.
(b) We have $a_{n+1} = 3 + \frac{a_n}{2} = \frac{6+a_n}{2} \geq \frac{a_n+a_n}{2} = a_n$ ■
(c) Since the sequence is increasing and bounded from above it is convergent with limit say equals to L. Thus, $L = 3 + \frac{L}{2}$ and solving for L we find $L = 6$ ■

Exercise 5.15
Give an example of two monotone sequences whose sum is not monotone.

Solution.
Let $\{a_n\}_{n=1}^{\infty} = \{1, 1, 2, 2, 3, 3, \cdots\}$ and $\{b_n\}_{n=1}^{\infty} = \{-1, -2, -2, -3, -3, \cdots\}$. The first sequence is increasing and the second sequence is decreasing. However the sum is the sequence $\{0, -1, 0, -1, 0, -1, \cdots\}$ which is not monotone ■

Solutions to Section 6

Exercise 6.1
Let $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence of a sequence $\{a_n\}_{n=1}^{\infty}$. Use induction on k to show that $n_k \geq k$ for all $k \in \mathbb{N}$.

Solution. For $k = 1$ we have $n_1 \in \mathbb{N}$ so that $n_1 \geq 1$. Suppose that $n_k \geq k$. Then $n_{k+1} > n_k \geq k$. Thus, $n_{k+1} \geq k + 1$ ■

Exercise 6.2
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that converges to a number L. Let $\{a_{n_k}\}_{k=1}^{\infty}$ be any subsequence of $\{a_n\}_{n=1}^{\infty}$.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N' such that if $n \geq N'$ then $|a_n - L| < \epsilon$.
(b) Let N be the first positive integer such that $n_N \geq N'$. Show that if $k \geq N$ then $|a_{n_k} - L| < \epsilon$. That is, the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ converges to L. Hence, every subsequence of a convergent sequence is convergent to the same limit of the original sequence.

Solution.
(a) This follows from the fact that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L.
(b) Let N be such that $n_N \geq N'$. Then if $k \geq N$ we have $n_k > n_N \geq N'$. Hence, $|a_{n_k} - L| < \epsilon$. This shows that the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ converges to L ■

Exercise 6.3
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let $S = \{n \in N : a_n > a_m \text{for all } m > n\}$.
(a) Suppose that S is infinite. Then there is a sequence $n_1 < n_2 < n_3 < \cdots$ such $n_k \in S$. Show that $a_{n_{k+1}} < a_{n_k}$. Thus, the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ is decreasing.
(b) Suppose that S is finite. Let n_1 be the first positive integer such that $n_1 \notin S$. Show that the subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ is increasing.

Solution.
(a) By the definition of S we have $a_{n_{k+1}} < a_{n_k}$ since $n_{k+1} > n_k$.
(b) Let n_1 be the first positive integer such that $n_1 \notin S$. This means that there is a positive integer $n_2 > n_1$ such that $a_{n_1} < a_{n_2}$. But $n_2 \notin S$ so that there is a positive integer $n_3 > n_2$ such that $a_{n_2} < a_{n_3}$. Continuing this process we find an increasing subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ ■

Exercise 6.4 *(Bolzano-Weierstrass)*
Every bounded sequence has a convergent subsequence. Hint: Exercise 5.8

Solution.
By the previous exercise, the sequence has a bounded monotone subsequence. By Exercise 5.8 this subsequence is convergent ■

Exercise 6.5
Show that the sequence $\{e^{\sin n}\}_{n=1}^{\infty}$ has a convergent subsequence.

Solution.
Since $|\sin n| \leq 1$ we must have $\frac{1}{e} \leq e^{\sin n} \leq e$ for all $n \in \mathbb{N}$. Thus, the given sequence is bounded so that by the Bolzano-Weierstrass theorem it has a convergent subsequence ■

Exercise 6.6
Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = \cos\frac{n\pi}{2}$ is divergent.

Solution.
Consider the two subsequences $\{a_{2n+1}\}_{n=1}^{\infty}$ that converges to 0 and $\{a_4\}_{n=1}^{\infty}$ that converges to 1. Thus, the original sequence must be divergent ■

Exercise 6.7
Prove that the sequence $\{a_n\}_{n=1}^{\infty}$ where

$$a_n = \frac{(n^2+20n+35)\sin n^3}{n^2+n+1}$$

has a convergent subsequence. Hint: Show that $\{a_n\}_{n=1}^{\infty}$ is bounded.

Solution.
We have

$$\begin{aligned}|a_n| =& \left|\frac{(n^2+20n+35)\sin n^3}{n^2+n+1}\right|\\ =&\frac{(n^2+20n+35)|\sin n^3|}{n^2+n+1}\\ \leq&\frac{n^2+20n+35}{n^2+n+1}\\ \leq&\frac{n^2+20n+35}{n^2} = 1+\frac{20}{n}+\frac{35}{n^2} \leq 56\end{aligned}$$

Hence, by the Bolzano Weierstraa theorem the given sequence has a convergent subsequence ■

Exercise 6.8
Show that the sequence defined by $a_n = 2\cos n - \sin n$ has a convergent subsequence.

Solution.
We have $-2 \leq 2\cos n \leq 2$ and $-1 \leq -\sin n \leq 1$. Adding we obtain $-3 \leq a_n \leq 3$ so that the given sequence is bounded. By the Bolzano-Weierstrass theorem, the given sequence has a convergent subsequence ■

Exercise 6.9
True or false: There is a sequence that converges to 6 but contains a subsequence converging to 0. Justify your answer.

Solution.
By Exercise 6.2, this cannot happen ■

Exercise 6.10
Give an example of an unbounded sequence with a bounded subsequence.

Solution.
Let

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$$

Then $\{a_n\}_{n=1}^{\infty}$ is unbounded. However, the subsequence $\{a_{2n+1}\}_{n=1}^{\infty} = \{0, 0, 0, \cdots\}$ is bounded ■

Exercise 6.11
Show that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent by using subsequences.

Solution.
This sequence has two subsequences $\{a_{2n}\}_{n=1}^{\infty}$ that converges to 1 and $\{a_{2n+1}\}_{n=1}^{\infty}$ that converges to -1. Thus, the original sequence cannot be convergent ■

Solutions to Section 7

Exercise 7.1
Consider the sequence whose $n-$th term is given by $a_n = \frac{1}{n}$. Let $\epsilon > 0$ be arbitrary and choose $N > \frac{2}{\epsilon}$. Show that for $m, n \geq N$ we have $|a_m - a_n| < \epsilon$. That is, the above sequence is a Cauchy sequence. Hint: Exercise 1.17.

Solution.
Let $n, m \geq N$. Then by Exercise 1.17 we have

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} == \epsilon \blacksquare$$

Exercise 7.2
Show that any Cauchy sequence is bounded. Hint: Let $\epsilon = 1$ and use Exercise 1.18.

Solution.
Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Let $\epsilon = 1$. There is a positive integer N such that whenever $n, m \geq N$ we have $|a_n - a_m| < 1$. In particular, letting $m = N$ we can write $|a_n - a_N| < 1$ for all $n \geq N$. By Exercise1.18 we can write $|a_n| - |a_N| < 1$ for all $n \geq N$ or $|a_n| < 1 + |a_N|$ for all $n \geq N$. Let $M = \max\{|a_1|, |a_2|, \cdots, |a_{N-1}|, |a_N| + 1\}$. Then $|a_n| \leq M$ for all $n \geq 1$. That is, $\{a_n\}_{n=1}^{\infty}$ is bounded $\blacksquare$

Exercise 7.3
Show that if $\lim_{n\to\infty} a_n = A$ then $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Thus, every convergent sequence is a Cauchy sequence.

Solution.
Let $\epsilon > 0$ be arbitrary. Then there is a positive integer N such that $|a_n - A| < \frac{\epsilon}{2}$ for all $n \geq N$. Thus, for all $n, m \geq N$ we have

$$\begin{aligned} |a_n - a_m| =& |(a_n - A) + (A - a_m)| \leq |a_n - A| + |a_m - A| \\ <& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This shows that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence $\blacksquare$

Exercise 7.4
(a) Using Exercise 7.2, show that for each $n \geq 1$, the sequence $\{a_n, a_{n+1}, \cdots\}$ is bounded.
(b) Show that for each $n \geq 1$ the infimum of $\{a_n, a_{n+1}, \cdots\}$ exists. Call it b_n.

Solution.
(a) By Exercise 7.2, there is a positive constant M such that $|a_n| \leq M$ for all $n \geq 1$. In particular, $|a_n| \leq M, |a_{n+1}| \leq M, |a_{n+2}| \leq M, \cdots$. That is, $\{a_n, a_{n+1}, \cdots\}$ is bounded.
(b) Since $\{a_n, a_{n+1}, \cdots\}$ is bounded, it is bounded from below. By the Completeness Axiom, $b_n = \inf\{a_m : m \geq n\}$ exists ■

Exercise 7.5
(a) Show that the sequence $\{b_n\}_{n=1}^{\infty}$ is bounded from above.
(b) Show that the sequence $\{b_n\}_{n=1}^{\infty}$ is increasing. Hint: Show that b_n is a lower bound of the sequence $\{a_{n+1}, a_{n+2}, \cdots\}$.

Solution.
(a) By the definition of infimum, we have $b_n \leq a_n$ for all $n \geq 1$. Thus, $b_n \leq a_n \leq |a_n| \leq M$. This, shows that $\{b_n\}_{n=1}^{\infty}$ is bounded from above.
(b) We know that $b_n \leq a_m$ for all $m \geq n$. In particular $b_n \leq a_m$ for all $m \geq n+1$. This shows that b_n is a lower bound of $\{a_{n+1}, a_{n+2}, \cdots\}$. But b_{n+1} is the greatest lower bound of $\{a_{n+1}, a_{n+2}, \cdots\}$. Hence, $b_n \leq b_{n+1}$ and the sequence $\{b_n\}_{n=1}^{\infty}$ is increasing ■

Exercise 7.6
Show that the sequence $\{b_n\}_{n=1}^{\infty}$ is convergent. Call the limit B.

Solution.
This follows from Exercise 5.5 ■

Exercise 7.7
(a) Let $\epsilon > 0$ be arbitrary. Using the definition of Cauchy sequences and Exercise **??**, show that there is a positive integer N such that $a_N - \frac{\epsilon}{2} < a_n < a_N + \frac{\epsilon}{2}$ for all $n \geq N$.
(b) Using (a), show that $a_N - \frac{\epsilon}{2}$ is a lower bound of the sequence $\{a_N, a_{N+1}, \cdots\}$. Thus, $a_N - \frac{\epsilon}{2} \leq b_n$ for all $n \geq N$.
(c) Again, using (a) show that $b_n \leq a_N + \frac{\epsilon}{2}$ for all $n \geq N$. Thus, combining

(b) and (c), we obtain $a_N - \frac{\epsilon}{2} \le b_n < a_N + \frac{\epsilon}{2}$.
(d) Using Exercise 4.11, show that $a_N - \frac{\epsilon}{2} \le B \le a_N + \frac{\epsilon}{2}$.
(e) Using (a), (d), and Exercise 1.17, show that $\lim_{n\to\infty} a_n = B$. Thus, every Cauchy sequence is convergent.

Solution.
(a) Let $\epsilon > 0$ be arbitrary. Since $\{a_n\}_{n=1}^\infty$ is Cauchy, there is a positive integer N such that $|a_n - a_m| < \frac{\epsilon}{2}$ for all $n, m \ge N$. In particular, for all $n \ge N$ we have $|a_n - a_N| < \frac{\epsilon}{2}$ which, by Exercise 1.14, we have $a_N - \frac{\epsilon}{2} < a_n < a_N + \frac{\epsilon}{2}$ for all $n \ge N$.
(b) From part (a), we have $a_N - \frac{\epsilon}{2} < a_n$ for all $n \ge N$. This shows, that $a_N - \frac{\epsilon}{2}$ is a lower bound of the sequence $\{a_n, a_{n+1}, \cdots\}_{n=N}^\infty$ for $n \ge N$. But b_n is the greatest lower bound of the sequence $\{a_n, a_{n+1}, \cdots\}_{n=N}^\infty$ for $n \ge N$. We conclude that $a_N - \frac{\epsilon}{2} \le b_n$ for all $n \ge N$.
(c) For $n \ge N$ we have $b_n \le a_n < a_N + \frac{\epsilon}{2}$.
(d) Taking the limit as $n \to \infty$ we obtain $a_N - \frac{\epsilon}{2} \le B \le a_N + \frac{\epsilon}{2}$.
(e) Using (a), (d) and Exercise 1.17, we have that for $n \ge N$

$$\begin{aligned} |a_n - B| =& |(a_n - a_N) + (a_N - B)|| \le |a_n - a_N| + |a_N - B| \\ <& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \blacksquare \end{aligned}$$

Exercise 7.8
(a) Show that if $\{a_n\}_{n=1}^\infty$ is Cauchy then $\{a_n^2\}_{n=1}^\infty$ is also Cauchy.
(b) Give an example of Cauchy sequence $\{a_n^2\}_{n=1}^\infty$ such that $\{a_n\}_{n=1}^\infty$ is not Cauchy.

Solution.
(a) Since $\{a_n\}_{n=1}^\infty$ is Cauchy then it is convergent. Since the product of two convergent sequences is convergentm the sequence $\{a_n^2\}_{n=1}^\infty$ is convergent and therefore is Cauchy.
(a) Let $a_n = (-1)^n$ for all $n \in \mathbb{N}$. The sequence $\{a_n\}_{n=1}^\infty$ is not Cauchy since it is divergent. However, the sequence $\{a_n^2\}_{n=1}^\infty = \{1, 1, \cdots\}$ converges to 1 so it is Cauchy $\blacksquare$

Exercise 7.9
Let $\{a_n\}_{n=1}^\infty$ be a Cauchy sequence such that a_n is an integer for all $n \in \mathbb{N}$. Show that there is a positive integer N such that $a_n = C$ for all $n \ge N$, where C is a constant.

Solution.
Let $\epsilon = \frac{1}{2}$. Since $\{a_n\}_{n=1}^{\infty}$ is Cauchy, there is a positive integer N such that if $m, n \geq N$ we have $|a_m - a_n| < \frac{1}{2}$. But $a_m - a_n$ is an integer so we must have $a_n = a_N$ for all $n \geq N$ ■

Exercise 7.10
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that satisfies

$$|a_{n+2} - a_{n+1}| < c^2|a_{n+1} - a_n| \text{ for all } n \in \mathbb{N}$$

where $0 < c < 1$.
(a) Show that $|a_{n+1} - a_n| < c^n|a_2 - a_1|$ for all $n \geq 2$.
(b) Show that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Solution.
(a) The proof is by induction on n. For $n = 2$ we have $|a_3 - a_2| < c^2|a_2 - a_1|$. Suppose that $|a_{n+2} - a_{n+1}| < c^{n+1}|a_2 - a_1|$. Then $|a_{n+3} - a_{n+2}| < c^2|a_{n+2} - a_{n+1}| < c^{n+2}|a_2 - a_1|$.
(b) Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} c^n = 0$ we can find a positive integer N such that if $n \geq N$ then $|c|^n < (1-c)\epsilon$. Thus, for $n > m \geq N$ we have

$$\begin{aligned}|a_n - a_m| \leq& |a_{m+1} - a_m| + |a_{m+2} - a_{m+1}| + \cdots + |a_n - a_{n-1}|\\ <& c^m|a_2 - a_1| + c^{m+1}|a_2 - a_1| + \cdots + c^{n-1}|a_2 - a_1|\\ <& c^m(1 + c + c^2 + \cdots)|a_2 - a_1|\\ =& \frac{c^m}{1-c}|a_2 - a_1| < \epsilon\end{aligned}$$

It follows that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence ■

Exercise 7.11
What does it mean for a sequence $\{a_n\}_{n=1}^{\infty}$ to not be Cauchy?

Solution.
A sequence $\{a_n\}_{n=1}^{\infty}$ is not a Cauchy sequence if there is a real number $\epsilon > 0$ such that for all positive integer N there exist $n, m \in \mathbb{N}$ such that $n, m \geq N$ and $|a_n - a_m| \geq \epsilon$ ■

Exercise 7.12
Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two Cauchy sequences. Define $c_n = |a_n - b_n|$. Show that $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Solution.
Let $\epsilon > 0$ be given. There exist positive integers N_1 and N_2 such that if $n, m \geq N_1$ and $n, m \geq N_2$ we have $|a_n - a_m| < \frac{\epsilon}{2}$ and $|b_n - b_m| < \frac{\epsilon}{2}$. Let $N = N_1 + N_2$. If $n, m \geq N$ then $|c_n - c_m| = ||a_n - b_n| - |a_m - b_m|| \leq |(a_n - b_n) + (a_m - b_m)| \leq |a_n - a_m| + |b_n - b_m| < \epsilon$. Hence, $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence ■

Exercise 7.13
Suppose $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Suppose $a_n \geq 0$ for infinitely many n and $a_n \leq 0$ for infinitely many n. Prove that $\lim_{n\to\infty} a_n = 0$.

Solution.
Let $\epsilon > 0$ be given. Since $\{a_n\}_{n=1}^{\infty}$ is Cauchy, there is a positive integer N such that if $n, m \geq N$ then $|a_n - a_m| < \epsilon$. Let $n \geq N$. Then the element a_n is positive, negative or zero.
Case 1: Suppose $a_n \geq 0$. Since $a_m \leq 0$ for infinitely many m, there is $m \geq N$ such that $a_m \leq 0$ (else, there would be less than N nonpositive terms in the sequence, which contradicts the assumption). Since $n, m \geq N$, $a_n \leq a_n - a_m = |a_n - a_m| < \epsilon$.
Case 2: Suppose $a_n < 0$. Since $a_m geq 0$ for infinitely many m, there is $m \geq N$ such that $a_m \geq 0$ (else, there would be less than N non-negative terms in the sequence, which contradicts the assumption). But then, since $n, m \geq N, a_n < -a_n \leq -a_n + a_m \leq |a_n - a_m| < \epsilon$. So, in any case, for any given ϵ we can find $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n| < \epsilon$, that is, $\lim_{n\to\infty} a_n = 0$ ■

Exercise 7.14
Explain why the sequence defined by $a_n = (-1)^n$ is not a Cauchy sequence.

Solution.
We know that every Cauchy sequence is convergent. We also know that the given sequence is divergent. Thus, it can not be Cauchy ■

Solutions to Section 8

Exercise 8.1
Show that $\lim_{x\to 1}\frac{x^2-1}{x-1}=2$.

Solution.
Let $\epsilon>0$ be arbitrary. We note first that

$$\left|\frac{x^2-1}{x-1}-2\right|=\left|\frac{(x-1)(x+1)}{x-1}-2\right|=|x+1-2|=|x-1|.$$

Thus, choose $\delta\le\epsilon$. If $0<|x-1|<\delta$ (which is $\le\epsilon$) we obtain

$$\left|\frac{x^2-1}{x-1}-2\right|=|x-1|<\epsilon$$

which is the required result ■

Exercise 8.2
Let $f(x)=\frac{|x|}{x}$. Suppose that $\lim_{x\to 0}f(x)=L$.
(a) Show that there is a positive number δ such that if $0<|x|<\delta$ then $\left|\frac{|x|}{x}-L\right|<\frac{1}{4}$.
(b) Let $x_1=\frac{\delta}{4}$ and $x_2=-\frac{\delta}{4}$. Compute the value of $|f(x_1)-f(x_2)|$.
(c) Use (a) to show that $|f(x_1)-f(x_2)|<\frac{1}{2}$.
(d) Conclude that L does not exist. That is, $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Solution.
(a) Since $\lim_{x\to 0}f(x)=L$, for $\epsilon=\frac{1}{4}$ we can find a $\delta>0$ such that $\left|\frac{|x|}{x}-L\right|<\frac{1}{4}$ whenever $0<|x-0|<\delta$.
(b) We have $|f(x_1)-f(x_2)|=|1-(-1)|=2$.
(c) Since both x_1 and x_2 satisfy $0<|x_1|<\delta$ and $0<|x_2|<\delta$, we have $|f(x_1)-L|<\frac{1}{4}$ and $|f(x_2)-L|<\frac{1}{4}$. Thus,

$$\begin{aligned}|f(x_1)-f(x_2)|=&|(f(x_1)-L)-(f(x_2)-L)| \qquad (1)\\ \le&|f(x_1)-L|+|f(x_2)-L|\\ <&\frac{1}{4}+\frac{1}{4}=\frac{1}{2}\end{aligned}$$

But by (b), we have that $|f(x_1)-f(x_2)|=2$. We conclude that $2<\frac{1}{2}$ which in impossible.
(d) The contradiction obtained in (c) shows that $\lim_{x\to 0}\frac{|x|}{x}$ does not exist ■

Exercise 8.3
Let $f(x) = \sin\left(\frac{1}{x}\right)$. Suppose that $\lim_{x\to 0} f(x) = L$.
(a) Show that there is a positive number δ such that if $0 < |x| < \delta$ then $\left|\sin\left(\frac{1}{x}\right) - L\right| < \frac{1}{4}$.
(b) Let n be a positive integer such that $x_1 = \frac{2}{(2n+1)\pi} < \delta$ and $x_2 = \frac{1}{(2n+1)\pi} < \delta$. Compute the value of $|f(x_1) - f(x_2)|$.
(c) Use (a) to show that $|f(x_1) - f(x_2)| < \frac{1}{2}$.
(d) Conclude that L does not exist. That is, $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist.

Solution.
(a) Since $\lim_{x\to 0} f(x) = L$, for $\epsilon = \frac{1}{4}$ we can find a $\delta > 0$ such that $\left|\sin\left(\frac{1}{x}\right) - L\right| < \frac{1}{4}$ whenever $0 < |x - 0| < \delta$.
(b) We have $|f(x_1) - f(x_2)| = |-1 - 0| = 1$.
(c) Since both x_1 and x_2 satisfy $0 < |x_1| < \delta$ and $0 < |x_2| < \delta$, we have $|f(x_1) - L| < \frac{1}{4}$ and $|f(x_2) - L| < \frac{1}{4}$. Thus,

$$\begin{aligned} |f(x_1) - f(x_2)| =& |(f(x_1) - L) - (f(x_2) - L)| \\ \leq& |f(x_1) - L| + |f(x_2) - L| \\ <& \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \end{aligned} \quad (2)$$

But by (b), we have that $|f(x_1) - f(x_2)| = 1$. We conclude that $1 < \frac{1}{2}$ which in impossible.
(d) The contradiction obtained in (c) shows that $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist ■

Exercise 8.4
Suppose that $\lim_{x\to a} f(x)$ exists. Also, suppose that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} f(x) = L_2$. So either $L_1 = L_2$ or $L_1 \neq L_2$.
(a) Suppose that $L_1 \neq L_2$. Show that there exist positive constants δ_1 and δ_2 such that if $0 < |x - a| < \delta_1$ then $|f(x) - L_1| < \frac{|L_1 - L_2|}{2}$ and if $0 < |x - a| < \delta_2$ then $|f(x) - L_2| < \frac{|L_1 - L_2|}{2}$.
(b) Let $\delta = \min\{\delta_1, \delta_2\}$ so that $\delta < \delta_1$ and $\delta < \delta_2$. Show that if $0 < |x - a| < \delta$ then $|L_1 - L_2| < |L_1 - L_2|$ which is impossible.
(c) Conclude that $L_1 = L_2$. That is, whenever a function has a limit, that limit is unique.

Solution.
(a) Since $L_1 \neq L_2$, we have $\epsilon = \frac{|L_1 - L_2|}{2} > 0$. Since $\lim_{x\to a} f(x) = L_1$, there

is $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$ then $|f(x) - L_1| < \frac{|L_1 - L_2|}{2}$. Similarly, since $\lim_{x\to a} f(x) = L_2$ there is $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$ then $|f(x) - L_2| < \frac{|L_1-L_2|}{2}$.
(b) Suppose $0 < |x-a| < \delta$. Then

$$\begin{aligned}|L_1 - L_2| =&|f(x) - L_2 + L_1 - f(x)|\\ \leq&|f(x) - L_1| + |f(x) - L_2|\\ <&\frac{|L_1 - L_2|}{2} + \frac{|L_1 - L_2|}{2} = |L_1 - L_2|\end{aligned}$$

(c) The contradiction obtained in (b) implies $L_1 = L_2$ ■

Exercise 8.5
Using the $\epsilon\delta$ definition of limit show that

$$\lim_{x\to -1}(2x^2 + x + 1) = 2.$$

Solution.
Let $\epsilon > 0$ be given. Note first that $|2x^2 + x + 1 - 2| = |(2x-1)(x+1)| = |2x-1||x+1| \leq (2|x|+1)|x+1|$. If $0 < |x+1| < 1$ then $|x| - 1 < 1$ and this implies $|x| < 2$. So choose $\delta = \min\{1, \frac{\epsilon}{5}\}$. Clearly, $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{5}$. So if $0 < |x+1| < \delta$ we have $|2x^2+x+1-2| \leq (2|x|+1)|x+1| < 5|x+1| \leq 5\cdot\frac{\epsilon}{5} = \epsilon$ ■

Exercise 8.6
Prove directly from the definition that $\lim_{x\to 1}\frac{x}{x+3} = \frac{1}{4}$.

Solution.
First note that

$$\left|\frac{x}{x+3} - \frac{1}{4}\right| = \frac{3}{4}\frac{|x-1|}{|x+3|}.$$

If $|x-1| < 1$ then $0 < x < 2$. Thus, $|x+3| = x+3 > 3$. Let $\epsilon > 0$ be given. Choose $\delta = \min\{1, 4\epsilon\}$. Then if $0 < |x-1| < \delta$ we have $\left|\frac{x}{x+3} - \frac{1}{4}\right| = \frac{3}{4}\frac{|x-1|}{|x+3|} < \frac{3}{4}\frac{|x-1|}{3} = \frac{|x-1|}{4} < \frac{4\epsilon}{4} = \epsilon$ ■

Exercise 8.7
In this exercise we discuss the concept of sided limits.
(a) We say that L is the **left side limit** of f as x approaches a from the left if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon$$

and we write $\lim_{x \to a^-} f(x) = L$. Show that $\lim_{x \to 0^-} \frac{|x|}{x} = -1$.
(b) We say that L is the **right side limit** of f as x approaches a from the right if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon$$

and we write $\lim_{x \to a^+} f(x) = L$. Show that $\lim_{x \to 0^+} \frac{|x|}{x} = 1$.

Exercise 8.8
Prove that $L = \lim_{x \to a} f(x)$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

Solution. Suppose first that $\lim_{x \to a} f(x) = L$. Let $\epsilon > 0$ be given. Then there is $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$. Suppose that $0 < a - x < \delta$. Then $0 < a - x = |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$. This shows that $L = \lim_{x \to a^-} f(x)$. Likewise, one can show that $L = \lim_{x \to a^+} f(x)$.
Conversely, suppose $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$. Let $\epsilon > 0$ be given. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $0 < a - x < \delta_1 \Rightarrow |f(x) - L| < \epsilon$ and $0 < x - a < \delta_2 \Rightarrow |f(x) - L| < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Suppose $0 < |x - a| < \delta$. Since $|x - a| = \pm(x - a)$ we must have $|f(x) - L| < \epsilon$. This shows that $\lim_{x \to a} f(x) = L$ ■

Exercise 8.9
Using ϵ and δ, what does it mean that $\lim_{x \to a} f(x) \neq L$?

Solution. If $\lim x \to af(x) \neq L$ then there is an $\epsilon > 0$ such that for all $\delta > 0$ there is an x in the domain of f such that $0 < |x - a| < \delta$ but $|f(x) - L| \geq \epsilon$ ■

Solutions to Section 9

Exercise 9.1
Suppose that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. Show that

$$\lim_{x\to a}[f(x) \pm g(x)] = L_1 \pm L_2.$$

Solution.
Let $\epsilon > 0$ be arbitrary. Since $\lim_{x\to a} f(x) = L_1$, we can find $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \Longrightarrow |f(x) - L_1| < \frac{\epsilon}{2}.$$

Similarly, since $\lim_{x\to a} g(x) = L_2$, we can find $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \Longrightarrow |g(x) - L_2| < \frac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Notice that $\delta \le \delta_1$ and that $\delta \le \delta_2$. Thus, if $0 < |x-a| < \delta$ then

$$\begin{aligned}|(f(x)+g(x)) - (L_1+L_2)| =&|(f(x)-L_1) + (g(x)-L_2)|\\ \le&|f(x)-L_1| + |g(x)-L_2|\\ <&\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

A similar argument holds for $f(x) - g(x)$ ■

Exercise 9.2
Suppose that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. Show the following:
(a) There is a $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \Longrightarrow |f(x)| < 1 + |L_1|.$$

Hint: Notice that $f(x) = (f(x) - L_1) + L_1$.
(b) Given $\epsilon > 0$, there is a $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \Longrightarrow |f(x) - L_1| < \frac{\epsilon}{2(1+|L_2|)}.$$

Solution.
(a) Let $\epsilon = 1$. Since $\lim_{x\to a} f(x) = L_1$, we can find $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \Longrightarrow |f(x) - L_1| < 1.$$

But then

$$|f(x)| = |(f(x) - L_1) + L_1| \leq |f(x) - L_1| + |L_1| < 1 + |L_1|.$$

(b) Since $\frac{\epsilon}{2(1+|L_2|)} > 0$ and $\lim_{x\to a} f(x) = L_1$, we can find $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \Longrightarrow |f(x) - L_1| < \frac{\epsilon}{2(1 + |L_2|)} \blacksquare$$

Exercise 9.3
Suppose that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$.
(a) Show that $f(x)g(x) - L_1L_2 = f(x)(g(x) - L_2) + L_2(f(x) - L_1)$.
(b) Show that $|f(x)g(x) - L_1L_2| \leq |f(x)||g(x) - L_2| + |L_2||f(x) - L_1|$.
(c) Show that $\lim_{x\to a} f(x)g(x) = L_1L_2$. Hint: Use the previous exercise.

Solution.
(a) We have

$$\begin{aligned} f(x)(g(x) - L_2) + L_2(f(x) - L_1) =& f(x)g(x) - L_2f(x) + L_2f(x) - L_1L_2 \\ =& f(x)g(x) - L_1L_2. \end{aligned}$$

(b) By (a) and Exercise 1.7/Exercise1.17, we have

$$\begin{aligned} |f(x)g(x) - L_1L_2| =& |f(x)(g(x) - L_2) + L_2(f(x) - L_1)| \\ \leq& |f(x)(g(x) - L_2)| + |L_2(f(x) - L_1)| \\ =& |f(x)||(g(x) - L_2)| + |L_2||(f(x) - L_1)| \end{aligned}$$

(c) Let $\epsilon > 0$ be arbitrary. By Exercise 9.2(a), there is $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \Longrightarrow |f(x)| < 1 + |L_1|.$$

Also, by Exercise 9.2(b), there exist $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$0 < |x - a| < \delta_2 \Longrightarrow |f(x) - L_1| < \frac{\epsilon}{2(1 + |L_2|)}$$

and

$$0 < |x - a| < \delta_3 \Longrightarrow |g(x) - L_2| < \frac{\epsilon}{2(1 + |L_1|)}.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Notice that $\delta \leq \delta_1, \delta \leq \delta_2$, and $\delta \leq \delta_3$. Suppose that $0 < |x - a| < \delta$. Using (b) and the above inequalities we find

$$\begin{aligned}
|f(x)g(x) - L_1L_2| \leq & |f(x)||g(x) - L_2| + |L_2||f(x) - L_1| \\
< & (1 + |L_1|) \cdot \frac{\epsilon}{2(1 + |L_1|)} + |L_2| \cdot \frac{\epsilon}{2(1 + |L_2|)} \\
< & (1 + |L_1|) \cdot \frac{\epsilon}{2(1 + |L_1|)} + (1 + |L_2|)| \cdot \frac{\epsilon}{2(1 + |L_2|)} \\
= & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

This establishes the required result ■

Exercise 9.4
(a) Suppose that $|f(x)| \leq M$ for all x in its domain and $\lim_{x \to a} g(x) = 0$. Show that

$$\lim_{x \to a} f(x)g(x) = 0.$$

Hint: Recall Exercise 4.5
(b) Show that $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

Solution.
(a) Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \to a} g(x) = 0$, there is a positive number δ such that if $0 < |x - a| < \delta$ then $|g(x) - 0| = |g(x)| < \frac{\epsilon}{M}$. Thus, for $0 < |x - a| < \delta$ we have

$$|f(x)g(x) - 0| = |f(x)g(x)| = |f(x)||g(x)| < M \cdot \frac{\epsilon}{M} = \epsilon.$$

This shows that $\lim_{x \to a} f(x)g(x) = 0$.
(b) Let $f(x) = x$ and $g(x) = \sin\left(\frac{1}{x}\right)$. Note that $\lim_{x \to 0} f(x) = 0$ and $|g(x)| = \left|\sin\left(\frac{1}{x}\right)\right| \leq 1$. It follows from (a) that

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = \lim_{x \to 0} f(x)g(x) = 0 \ ■$$

Exercise 9.5
Suppose that $\lim_{x\to a} f(x) = L$ with $L \neq 0$. Show that there exists a $\delta > 0$ such that
$$0 < |x-a| < \delta \Longrightarrow |f(x)| > \frac{|L|}{2} > 0.$$
Hint: Recall the solution to Exercise 4.7

Solution.
Let $\epsilon = \frac{|L|}{2} > 0$. Then there exists $\delta > 0$ such that $|f(x) - L| < \frac{|L|}{2}$ whenever $0 < |x-a| < \delta$. But by Exercise 1.18, we have $|L| - |f(x)| < \frac{|L|}{2}$ or $|f(x)| > \frac{|L|}{2}$ whenever $0 < |x-a| < \delta$ ■

Exercise 9.6
Let $g(x)$ be a function with the following conditions:
(1) $g(x) \neq 0$ for all x in the domain of g.
(2) $\lim_{x\to a} g(x) = L_2$, with $L_2 \neq 0$.
(a) Show that there is a $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$ then
$$\left|\frac{1}{g(x)} - \frac{1}{L_2}\right| < \frac{2}{|L_2|^2}|g(x) - L_2|.$$
(b) Let $\epsilon > 0$ be arbitrary. Show that there is $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$ then
$$|g(x) - L_2| < \frac{|L_2|^2}{2}\epsilon.$$
(c) Using (a) and (b), show that
$$\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{L_2}.$$
Hint: Recall Exercise 4.8

Solution.
(a) From Exercise 9.5, we can find $\delta_1 > 0$ such that $|g(x)| > \frac{|L_2|}{2}$ whenever $0 < |x-a| < \delta_1$. Thus, for $0 < |x-a| < \delta_1$ we obtain
$$\left|\frac{1}{g(x)} - \frac{1}{L_2}\right| = \frac{|g(x) - L_2|}{|g(x)||L_2|} < \frac{2}{|L_2|^2}|g(x) - L_2|.$$

(b) Let $\epsilon > 0$ be arbitrary. Since $\lim_{x\to a} g(x) = L_2$, we can find a positive number δ_2 such that if $0 < |x-a| < \delta_2$ then

$$|g(x) - L_2| < \frac{|L_2|^2}{2}\epsilon.$$

(c) Let $\epsilon > 0$ be arbitrary. Let $\delta = \min\{\delta_1, \delta_2\}$. Notice that $\delta \le \delta_1$ and $\delta \le \delta_2$. Now, if $0 < |x-a| < \delta$ then

$$\left|\frac{1}{g(x)} - \frac{1}{L_2}\right| = \frac{|g(x) - L_2|}{|g(x)||L_2|} < \frac{2}{|L_2|^2}|g(x) - L_2| < \frac{2}{|L_2|^2}\frac{|L_2|^2}{2}\epsilon = \epsilon.$$

This establishes the required result ■

Exercise 9.7
Show that if $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ where $g(x) \neq 0$ in its domain and $L_2 \neq 0$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

Hint: Recall Exercise 4.10.

Solution.
Using Exercise 9.6 and Exercise 9.3 we can write

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a}\left[f(x) \cdot \frac{1}{g(x)}\right] = L_1 \cdot \frac{1}{L_2} = \frac{L_1}{L_2} \ \blacksquare$$

Exercise 9.8
Let $f(x)$ and $g(x)$ be two functions with a common domain D and a a point in D. Suppose that $f(x) \le g(x)$ for all x in D. Show that if $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ then $L_1 \le L_2$. Hint: Recall Exercise 4.11

Solution.
Since $\lim_{x\to a} f(x) = L_1$, there exists a positive number δ_1 such that $|f(x) - L_1| < \epsilon$ whenever $0 < |x-a| < \delta_1$. By Exercise 1.14, this is equivalent to $L_1 - \epsilon < f(x) < L_1 + \epsilon$ whenever $0 < |x-a| < \delta_1$. Similarly, since $\lim_{x\to a} g(x) = L_2$, there exists a positive number δ_2 such that $|g(x) - L_2| < \epsilon$ whenever $0 < |x-a| < \delta_2$. By Exercise 1.14, this is equivalent to $L_2 - \epsilon <$

$g(x) < L_2 + \epsilon$ whenever $0 < |x - a| < \delta_2$.
Suppose that $L_1 > L_2$. Let $\epsilon = \frac{L_1 - L_2}{2}$. We have

$$\begin{aligned} g(x) <& L_2 + \epsilon = L_2 + \frac{L_1 - L_2}{2} = \frac{L_1 + L_2}{2} \\ =& L_1 - \frac{L_1 - L_2}{2} = L_1 - \epsilon < f(x) \end{aligned}$$

This contradicts the fact that $f(x) \leq g(x)$ for all x in D. Hence, we conclude that $L_1 \leq L_2$ ■

Exercise 9.9
Let D be the domain of a function $f(x)$. Suppose that $f(x) \geq 0$ for all x in D and $\lim_{x \to a} f(x) = L$ with $L > 0$.
(a) Show that

$$\sqrt{f(x)} - \sqrt{L} = \frac{f(x) - L}{\sqrt{f(x)} + \sqrt{L}}.$$

(b) Let $\epsilon > 0$. Show that there exists $\delta > 0$ such that $|f(x) - L| < \epsilon\sqrt{L}$ whenever $0 < |x - a| < \delta$.
(c) Show that

$$\lim_{x \to a} \sqrt{f(x)} = \sqrt{L}.$$

Solution.
(a) We have

$$\sqrt{f(x)} - \sqrt{L} = \frac{(\sqrt{f(x)} - \sqrt{L})(\sqrt{f(x)} + \sqrt{L})}{\sqrt{f(x)} + \sqrt{L}} = \frac{f(x) - L}{\sqrt{f(x)} + \sqrt{L}}.$$

(b) Since $\lim_{x \to a} f(x) = L$ and $\epsilon\sqrt{L} > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon\sqrt{L}$ whenever $0 < |x - a| < \delta$.
(c) For $0 < |x - a| < \delta$ we have

$$|\sqrt{f(x)} - \sqrt{L}| = \left| \frac{f(x) - L}{\sqrt{f(x)} + \sqrt{L}} \right| \leq \frac{|f(x) - L|}{\sqrt{L}} < \frac{\epsilon\sqrt{L}}{\sqrt{L}} = \epsilon.$$

This establishes a proof of the required result ■

Exercise 9.10 *(Squeeze Rule)*
Let $f(x), g(x)$ and $h(x)$ be three functions with common domain D and a be a point in D. Suppose that
(1) $g(x) \leq f(x) \leq h(x)$ for all x in D.
(2) $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$.
Show that $\lim_{x\to a} f(x) = L$. Hint: Recall Exercise 3.11

Solution.
Let $\epsilon > 0$. From (2) we can find $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then

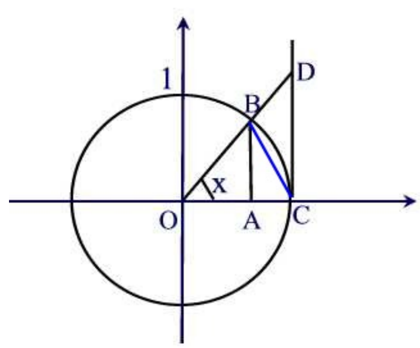
$$|g(x) - L| < \epsilon \Leftrightarrow L - \epsilon < g(x) < L + \epsilon.$$

Similarly, we can find $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then

$$|h(x) - L| < \epsilon \Leftrightarrow L - \epsilon < h(x) < L + \epsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Suppose $0 < |x - a| < \delta$. Then $L - \epsilon < g(x) \leq f(x) \leq h(x) < L + \epsilon$. That is $L - \epsilon < f(x) < L + \epsilon$ whenever $0 < |x - a| < \delta$. This implies that $\lim_{x\to a} f(x) = L$ ■

Exercise 9.11
Consider the following figure.



where $0 < x < \frac{\pi}{2}$.
(a) Using geometry, establish the inequality

$$0 < \sin x < x.$$

Hint: The area of a circular sector with radius r and central angle θ is given by the formula $\frac{1}{2}r^2\theta$.

(b) Show that $\lim_{x\to 0^+} \sin x = 0$.
(c) Show that $\lim_{x\to 0^-} \sin x = 0$. Thus, we conclude that $\lim_{x\to 0} \sin x = 0$.
Hint: Recall that the sine function is an odd function.
(d) Show that $\lim_{x\to 0} \cos x = 1$. Hint: $\cos^2 x + \sin^2 x = 1$.
(e) Using geometry, establish the double inequality

$$\frac{\sin x \cos x}{2} < \frac{x}{2} < \frac{\tan x}{2}.$$

(f) Using (a) show that

$$\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}.$$

(g) Show that

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

(h) Show that for $-\frac{\pi}{2} < x < 0$ we have also

$$\lim_{x\to 0^-} \frac{\sin x}{x} = 1.$$

Solution.
(a) The area of the triangle OBC is smaller than the circular area. Thus

$$0 < \frac{\sin x}{2} < \frac{x}{2}$$

or

$$0 < \sin x < x.$$

(b) This follows from (a) and the squeeze rule.
(c) For $x < 0$ we can write

$$\lim_{x\to 0^-} \sin x = \lim_{x\to 0^-} -\sin(-x) = -\lim_{-x\to 0^+} \sin(-x) = 0.$$

(d) Since $\cos x \geq 0$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ we have $\cos x = \sqrt{1 - \sin^2 x}$. Thus, using Exercise 9.9, we find

$$\lim_{x\to 0} \cos x = \lim_{x\to 0} \sqrt{1 - \sin^2 x} = \sqrt{\lim_{x\to 0}(1 - \sin^2 x)} = \sqrt{1 - 0} = 1.$$

(e) The area of the triangle OAB is $\frac{\sin x \cos x}{2}$. The area of the triangle OCD is $\frac{\tan x}{2}$. From the graph we see that

Area of triangle OAB < Area of circular sector < Area of triangle OCD.
Thus,

$$\frac{\sin x \cos x}{2} < \frac{x}{2} < \frac{\tan x}{2}.$$

(f) Since $0 < x < \frac{\pi}{2}$, $\sin x > 0$. Simple algebra leads to

$$\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}.$$

(g) Notice that $\lim_{x\to 0} \cos x = 1$ and $\lim_{x\to 0} \frac{1}{\cos x} = 1$. Now the result follows by applying the squeeze rule.
(h) Since $\sin x = -\sin(-x)$ and $0 < -x < \frac{\pi}{2}$ for $-\frac{\pi}{2} < x < 0$, we can use (g) to obtain

$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{-x\to 0^+} \frac{\sin(-x)}{-x} = 1 \blacksquare$$

Exercise 9.12
Find each of the following limits:
(1) $\lim_{x\to 1} \frac{\sqrt{x^2+3}-2\sqrt{x}}{x^2-1}$.
(2) $\lim_{x\to 2^-} \frac{x-2}{|x^2-5x+6|}$.
Solution.
(a) We have

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{x^2+3}-2\sqrt{x}}{x^2-1} &= \lim_{x\to 1} \frac{(\sqrt{x^2+3}-2\sqrt{x})(\sqrt{x^2+3}+2\sqrt{x}}{(\sqrt{x^2+3}+2\sqrt{x})(x^2-1)} \\
&= \lim_{x\to 1} \frac{x^2-4x+3}{(\sqrt{x^2+3}+2\sqrt{x})(x^2-1)} \\
&= \lim_{x\to 1} \frac{(x-1)(x-3)}{(\sqrt{x^2+3}+2\sqrt{x})(x-1)(x+1)} \\
&= \lim_{x\to 1} \frac{x-3}{(\sqrt{x^2+3}+2\sqrt{x})(x+1)} = -\frac{1}{4}
\end{aligned}$$

(2) We have

$$\begin{aligned}
\lim_{x\to 2^-} \frac{x-2}{|x^2-5x+6|} &= \lim_{x\to 2^-} \frac{x-2}{|(x-2)(x-3)|} \\
&= \lim_{x\to 2^-} \frac{x-2}{(x-2)(x-3)} \\
&= \lim_{x\to 2^-} \frac{1}{x-3} = -1 \blacksquare
\end{aligned}$$

Exercise 9.13
Find $\lim_{x\to\infty} \frac{x^2+x}{x^2-x}$ by using the change of variable $u = \frac{1}{x}$.

Solution.
By letting $u = \frac{1}{x}$ we find

$$\lim_{x\to\infty} \frac{x^2+x}{x^2-x} = \lim_{u\to 0} \frac{1+u}{1-u} = 1 \blacksquare$$

Exercise 9.14
Find $\lim_{x\to 0} \sqrt[3]{x} \sin \frac{1}{x}$.

Solution.
We have for $x > 0$

$$-\sqrt[3]{x} \le \sqrt[3]{x} \sin \frac{1}{x} \le \sqrt[3]{x}$$

and for $x < 0$

$$-\sqrt[3]{x} \ge \sqrt[3]{x} \sin \frac{1}{x} \ge \sqrt[3]{x}$$

By the squeeze rule we conclude that $\lim_{x\to 0} \sqrt[3]{x} \sin \frac{1}{x} = 0$ $\blacksquare$

Exercise 9.15
Find $\lim_{x\to 0} x^2 \tan x$.

Solution.
For $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ we have $-1 \le \tan x \le 1$. Thus, $-x^2 \le x^2 \tan x \le x^2$. By the squeeze rule we conclude that $\lim_{x\to 0} x^2 \tan x = 0$ $\blacksquare$

Exercise 9.16
Let n be a positive integer. Prove that $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$.

Solution.
Follows by a simple induction on n in Exercise 9.3 $\blacksquare$

Solutions to Section 10

Exercise 10.1
Suppose that $\lim_{x\to a} f(x) = L$, where a is in the domain of f. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence whose terms belong to the domain of f and are different from a and suppose that $\lim_{n\to\infty} a_n = a$.
(a) Let $\epsilon > 0$ be arbitrary. Show that there exist a positive integer N and a positive number δ such that for $n \geq N$ we have $|a_n - a| < \delta$ and for $0 < |x - a| < \delta$ we have $|f(x) - L| < \epsilon$.
(b) Use (a) to conclude that for a given $\epsilon > 0$ there is a positive integer N such that if $n \geq N$ then $|f(a_n) - L| < \epsilon$. That is, $\lim_{n\to\infty} f(a_n) = L$.

Solution.
(a) Let $\epsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, we can find a positive constant δ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$. But for $\delta > 0$ we can find a positive integer N such that for $n \geq N$ we have $|a_n - a| < \delta$ due to the fact that $\lim_{n\to\infty} a_n = a$. Since $a_n \neq a$ for all $n \geq 1$ we conclude that $0 < |a_n - a| < \delta$ for all $n \geq N$.
(b) Using (a), for a given $\epsilon > 0$ we can find a positive integer N such that if $n \geq N$ then $0 < |a_n - a| < \delta$ which implies that $|f(a_n) - L| < \epsilon$. This establishes that

$$\lim_{n\to\infty} f(a_n) = L \blacksquare$$

Exercise 10.2
Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence of terms in the domain of f with $a_n \neq a$ for all $n \geq 1$. Suppose that if $\lim_{n\to\infty} a_n = a$ then $\lim_{n\to\infty} f(a_n) = L$. Clearly, either

$$\lim_{x\to a} f(x) = L \text{ or } \lim_{x\to a} f(x) \neq L.$$

(a) Suppose first that $\lim_{x\to a} f(x) \neq L$. Show that there is an $\epsilon > 0$ and a sequence $\{a_n\}_{n=1}^{\infty}$ of terms in the domain of f such that $0 < |a_n - a| < \frac{1}{n}$ and $|f(a_n) - L| \geq \epsilon$.
(b) Use the squeeze rule to show that $\lim_{n\to\infty} |a_n - a| = 0$.
(c) Use the fact that $-|a| \leq a \leq |a|$ for any number a and the squeeze rule to show that $\lim_{n\to\infty}(a_n - a) = 0$.
(d) Use Exercise 4.1 to show that $\lim_{n\to\infty} a_n = a$.
(e) Using (a), (d), the given hypothesis and Exercise 4.11, show that $\epsilon \leq 0$.

Thus, this contradiction shows that $\lim_{x\to a} f(x) \neq L$ cannot happen. We conclude that

$$\lim_{x\to a} f(x) = L$$

Solution.
(a) Since $\lim_{x\to a} f(x) \neq L$, there is $\epsilon > 0$ such that for every $n \geq 1$, there is a_n in the domain of f such that $0 < |a_n - a| < \frac{1}{n}$ but $|f(a_n) - L| \geq \epsilon$.
(b) Let $b_n = 0, c_n = |a_n - a|$, and $d_n = \frac{1}{n}$. Then we have $b_n < c_n < d_n$. Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$. By the squeeze rule we obtain $\lim_{n\to\infty} |a_n - a| = 0$.
(c) Since $-|a_n - a| \leq a_n - a \leq |a_n - a|$, by the squeeze rule we find $\lim_{n\to\infty}(a_n - a) = 0$.
(d) Since $a_n = (a_n - a) + a$, we can use Exercise 4.1 to obtain $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(a_n - a) + \lim_{n\to\infty} a = 0 + a = a$.
(e) According to (a) and (d), we can find a sequence $\{a_n\}_{n=1}^{\infty}$ of terms in the domain of f such that $a_n \neq a$ for all $n \geq 1$, $|f(a_n) - L| \geq \epsilon$. By hypothesis, $\lim_{n\to\infty} f(a_n) = L$. By Exercise 4.11, $\epsilon \leq 0$, a contradiction. Hence, we must have

$$\lim_{x\to a} f(x) = L \blacksquare$$

Exercise 10.3
Let f be a function with domain D and a be a point in D. Suppose that f satisfies the following Property:

(P) If $\{a_n\}_{n=1}^{\infty}$, with a_n in D, $a_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} a_n = a$ then $\{f(a_n)\}_{n=1}^{\infty}$ is a Cauchy sequence.

(a) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements of D such that $a_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} a_n = a$. Show that the sequence $\{f(a_n)\}_{n=1}^{\infty}$ is convergent. Call the limit L. Hint: See Exercise 7.7
(b) Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of elements of D such that $b_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} b_n = a$. Show that the sequence $\{f(b_n)\}_{n=1}^{\infty}$ converges to L'.

Solution.
(a) By Property (P) the sequence $\{f(a_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. By Exerccise 7.7, the sequence $\{f(a_n)\}_{n=1}^{\infty}$ converges, say with limit L.
(b) By an argument similar to (a), there is a number L' such that $\lim_{n\to\infty} f(b_n) = L'$ $\blacksquare$

Exercise 10.4

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be the two sequences of the previous exercise. Define the sequence

$$\{c_n\} = \{b_1, a_1, b_2, a_2, b_3, a_3, \cdots\}.$$

That is, $c_n = a_k$ if $n = 2k$ and $c_n = b_k$ if $n = 2k + 1$ where $k \geq 0$.
(a) Show that for all $n \geq$ we have $c_n \in D$ and $c_n \neq a$.
(b) Let $\epsilon > 0$. Show that there exist positive integers N_1 and N_2 such that if $n \geq N_1$ then $|a_n - a| < \epsilon$ and if $n \geq N_2$ then $|b_n - a| < \epsilon$.
(c) Let $N = 2N_1 + 2N_2 + 1$. Show that if $n \geq N$ then $|c_n - a| < \epsilon$. Hence, $\lim_{n\to\infty} c_n = a$. Hint: Consider the cases $n = 2k$ or $n = 2k + 1$.
(d) Show that $\lim_{n\to\infty} f(c_n) = L''$ for some number L''.

Solution.

(a) Since c_n is either a_n or b_n and both belong to D and are different from a, we conclude that c_n belongs to D and $c_n \neq a$ for all $n \geq 1$.
(b) Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = a$ we can find positive integers N_1 and N_2 such that if $n \geq N_1$ we have $|a_n - a| < \epsilon$ and if $n \geq N_2$ we have $|b_n - a| < \epsilon$.
(c) Let $N = 2N_1 + 2N_2 + 1$. Let $n \geq N$. If $n = 2k + 1$ then $2k + 1 \geq N \geq 2N_2 + 1 \to k \geq N_2$ so that $|c_n - a| = |b_k - a| < \epsilon$. If $n = 2k \geq N \geq 2N_1$ then $k \geq N_1$ and in this case $|c_n - a| = |a_k - a| < \epsilon$. Thus, whether n is even or odd we have $|c_n - a| < \epsilon$. It follows that

$$\lim_{n\to\infty} c_n = a.$$

(d) With an argument similar to Exercise 10.2, we conclude that $\lim_{n\to\infty} f(c_n) = L''$ for some number L'' ■

Exercise 10.5

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be as in the previous exercise.
(a) Compare $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$.
(b) Let $\epsilon > 0$ be arbitrary. Show that there is a positive integer N such that if $n \geq N$ then $|f(c_n) - L''| < \epsilon$.
(c) Let N_1 be a positive integer such that $N_1 \geq \frac{N}{2}$. Show that if $n \geq N_1$ then $|f(a_n) - L''| < \epsilon$. Hence, $\lim_{n\to\infty} f(a_n) = L''$.
(d) Show that $\lim_{n\to\infty} f(b_n) = L''$. Thus, by Exercise 3.6, we must have $L = L' = L''$.

Solution.
(a) First note that $\{a_n\}_{n=1}^{\infty} = \{c_{2n}\}_{n=1}^{\infty}$.
(b) Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} f(c_n) = L''$, there is a positive integer N such that if $n \geq N$ we have $|f(c_n) - L''| < \epsilon$.
(c) Let N_1 be a positive integer such that $N_1 \geq \frac{N}{2}$. For $n \geq N_1 \to 2n \geq N \to |f(a_n) - L''| = |f(c_{2n}) - L''| < \epsilon$. Hence, $\lim_{n\to\infty} f(a_n) = L''$. by Exercise 3.6, we obtain $L = L''$.
(d) First note that $\{b_n\}_{n=1}^{\infty} = \{c_{2n-1}\}_{n=1}^{\infty}$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} f(c_n) = L''$, there is a positive integer N such that if $n \geq N$ we have $|f(c_n) - L''| < \epsilon$. Let N_2 be a positive integer such that $N_2 \geq \frac{N+1}{2}$. For $n \geq N_2 \to 2n - 1 \geq N \to |f(b_n) - L''| = |f(c_{2n-1}) - L''| < \epsilon$. Hence, $\lim_{n\to\infty} f(b_n) = L''$. by Exercise 3.6, we obtain $L = L''$ ■

Exercise 10.6
Prove that if a function f satisfies property (P) then $\lim_{x\to a} f(x)$ exists. Hint: Use Exercise 10.2.

Solution.
This follows from Exercise 10.2 and Exercise 10.5 ■

Exercise 10.7
Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be the two sequences defined by $a_n = \frac{1}{2n\pi}$ and $b_n = \frac{1}{(2n+\frac{1}{2})\pi}$. Clearly, $a_n, b_n \neq 0$ for all $n \in \mathbb{N}$, $a_n \to 0$ and $b_n \to 0$. Show that $\lim_{x\to 0} f(x)$ does not exist.

Solution.
Since $\lim_{n\to\infty} f(a_n) = 0$ and $\lim_{n\to\infty} f(b_n) = 1$, by Exercise 10.1, $\lim_{x\to 0} f(x)$ does not exist ■

Exercise 10.8
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_n \neq 2$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 2$.
(a) Find $\lim_{n\to\infty} \frac{a_n^2-4}{a_n-2} = 4$.
(b) Find $\lim_{x\to 2} \frac{x^2-4}{x-2}$.

Solution.
(a) We have

$$\lim_{n\to\infty} \frac{a_n^2 - 4}{a_n - 2} = \lim_{n\to\infty} (a_n + 2) = 4.$$

(b) By Exercise 10.2, we have

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = 4 \blacksquare$$

Exercise 10.9
Consider the floor function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ denote the largest integer less than or equal to x. Find $\lim_{x\to 1}\lfloor x \rfloor$ using sequences.

Solution.
Let $\{a_n\}_{n=1}^{\infty} \subset [0, 1]$ such that $a_n \neq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = 1$. Since $a_n \in (0, 1)$ we have $f(a_n) = 0$ for all $n \in \mathbb{N}$. Thus, $\lim_{n\to\infty} f(a_n) = 0$. We conclude that $\lim_{x\to 1} f(x) = 0 \blacksquare$

Exercise 10.10
Consider the floor function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ denote the largest integer less than or equal to x.
(a) Let $a_n = 1 - \frac{1}{n}$ and $b_n = 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$. Find $\lim_{n\to\infty} f(a_n)$ and $\lim_{n\to\infty} f(b_n)$.
(b) Does $\lim_{x\to 1}\lfloor x \rfloor$ exist?

Solution.
(a) $\lfloor f(a_n) \rfloor = 0$ for all $n \in \mathbb{N}$ so $\lim_{n\to\infty} f(a_n) = 0$. Likewise, $\lfloor f(b_n) \rfloor = 1$ for all $n \in \mathbb{N}$ so $\lim_{n\to\infty} f(b_n) = 1$.
(b) By Exercise **??**, $\lim_{x\to 1}\lfloor x \rfloor$ does not exist $\blacksquare$

Solutions to Section 11

Exercise 11.1
Show that the function $f(x) = x^2$ is continuous at $x = 0$.

Solution.
Let $\epsilon > 0$ be arbitrary. Let $0 < \delta < \sqrt{\epsilon}$. Then for any x such that $|x - 0| < \delta$ we have $|x^2 - 0| = |x|^2 < \delta^2 < \epsilon$. This shows that $f(x) = x^2$ is continuous at $x = 0$ ■

Exercise 11.2
Show that f is continuous at $x = a$ if and only if $\lim_{x\to a} f(x) = f(a)$.

Solution.
Suppose first that f is continuous at $x = a$. Let $\epsilon > 0$ be arbitrary. Then, there is a $\delta > 0$ such that for all x in D, if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$. In particular, for all x in D such that $x \neq a$ if $0 < |x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$. But this is the same as

$$\lim_{x\to a} f(x) = f(a).$$

Conversely, suppose that $\lim_{x\to a} f(x) = f(a)$. Let $\epsilon > 0$ be given. There is a $\delta > 0$ such that for all x in D, if $0 < |x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$. Since $|a - a| = 0 < \delta$ and $|f(a) - f(a)| = 0 < \epsilon$ then for all $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$. This shows that f is continuous at $x = a$ ■

Exercise 11.3
Consider the function

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$$

Show that f is discontinuous at $x = 2$.

Solution.
We have $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x + 2) = 4 \neq f(2)$. By Exercise 11.2, the function is discontinuous at 2 ■

Exercise 11.4
Suppose that f is discontinuous at $x = a$.
(a) Show that there is and $\epsilon > 0$ and a sequence $\{a_n\}_{n=1}^{\infty}$ of elements in D such that $0 \le |a_n - a| < \frac{1}{n}$ and $|f(a_n) - f(a)| \ge \epsilon$.
(b) Show that $\lim_{n\to\infty} |a_n - a| = 0$.
(c) Show that $\lim_{n\to\infty} a_n = a$

Solution.
(a) By Definition 11, there is an $\epsilon > 0$ such that for each $\delta_n = \frac{1}{n}$ we can find a_n in D with $0 \le |a_n - a| < \frac{1}{n}$ and $|f(a_n) - f(a)| \ge \epsilon$.
(b) Using the Squeeze rule we conclude that $\lim_{n\to\infty} |a_n - a| = 0$.
(c) Since $-|a_n - a| \le a_n - a \le |a_n - a|$, by the Squeeze rule we conclude that $\lim_{n\to\infty}(a_n - a) = 0$. But $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(a_n - a) + a = \lim_{n\to\infty}(a_n - a) + \lim_{n\to\infty} a = 0 + a = a$ ■

Exercise 11.5
Suppose that f is continuous at $x = a$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements in D converging to a.
(a) Let $\epsilon > 0$ be given. Show that there is a $\delta > 0$ such that for any x in D such that $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$.
(b) With the ϵ and δ as in (a), show that there is a positive integer N such that if $n \ge N$ then $|a_n - a| < \delta$.
(c) Conclude that $\lim_{n\to\infty} f(a_n) = f(a)$.

Solution.
(a) Let $\epsilon > 0$ be given. Since f is continuous at $x = a$, there is a $\delta > 0$ such that for any x in D with $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$.
(b) Since $\lim_{n\to\infty} a_n = a$, for the $\delta > 0$ we can find a positive integer N such that if $n \ge N$ then $|a_n - a| < \delta$.
(c) Using (a) and (b) we see that for any $\epsilon > 0$ we can find a positive integer N such that if $n \ge N$ we have $|a_n - a| < \delta$ which implies that $|f(a_n) - f(a)| < \epsilon$. This establishes the result

$$\lim_{n\to\infty} f(a_n) = f(a) \blacksquare$$

Exercise 11.6
Suppose that for any sequence $\{a_n\}_{n=1}^{\infty}$ of elements in D that converges to a, the sequence $\{f(a_n)\}_{n=1}^{\infty}$ converges to $f(a)$. Then either f is continuous at a

or f is discontinuous at a.
(a) Suppose that f is discontinuous at a. Show that there is an $\epsilon > 0$ and a sequence $\{a_n\}_{n=1}^{\infty}$ of elements in D such that $\lim_{n\to\infty} a_n = a$ and $|f(a_n) - f(a)| \geq \epsilon$ for all $n \geq 1$.
(b) Show that $\lim_{n\to\infty} f(a_n) = f(a)$.
(c) Show that by (a) and (b) we conclude that $\epsilon \leq 0$, a contradiction. Thus, f must be continuous at $x = a$.

Solution.
(a) This follows from Exercise **??**.
(b) By hypothesis, since $\lim_{n\to\infty} a_n = a$ we must have $\lim_{n\to\infty} f(a_n) = f(a)$ or $\lim_{n\to\infty}(f(a_n) - f(a)) = \lim_{n\to\infty} f(a_n) - \lim_{n\to\infty} f(a) = f(a) - f(a) = 0$.
(c) Since $0 \leq -(f(a_n) - f(a)) \leq |f(a_n) - f(a)| \leq (f(a_n) - f(a))$ so that by the Squeeze rule $\lim_{n\to\infty} |f(a_n) - f(a)| = 0$. Since, $0 < \epsilon \leq |f(a_n) - f(a)|$ for all $n \geq 1$, we can apply the Squeeze rule and get $\epsilon \leq 0$, a contradiction. We conclude that f must be continuous at $x = a$ ■

Exercise 11.7
Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(a) Let $a_n = -\frac{1}{n}$. Find $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} f(a_n)$.
(b) Is f continuous at $x = 0$?

Solution.
(a) We have $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} f(a_n) = 0$.
(b) From (a) and Exercise 11.5, the function is discontinuous at $x = 0$ ■

Exercise 11.8
Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} f(a_n)$ exists, but $\lim_{n\to\infty} a_n$ does not exist.

Solution.
Let $a_n = (-1)^n$. We know that this sequence is divergent so that $\lim_{n\to\infty} a_n$ does not exist. However, $f(a_n) = 1$ for all $n \in \mathbb{N}$ so that $\lim_{n\to\infty} f(a_n) = 1$ ■

Exercise 11.9
Determine the values of a and b that makes the function f continuous everywhere.

$$f(x) = \begin{cases} 2\frac{\sin x}{x} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ b\cos x & \text{if } x > 0 \end{cases}$$

Solution.
We must have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = a$. But $\lim_{x\to 0^-} f(x) = 2$ so that $a = 2$. Also, $\lim_{x\to 0^+} f(x) = b = 2$ ■

Exercise 11.10
Using the ϵ-δ definition of continuity show that $f(x) = x^3$ is continuous at $x = 1$. Hint: $x^3 - 1 = (x-1)(x^2+x+1)$.

Solution.
We first note that $|x^3-1| = |(x-1)(x^2+x+1)| = |x-1||x^2+x+1| \le 3|x-1|$. Let $\epsilon > 0$ be given. Let $\delta < \frac{\epsilon}{5}$. Then $|x-1| < \delta \Rightarrow |f(x)-1| < \epsilon$. This shows that $f(x)$ is continuous at $x = 1$ ■

Exercise 11.11
Consider the function $f(x) = \cos\left(\frac{1}{x}\right)$.
(a) Let $a_n = \frac{1}{2n\pi}$ and $b_n = \frac{1}{(n+\frac{1}{2})\pi}$. Find $\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n, \lim_{n\to\infty} f(a_n)$, and $\lim_{n\to\infty} f(b_n)$.
(b) Is f continuous at $x = 0$?

Solution.
(a) we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0, \lim_{n\to\infty} f(a_n) = 1$, and $\lim_{n\to\infty} f(b_n) = 0$. Thus, $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ does not exist and therefore the function is not continuous at $x = 0$ ■

Exercise 11.12
Consider the function

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that this function is continuous at $x = 0$ by using the ϵ-δ definition.

Solution.
Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Then $|x| < \delta \Rightarrow \left|x\sin\left(\frac{1}{x}\right) - 0\right| \le |x| < \epsilon$ ■

Exercise 11.13
Prove that if f is continuous at $x = a$ so does $|f|$. Hint: Exercise 1.23.

Solution.
Let $\epsilon > 0$ be given. Then there is a $\delta > 0$ such that $|x - a| < delta \Rightarrow |f(x) - f(a)| < \epsilon$. But $||f(x)| - |f(a)|| \leq |f(x) - f(a)| < \epsilon$. This shows that $|f|$ is continuous at $x = a$ ■

Exercise 11.14
Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous on $\mathbb{R}$. Suppose $h : \mathbb{R} \to \mathbb{R}$ satisfies $f(x) \leq h(x) \leq g(x)$ for all $x \in \mathbb{R}$. If $f(c) = g(c)$, prove that h is continuous at c.

Solution.
Let $\{a_n\}_{n=1}^{\infty}$ be any sequence in $\mathbb{R}$ that converges to c. We must show that the sequence $\{h(a_n)\}_{n=1}^{\infty}$ converges to $h(c)$. By the continuity of f at c, we know that $\{f(a_n)\}_{n=1}^{\infty}$ converges to $f(c)$, and similarly we know that $\{g(a_n)\}_{n=1}^{\infty}$ converges to $g(c)$. We also have the inequality $f(a_n) \leq h(a_n) \leq g(a_n)$. Since $\lim_{n\to\infty} f(a_n) = f(c) = g(c) = \lim_{n\to\infty} g(a_n)$, it follows from the Squeeze theorem that $\{h(a_n)\}_{n=1}^{\infty}$ converges, and $\lim_{n\to\infty} h(a_n) = f(c) = h(c)$. Thus h is continuous at c ■

Exercise 11.15
Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. Show that f is continuous on $[0, \infty)$.

Solution.
First we show that f is right continuous at $x = 0$. Let $\epsilon > 0$ be given. Choose $\delta = \epsilon^2$. Then if $0 < x < \delta$ we have $|\sqrt{x} - 0| = \sqrt{x} < \epsilon$.
Now let $c > 0$. Let $\epsilon > 0$ be given. Let $\delta < \epsilon\sqrt{c}$. Then

$$\begin{aligned}|f(x) - f(c)| =& \left|\frac{(\sqrt{x} - \sqrt{c})(\sqrt{x} - \sqrt{c})}{\sqrt{x} + \sqrt{c}}\right| \\ =& \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \\ < \sqrt{c}\epsilon \cdot \frac{1}{\sqrt{c}} = \epsilon \ ■\end{aligned}$$

Solutions to Section 12

Exercise 12.1
Let $f(x)$ and $g(x)$ be two functions with common domain D. Suppose that f and g are continuous at a point a in D. Show the following properties:
(i) $f \pm g$ is continuous at a.
(ii) $f \cdot g$ is continuous at a.
(iii) $\frac{f}{g}$ is continuous at a provided that $g(a) \neq 0$.

Solution.
(i) By Exercise 9.1, we have

$$\lim_{x \to a}(f \pm g)(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = f(a) \pm g(a) = (f \pm g)(a).$$

By Exercise 11.2, $(f \pm g)$ is continuous at a.
(ii) By Exercise 9.3, we have

$$\lim_{x \to a}(f \cdot g)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = f(a) \cdot g(a) = (f \cdot g)(a).$$

By Exercise 11.2, $(f \cdot g)$ is contiuous at a.
(iii) By Exercise 9.7, we have

$$\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a).$$

By Exercise 11.2, $\frac{f}{g}$ is contiuous at a ∎

Exercise 12.2
Let f be continuous at a point a in its domain with $f(a) \neq 0$. Show that there exists a $\delta > 0$ such that

$$|x - a| < \delta \Longrightarrow |f(x)| > \tfrac{|f(a)|}{2}.$$

That is, there is an open interval centered at a where the function is always different from zero there. Hint: Look at Exercise 4.7

Solution.
Let $\epsilon = \frac{|f(a)|}{2} > 0$. By the definition of continuity, there is a $\delta > 0$ such that

$$|x - a| < \delta \Longrightarrow |f(x) - f(a)| < \tfrac{|f(a)|}{2}.$$

By Exercise 1.18, we have $|f(a)| - |f(x)| < \frac{|f(a)|}{2}$ or $|f(x)| > \frac{|f(a)|}{2}$ ■

Exercise 12.3
Let $f : D \to \mathbb{R}$ and $g : D' \to \mathbb{R}$ with the range of f contained in D'. Thus, $g \circ f : D \to \mathbb{R}$ is a function with domain D. Suppose that f is continuous at a and g is continuous at $f(a)$.
(a) Let $\epsilon > 0$ be given. Show that there is a $\delta' > 0$ such that for all y in D' satisfying $|y - f(a)| < \delta'$ we have $|g(y) - g(f(a))| < \epsilon$.
(b) Show that there is a $\delta'' > 0$ such that if $|x-a| < \delta''$ then $|f(x)-f(a)| < \delta'$.
(c) Show that there is a $\delta > 0$ such that if $|x-a| < \delta$ then $|g(f(x))-g(f(a))| < \epsilon$. In other words, the composite function $g(f(x))$ is continuous at a. Hence, the composition of two continuous functions is a continuous function.

Solution.
(a) Let $\epsilon > 0$ be given. Since g is continuous at $f(a)$, that is

$$\lim_{y \to f(a)} g(y) = g(f(a))$$

there is a $\delta' > 0$ such that if $|y - f(a)| < \delta'$ then $|g(y) - g(f(a))| < \epsilon$, where y is in D'.
(b) Now, since f is continuous at a, there is a $\delta'' > 0$ such that if $|x - a| < \delta''$ then $|f(x) - f(a)| < \delta'$.
(c) Let $\delta = \min\{\delta', \delta''\}$. If $|x - a| < \delta$ then $|x - a| < \delta''$ which implies that $|f(x) - f(a)| < \delta'$. Letting $y = f(x)$ we find that $|y - f(a)| < \delta'$. But then $|g(y) - g(f(a))| = |g(f(x)) - g(f(a))| < \epsilon$. This establishes the fact that $g(f(x))$ is continuous at a ■

Exercise 12.4
In Exercise 9.11, we established that $\lim_{x\to 0} \sin x = 0 = \sin 0$. That is, the sine function is continuous at 0.
(a) Using the trigonometric identity

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

show that the sine function is continuous at every number a. Hint: Use the substitution $u = x - a$ and note that $u \to 0$ as $x \to a$.
(b) Show that the cosine function is continuous for every number a. Hint: Note that $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ and use Exercise 12.3.

Solution.
(a) Letting $u = x - a$ we have

$$\begin{aligned}\lim_{x\to a}\sin x =& \lim_{u\to 0}\sin(u+a)\\ =& \lim_{u\to 0}(\sin u\cos a + \cos u \sin a)\\ =& \lim_{u\to 0}\sin u\cos a + \lim_{u\to 0}\cos u\sin a\\ =&0 + (1)\sin a = \sin a\end{aligned}$$

(b) Let $f(x) = \frac{\pi}{2} - x$, $g(x) = \sin x$, and $h(x) = \cos x$. Then $h(x) = \sin\left(\frac{\pi}{2} - x\right) = g(f(x))$. Since f and g are continuous for every real number, by the previous exercise, h is continuous for every real number ■

Exercise 12.5
Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous such that $f(x) = 0$ for all $x \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$. Hint: Exercise 3.21

Solution.
Let a be an irrational number. By Exercise 3.21, there is a sequence of rational numbers $\{a_n\}_{n=1}^{\infty}$ that converges to a. By continuity, we must have $f(a) = \lim_{n\to\infty} f(a_n) = 0$. Thus, $f(x) = 0$ for all $x \in \mathbb{R}$ ■

Exercise 12.6
Consider the function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}\end{cases}$$

(a) Prove that f is continuous at $x = 0$.
(b) Let $a \neq 0$. Prove that f is discontinuous at $x = a$.

Solution.
(a) Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. If x is rational such that $|x| < \delta$ then $|f(x) - f(0)| = |x| < \epsilon$. If x is irrational such that $|x| < \delta$ then $|f(x) - f(0)| = 0 < \epsilon$. Hence, f is continuous at $x = 0$.
(b) Let $a \neq 0$. We can find a sequence of rationals $\{a_n\}_{n=1}^{\infty}$ that converges to a. In this case, $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_n = a$. Also, we can find a sequence of irrationals $\{b_n\}_{n=1}^{\infty}$ that converges to a. In this case, $\lim_{n\to\infty} f(b_n) = 0$. Thus, $lim_{x\to a} f(x)$ does not exist and so the function is discontinuous at $x = a$ ■

Exercise 12.7
Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions and $f(x) = g(x)$ for every $x \in \mathbb{Q}$. Show that $f(x) = g(x)$ for every $x \in \mathbb{R}$.

Solution.
Apply Exercise 12.5 to the function $h(x) = f(x) - g(x)$ ■

Exercise 12.8
Use continuity to evaluate $\lim_{x\to\pi} \sin(x + \sin x)$.

Solution.
We have $\lim_{x\to\pi} \sin(x + \sin x) = \sin(\pi + 0) = 0$ ■

Exercise 12.9
Give an example of two functions f and g that are not continuous on the interval $(0, 1)$ but their sum $f + g$ is continuous on $(0, 1)$.

Solution.
Let

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2} \\ 2 & \text{if } \frac{1}{2} \le x < 1 \end{cases}$$

and

$$g(x) = \begin{cases} 2 & \text{if } 0 < x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le x < 1 \end{cases}$$

Thus, f and g are discontinuous at $x = \frac{1}{2}$. However, $(f + g)(x) = 3$ for all $x \in (0, 1)$ which is a continuous function ■

Exercise 12.10
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$.
(a) Show that $f(0) = 0$ and $f(n) = an$ and $f(-n) = a(-n)$ for all $n \in \mathbb{N}$ where $a = f(1)$.
(b) Show $f\left(\frac{m}{n}\right) = a \cdot \frac{m}{n}$ where m and n are integers with $n \neq 0$. That is, $f(x) = ax$ for all $x \in \mathbb{Q}$.
(c) Show that $f(x) = ax$ for all $x \in \mathbb{R}$. Hint: Exercise 12.5 applied to the function $g(x) = f(x) - ax$.

Solution.
(a) We have $f(0) + f(0) = f(0 + 0) = f(0)$. Subtract $f(0)$ from both sides to obtain $f(0) = 0$. Let $a = f(1)$ and $n \in \mathbb{N}$. Write $n = 1 + 1 + \cdots + 1$. Then $f(n) = f(1 + 1 + \cdots + 1) = f(1) + f(1) + \cdots + f(1) = nf(1) = an$. Also, $f(-n) + f(n) = f(n - n) = f(0) = 0$ so that $f(-n) = -f(n) = a(-n)$.
(b) Let m and n be integers such that $n \neq 0$. Then $f\left(\frac{m}{n}\right) = f\left(m \cdot \frac{1}{n}\right) = mf\left(\frac{1}{n}\right)$. Since $nf\left(\frac{1}{n}\right) = f\left(\frac{n}{n}\right) = f(1) = a$ we obtain $f\left(\frac{m}{n}\right) = a\frac{m}{n}$. Thus, $f(x) = ax$ for all $x \in \mathbb{Q}$.
(c) Let $g(x) = f(x) - ax$. g is continuous on $\mathbb{R}$ and $g(x) = 0$ for all $x \in \mathbb{Q}$. by Exercise 12.5, we conclude that $g(x) = 0$ for all $x \in \mathbb{R}$. That is, $f(x) = ax$ for all $x \in \mathbb{R}$ ■

Exercise 12.11
Prove that if f is continuous on $[a, b]$, then either $f(x) = 0$ for some $x \in [a, b]$, or there is a number $\epsilon > 0$ such that $|f(x)| \geq \epsilon$ for all $x \in [a, b]$.

Solution.
Suppose that for every $\epsilon > 0$ there is an $x \in [a, b]$ such that $|f(x)| < \epsilon$. By letting $\epsilon = \frac{1}{n}$ we find a sequence $\{a_n\}_{n=1}^{\infty}$ such that $|f(a_n)| < \frac{1}{n}$. Since $\{a_n\}_{n=1}^{\infty}$ is bounded (since all the terms are in $[a, b]$) we can find a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ that converges to some number $x \in [a, b]$. Since $|f(a_{n_k})| < \frac{1}{n_k}$ and f is continuous we conclude that $f(x) = \lim_{n\to\infty} f(a_{n_k}) = 0$. Thus either there is some $\epsilon > 0$ with $|f(x)| \geq \epsilon$ for all $x \in [a, b]$, or there is some $x \in [a, b]$ with $f(x) = 0$ ■

Solutions to Section 13

Exercise 13.1
Show that the function $f(x) = x$ is uniformly continuous.

Solution.
Let $\epsilon > 0$ be given. Let $\delta = \epsilon$. Then for all x_1 and x_2, if $|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| = |x_1 - x_2| < \delta = \epsilon$. Hence, the given function is uniformly continuous ■

Exercise 13.2
Consider the function $f(x) = \frac{1}{x}$ on the set $x > 0$. Let $\delta > 0$ be any number and define $\alpha = \min\{2, \delta\}$. Then $\alpha \le 2$ and $\alpha \le \delta$. Let $x_1 = \frac{\alpha}{3} > 0$ and $x_2 = \frac{\alpha}{6} > 0$.
(a) Show that $|x_1 - x_2| < \delta$ but $|f(x_1) - f(x_2)| \ge \frac{3}{2}$.
(b) Conclude from (a) that f is not uniformly continuous on the interval $0 < x < \infty$.

Solution.
(a) We have $|x_1 - x_2| = \left|\frac{\alpha}{3} - \frac{\alpha}{6}\right| = \frac{\alpha}{6} \le \frac{\delta}{6} < \delta$. Moreover, $|f(x_1) - f(x_2)| = \left|\frac{3}{\alpha} - \frac{6}{\alpha}\right| = \frac{3}{\alpha} \ge \frac{3}{2}$.
(b) By letting $\epsilon = \frac{3}{2}$ there is no $\delta > 0$ where the definition of uniform continuity is satisfied. Hence, $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, \infty)$ ■

Exercise 13.3
(a) Show that if f is uniformly continuous on D then f is continuous at every point in D.
(b) Using properties of continuous functions, show that the function $f(x) = \frac{1}{x}$ is continuous on the interval $0 < x < \infty$.
(c) Is the converse of (a) always true? That is, every continuous function is uniformly continuous.

Solution.
(a) Suppose f is uniformly continuous on D. Let a be a point in D. Let $\epsilon > 0$ be given. Then by uniform continuity, there is a $\delta > 0$ such that for all x, u in D,

$$\text{if } |x - u| < \delta \Longrightarrow |f(x) - f(u)| < \epsilon.$$

In particular, the above is true if $u = a$. Hence, f is continuous at a.
(b) Since $g(x) = 1$ and $h(x) = x$ are both continuous in the interval $0 < x < \infty$, the ratio function $f(x) = \frac{1}{x}$ is also continuous there.
(c) No. The function $f(x) = \frac{1}{x}$ is continuous in the interval $0 < x < \infty$ (by (b)) but not uniformly continuous there (by Exercise 13.2(a)) ■

Exercise 13.4
Show that if $f, g : D \to \mathbb{R}$ are uniformly continuous then $f + g : D \to \mathbb{R}$ is also uniformly continuous.

Solution.
Let $\epsilon > 0$. Since f is uniformly continuous, there is a $\delta_1 > 0$ such that

$$\text{if } |x - u| < \delta_1 \Longrightarrow |f(x) - f(u)| < \tfrac{\epsilon}{2} \text{ for all } x, u \text{ in } D.$$

Likewise, since g is uniformly continuous, there is a $\delta_2 > 0$ such that

$$\text{if } |x - u| < \delta_2 \Longrightarrow |g(x) - g(u)| < \tfrac{\epsilon}{2} \text{ for all } x, u \text{ in } D.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If x, u are in D such that $|x - u| < \delta$ then $|x - u| < \delta_1$ and $|x-u| < \delta_2$. But then $|f(x)-f(u)| < \frac{\epsilon}{2}$ and $|g(x)-g(u)| < \frac{\epsilon}{2}$. Therefore,

$$\begin{aligned}|(f+g)(x) - (f+g)(u)| =&|(f(x) - f(u)) + (g(x) - g(u))|\\ \leq&|f(x) - f(u)| + |g(x) - g(u)|\\ <&\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

It follows that $f + g$ is also uniformly continuous ■

Exercise 13.5
Let $f(x) = x^2$. Suppose that there is a $\delta > 0$ such that $|x_1 - x_2| < \delta$ for all real numbers x_1 and x_2. In addition, suppose we want $|x_1^2 - x_2^2| = 1$. That is, $|x_1 - x_2||x_1 + x_2| = 1$. One way to achieve that is by setting $x_1 - x_2 = \frac{\delta}{2}$ and $x_1 + x_2 = \frac{2}{\delta}$.
(a) Find x_1 and x_2 in terms of δ.
(b) Show that f is not uniformly continuous. Hint: Let $\epsilon = \frac{1}{2}$ in Definition 12.

Solution.
(a) Solving the system of equations by the method of elimination we find $x_1 = \frac{\delta}{4} + \frac{1}{\delta}$ and $x_2 = \frac{1}{\delta} - \frac{\delta}{4}$.
(b) By (a), letting $\epsilon = \frac{1}{2}$ we see that $|x_1 - x_2| = \frac{\delta}{2} < \delta$ but $|f(x_1) - f(x_2)| = 1 > \frac{1}{2} = \epsilon$. This shows that the given function is not uniformly continuous ■

Exercise 13.6
Give an example of two functions $f, g : D \to \mathbb{R}$ that are uniformly continuous but the product function $f \cdot g$ is not.

Solution.
Let $f(x) = g(x) = x$. Both functions are uniformly continuous by Exercise **??**. However, the product $(f \cdot g)(x) = x^2$ is not uniformly continuous by the previous exercise ■

Exercise 13.7
Let $f, g : D \to \mathbb{R}$ be uniformly continuous and bounded, say $|f(x)| \le M_1$ and $|g(x)| \le M_2$ for all x in D. Let $\epsilon > 0$ be arbitrary.
(a) Show that there is a $\delta_1 > 0$ such that

$$\text{if } |x - u| < \delta_1 \Longrightarrow |f(x) - f(u)| < \tfrac{\epsilon}{2M_2} \text{ for all } x, u \text{ in } D.$$

(b) Show that there is a $\delta_2 > 0$ such that

$$\text{if } |x - u| < \delta_2 \Longrightarrow |g(x) - g(u)| < \tfrac{\epsilon}{2M_1} \text{ for all } x, u \text{ in } D.$$

(c) Show that $f \cdot g : D \to \mathbb{R}$ is also uniformly continuous. Note that boundedness is crucial in this result. Hint: Note that $f(x)g(x) - f(u)g(u) = (f(x) - f(u)g(x) + f(u)(g(x) - g(u))$.

Solution.
(a) Since f is uniformly continuous, there is a $\delta_1 > 0$ such that

$$\text{if } |x - u| < \delta_1 \Longrightarrow |f(x) - f(u)| < \tfrac{\epsilon}{2M_2} \text{ for all } x, u \text{ in } D.$$

(b) Likewise, since g is uniformly continuous, there is a $\delta_2 > 0$ such that

$$\text{if } |x - u| < \delta_2 \Longrightarrow |g(x) - g(u)| < \tfrac{\epsilon}{2M_1} \text{ for all } x, u \text{ in } D.$$

(c)Let $\delta = \min\{\delta_1, \delta_2\}$. If x, u are in D such that $|x-u| < \delta$ then $|x-u| < \delta_1$ and $|x - u| < \delta_2$. But then $|f(x) - f(u)| < \frac{\epsilon}{2M_2}$ and $|g(x) - g(u)| < \frac{\epsilon}{2M_1}$. Therefore,

$$\begin{aligned}|f(x)g(x) - f(u)g(u)| =&|(f(x) - f(u))g(x) + f(u)(g(x) - g(u))| \\ \le&|f(x) - f(u)||g(x)| + |f(u)||g(x) - g(u)| \\ <&\frac{\epsilon}{2M_2}M_2 + M_1\frac{\epsilon}{2M_1} = \epsilon.\end{aligned}$$

It follows that $f \cdot g$ is also uniformly continuous ■

Exercise 13.8
Suppose that $f : D \to \mathbb{R}$ is uniformly continuous. Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence of terms in D.
(a) Let $\epsilon > 0$ be arbitrary. Show that there is a $\delta > 0$ such that

$$\text{If } |x_1 - x_2| < \delta \Longrightarrow |f(x_1) - f(x_2)| < \epsilon \text{ for all } x_1, x_2 \text{ in } D.$$

(b) Show that there is a positive integer N such that

$$\text{If } n, m \geq N \Longrightarrow |a_n - a_m| < \delta.$$

(c) Show that $\{f(a_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$ (and therefore by Exercise 7.7 is convergent).

Solution.
(a) This is just the definition of uniform continuity.
(b) This is just the definition of Cauchy sequence.
(c) For the given $\epsilon > 0$ we can find a positive integer N such that if $n, m \geq N$ we have $|a_n - a_m| < \delta$ by (b). Using (a), $|a_n - a_m| < \delta$ implies that $|f(a_n) - f(a_m)| < \epsilon$. Thus, $\{f(a_n)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$ and therefore is convergent $\blacksquare$

Exercise 13.9
Consider the function $f(x) = \tan x$ on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
(a) Show that the sequence $\{\frac{\pi}{2} - \frac{1}{n}\}_{n=1}^{\infty}$ is convergent.
(b) Show that the sequence in (a) is also Cauchy.
(c) Show that the sequence $\{f\left(\frac{\pi}{2} - \frac{1}{n}\right)\}_{n=1}^{\infty}$ is not Cauchy.
(d) Show that the function $f(x) = \tan x$ is not uniformly continuous on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Solution.
(a) $\lim_{n\to\infty} \left(\frac{\pi}{2} - \frac{1}{n}\right) = \frac{\pi}{2}$.
(b) This follows from Exercise 5.2.
(c) Since $\lim_{n\to\infty} f\left(\frac{\pi}{2} - \frac{1}{n}\right) = \lim_{n\to\infty} \tan\left(\frac{\pi}{2} - \frac{1}{n}\right) = \lim_{n\to\infty} \cot\frac{1}{n} = \cot\left(\lim_{n\to\infty}\frac{1}{n}\right) = \cot 0$ which is undefined, the sequence $\{f\left(\frac{\pi}{2} - \frac{1}{n}\right)\}_{n=1}^{\infty}$ is not convergent and therefore is not Cauchy by Exercise 7.7.
(d) This follows from Exercise 13.8 $\blacksquare$

Exercise 13.10
Let $f : D \to \mathbb{R}$ and $g : D' \to \mathbb{R}$ be two uniformly continuous functions with the range of f contained in D'. Looking closely at Exercise 12.3, show that the composite function $g(f(x))$ is also uniformly continuous.

Solution.
Let $\epsilon > 0$ be given. Since g is uniformly continuous, there is a $\delta' > 0$ such that if $|u - v| < \delta'$ then $|g(u) - g(v)| < \epsilon$, where u and v are in D'.
Now, since f is uniformly continuous, there is a $\delta'' > 0$ such that if $|x_1 - x_2| < \delta''$ then $|f(x_1) - f(x_2)| < \delta'$, where x_1 and x_2 are in D. Let $\delta = \min\{\delta', \delta''\}$. Let x_1 and x_2 be in D such that $|x_1 - x_2| < \delta$. Then $|x_1 - x_2| < \delta''$ which implies that $|f(x_1) - f(x_2)| < \delta'$. But $f(x_1)$ and $f(x_2)$ are in D' with $|f(x_1) - f(x_2)| < \delta'$. Then $|g(f(x_1)) - g(f(x_2))| < \epsilon$. This establishes the fact that $g(f(x))$ is uniformly continuous ■

Exercise 13.11
Consider the function $f(x) = \sin x$ defined on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
(a) Use Exercise 9.11(a) to show that $|\sin x| \leq |x|$ on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
(b) Using the trigonometric identity $\sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$ and (a) to show that

$$|\sin a - \sin b| \leq |a - b|.$$

(c) Show that f is uniformly continuous on the $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Solution.
(a) For $0 \leq x < \frac{\pi}{2}$ we know that $\sin x \geq 0$ so that $-|x| \leq \sin x \leq x = |x|$. That is, $|\sin x| \leq |x|$. For $-\frac{\pi}{2} < x < 0$ we have $|\sin(-x)| \leq |-x|$ which is the same as $|\sin x| \leq |x|$.
(b) Using (a) and the fact that $|\cos x| \leq 1$ we can write

$$\begin{aligned}
|\sin a - \sin b| =& \left|2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)\right| \\
=& 2\left|\sin\left(\frac{a-b}{2}\right)\right|\left|\cos\left(\frac{a+b}{2}\right)\right| \\
\leq& 2\left|\sin\left(\frac{a-b}{2}\right)\right| \leq 2 \cdot \frac{|a-b|}{2} = |a-b|
\end{aligned}$$

(c)Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Then for all a and b on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ we have

$$\text{if } |a - b| < \delta \Longrightarrow |\sin a - \sin b| \leq |a - b| < \delta = \epsilon$$

Thus, $f(x) = \sin x$ is uniformly continuous on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ ■

Exercise 13.12
Using Exercise 13.10 and Exercise 13.11, show that the function $g(x) = \cos x$ is uniformly continuous in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Solution.
First, recall that $\cos x = \sin\left(\frac{\pi}{2} - x\right)$. Let $f(x) = \frac{\pi}{2} - x$ and $g(x) = \sin x$. Then $\cos x = g(f(x))$ is uniformly continuous (by Exercise 13.10) since both f and g are uniformly continuous(by Exercise 13.11 and Exercise 13.1) ■

Exercise 13.13
Give an example of two uniformly continuous functions f and g such that $\frac{f(x)}{g(x)}$ is not uniformly continuous.

Solution.
The functions $f(x) = \sin x$ and $g(x) = \cos x$ are uniformly continuous on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ (See Exercise 13.11 and Exercise 13.12.) However, the function $\frac{f(x)}{g(x)} = \tan x$ is not uniformly continuous in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ (See Exercise 13.9) ■

Exercise 13.14
Let $g : D \to \mathbb{R}$ be a uniformly continuous function with $|g(x)| \geq M > 0$ for all $x \in D$. Hence, the function $\frac{1}{g(x)}$ is bounded and $g(x) \neq 0$ for all x in D. Show that $\frac{1}{g(x)}$ is uniformly continuous.

Solution.
Let $\epsilon > 0$ be given. By uniform continuity, there is a $\delta > 0$ such that

$$\text{if } |a - b| < \delta \Longrightarrow |g(a) - g(b)| < \epsilon M^2.$$

Thus, if $|a - b| < \delta$ we can also have

$$\begin{aligned}
\left|\frac{1}{g(a)} - \frac{1}{g(b)}\right| =& \left|\frac{g(b) - g(a)}{g(a)g(b)}\right| \\
=& \frac{|g(a) - g(b)|}{|g(a)||g(b)|} \\
\leq& \frac{|g(a) - g(b)|}{M^2} < \frac{M^2\epsilon}{M^2} = \epsilon
\end{aligned}$$

This shows that $\frac{1}{g(x)}$ is uniformly continuous ■

Exercise 13.15
Let $f, g : D \to \mathbb{R}$ be two uniformly continuous functions such that $f(x)$ is bounded and $|g(x)| \geq M > 0$ for all $x \in D$. Show that the function $\frac{f(x)}{g(x)}$ is uniformly continuous on D.

Solution.
Note that $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ which is the product of two uniformly continuous functions with $f(x)$ and $\frac{1}{g(x)}$ bounded. By Exercise 13.7(c), the function $\frac{f(x)}{g(x)}$ is uniformly continuous $\blacksquare$

Exercise 13.16
A function $f : D \to \mathbb{R}$ is said to be **Lipschitz** if there is a constant $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in D$. Show that a Lipschitz function is uniformly continuous.

Solution.
Let $\epsilon > 0$ be given. Let $\delta = \frac{\epsilon}{K}$. Then for all $x, y \in D$ such that $|x - y| < \delta$ we have

$$|f(x) - f(y)| \leq K|x - y| < K\frac{\epsilon}{K} = \epsilon.$$

This shows that f is uniformly continuous $\blacksquare$

Solutions to Section 14

Exercise 14.1
(a) Let $c_0 = \frac{a+b}{2}$. Then either $[a, c_0]$ or $[c_0, b]$ contains an infinite members of $\{v_n\}_{n=1}^{\infty}$. Let's call the interval $[a_1, b_1]$. Show that $b_1 - a_1 = \frac{b-a}{2}$.
(b) Let $c_1 = \frac{a_1+b_1}{2}$. Then either $[a_1, c_1]$ or $[c_1, b_1]$ contains an infinite members of $\{v_n\}_{n=1}^{\infty}$. Let's call the interval $[a_2, b_2]$. Show that $b_2 - a_2 = \frac{b-a}{2^2}$. Compare a_1 and a_2. Compare b_1 and b_2.
(c) Let $c_2 = \frac{a_2+b_2}{2}$. Then either $[a_2, c_2]$ or $[c_2, b_2]$ contains an infinite members of $\{v_n\}_{n=1}^{\infty}$. Let's call the interval $[a_3, b_3]$. Show that $b_3 - a_3 = \frac{b-a}{2^3}$. Compare a_1, a_2 and a_3. Compare b_1, b_2 and b_3.

Solution.
(a) If $a_1 = a$ and $b_1 = c_0$ then $b_1 - a_1 = a - \frac{a+b}{2} = \frac{b-a}{2}$. If $a_1 = c_0$ and $b_1 = b$ then $b_1 - a_1 = b - \frac{a+b}{2} = \frac{b-a}{2}$.
(b) If $a_2 = a_1$ and $b_2 = c_1$ then $b_2 - a_2 = \frac{a_1+b_1}{2} - a_1 = \frac{b_1-a_1}{2} = \frac{b-a}{2^2}$. If $a_2 = c_1$ and $b_2 = b_1$ then $b_2 - a_2 = b_1 - \frac{a_1+b_1}{2} = \frac{b_1-a_1}{2} = \frac{b-a}{2^2}$. Note that $a_2 \geq a_1$ and $b_2 \leq b_1$.
(c) If $a_3 = a_2$ and $b_3 = c_2$ then $b_3 - a_3 = \frac{a_2+b_2}{2} - a_2 = \frac{b_2-a_2}{2} = \frac{b-a}{2^3}$. If $a_3 = c_2$ and $b_3 = b_2$ then $b_3 - a_3 = b_2 - \frac{a_2+b_2}{2} = \frac{b_2-a_2}{2} = \frac{b-a}{2^3}$. Note that $a_3 \geq a_2 \geq a_1$ and $b_3 \leq b_2 \leq b_1$ ■

Exercise 14.2
(a) Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded from above. What is an upper bound?
(b) Show that there is a constant M such that $M = \sup\{a_1, a_2, \cdots\}$.
(c) Show that $a \leq M \leq b$.

Solution.
(a) For all $n \geq 1$, we have $a_n \leq b$. That is the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded from above with an upper bound b.
(b) By the Completeness axiom of $\mathbb{R}$ there is a finite number M such that $M = \sup\{a_1, a_2, \cdots\}$.
(c) Since b is an upper bound of $\{a_n\}_{n=1}^{\infty}$ and M is the smallest upper bound, we must have $M \leq b$. Since M is an upper bound of $\{a_n\}_{n=1}^{\infty}$ we must have $a_n \leq M$ for all $n \geq 1$. Since $a \leq a_n$ for all $n \geq 1$ we conclude that $a \leq M \leq b$ ■

Exercise 14.3
(a) Show that there is $\delta > 0$ such that for any $a \le x \le b$ if $|x - M| < \delta$ then $|f(x) - f(M)| < \frac{\epsilon}{2}$.
(b) Show that for all u and v in $[a, b]$ if $|u - M| < \delta$ and $|v - M| < \delta$ then $|f(u) - f(v)| < \epsilon$.

Solution.
(a) This follows from the fact that f is continuous at M.
(b) If u and v are in $[a, b]$ such $|u - M| < \delta$ and $|v - M| < \delta$ then

$$\begin{aligned} |f(u) - f(v)| =&|(f(u) - f(M)) - (f(v) - f(M))| \\ \le&|f(u) - f(M)| + |f(v) - f(M)| \\ <&\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \blacksquare \end{aligned}$$

Exercise 14.4
(a) Let $w_n = \frac{b-a}{2^n}$. Show that $\lim_{n\to\infty} w_n = 0$. Hint: Squeeze rule.
(b) Show that there is a positive integer N such that $\frac{b-a}{2^N} < \frac{\delta}{2}$ and $|x - M| < \frac{\delta}{2}$ for all $a_N \le x \le b_N$.

Solution.
(a) For $n \ge 1$, we have $2^n \ge n$. Thus, $0 \le \frac{1}{2^n} \le \frac{1}{n}$ and $0 \le w_n \le \frac{b-a}{n}$. By the Squeeze rule we conclude that $\lim_{n\to\infty} \frac{b-a}{2^n} = 0$.
(b) From (a), there is a positive integer K such that if $n \ge K$ then $\frac{b-a}{2^n} < \frac{\delta}{2}$. Let N be large enough so that $N \ge K$ and $|x - M| < \frac{\delta}{2}$ for all $a_N \le x \le b_N$ $\blacksquare$

Exercise 14.5
(a) Using Exercise 14.4, show that there is a large n such that $\frac{1}{n} < \frac{\delta}{2}$ and $a_N \le v_n \le b_N$.
(b) For the n found in (a), show that $|u_n - v_n| < \frac{1}{n} < \frac{\delta}{2}$ and $|v_n - M| < \frac{\delta}{2}$.
(c) For the n found in (a), Show that $|u_n - M| < \delta$.
(d) Using (b), (c), and Exericse 14.3(b), show that $|f(u_n) - f(v_n)| < \epsilon$.
Conclusion: The result in (d), contradicts (1). Hence, f must be uniformly continuous.

Solution.
(a) Since $[a_N, b_N]$ contains an infinite number of v_n, we can choose n large enough so that $\frac{1}{n} < \frac{\delta}{2}$ and $a_N \le v_n \le b_N$.
(b) Since $a_N \le v_n \le b_N$, by the previous exercise we obtain $|v_n - M| < \frac{\delta}{2} < \delta$.

Moreover, from (**??**), we have $|u_n - v_n| < \frac{1}{n} < \frac{\delta}{2}$.
(c) We have $|u_n - M| = |(u_n - v_n) + (v_n - M)| \le |u_n - v_n| + |v_n - M| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$.
(d) Since $|u_n - M| < \delta$ and $|v_n - M| < \frac{\delta}{2} < \delta$, by Exericse 14.3(b) we obtain $|f(u_n) - f(v_n)| < \epsilon$. But this contradicts (1). Hence, f must be uniformly continuous ■

Exercise 14.6
Show that the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is uniformly continuous.

Solution.
By Exercise 11.15, f is continuous on $[0, 1]$. Thus, f is uniformly continuous on $[0, 1]$ ■

Exercise 14.7
(a) A function $f : D \to \mathbb{R}$ is said to be **Lipschitz** if there is a constant $K > 0$ such that $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in D$. Show that the function $f(x) = \sqrt{x}$ is not Lipschitz on $[0, 1]$. Hint: Assume the contrary and get a contradiction.
(b) Give an example of a uniformly continuous function that is not Lipschitz. Thus, the converse to Exercise 13.16 is false.

Solution.
(a) Suppose that f is Lipschitz on $[0, 1]$. Then there is a positive constant K such that $|\sqrt{x} - \sqrt{y}| \le K|x - y|$ for all $x, y \in [0, 1]$. Letting $y = 0$ we obtain $\sqrt{x} \le Kx$ for all $x \in [0, 1]$. In particular, $\sqrt{x} \le Kx$ for all $0 < x \le 1$ and therefore $K \ge \frac{1}{\sqrt{x}}$ for all $x \in (0, 1]$. Letting $x \to 0^+$ we see that $K \ge \infty$, a contradiction.
(b) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. We know from (a) that f is not Lipschitz. We also know by Exercise 14.6 that f is uniformly continuous ■

Exercise 14.8
Show, using the definition of uniform continuity (epsilon-delta definition) the function $f(x) = \frac{x}{x+1}$ is uniformly continuous on $[0, 2]$.

Solution.
Let $x, y \in [0, 2]$. Note that

$$\left|\frac{x}{x+1} - \frac{y}{y+1}\right| = \frac{|x - y|}{(x + 1)(y + 1)}.$$

Let $\epsilon > 0$ be given. Choose $\delta < \epsilon$. Since $(x+1)(y+1) \geq 1$ for $|x-y| < \delta$ we have

$$\left|\frac{x}{x+1} - \frac{y}{y+1}\right| = \frac{|x-y|}{(x+1)(y+1)} \leq |x-y| < \epsilon.$$

Hence, the given function is uniformly continuous on $[0,2]$ ■

Exercise 14.9
Conisder the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x = 0 \end{cases}$$

Show that f is uniformly continuous on $[0,1]$.

Solution.
$f(x)$ is continuous for all $x \in (0,1]$. Since $\lim_{x\to 0^+} f(x) = 1 = f(0)$ the function is continuous at $x = 0$. Thus, the function f is continuous on $[0,1]$ and therefore uniformly continuous ■

Exercise 14.10
Show that the function $f : (-2,1] \to \mathbb{R}$ defined by $f(x) = x^2$ is Lipschitz on $(-2,1]$.

Solution.
We have

$$|x^2 - y^2| = |x+y||x-y| \leq (|x| + |y|)|x-y| \leq 4|x-y| \text{ ■}$$

Solutions to Section 15

Exercise 15.1
Give an example of a continuous $f : D \to \mathbb{R}$ with D a bounded set (i.e. $|x| \leq M$ for some $M > 0$ and for all x in D) but $f(D)$ is not bounded.

Solution.
Let D be the open interval $(0, 1)$. Then D is bounded. Define $f : D \to \mathbb{R}$ by $f(x) = \frac{1}{x}$. We know from Exercise 13.3(b) that f is continuous on D. Moreover, $f(D)$ is the open interval $(0, \infty)$ which is not bounded ∎

Exercise 15.2
Let D be a bounded subset of $\mathbb{R}$ with $|x| \leq M$ for all $x \in D$. Suppose that $f : D \to \mathbb{R}$ is uniformly continuous.
(a) Show that there is a $\delta > 0$ such that if u and v belong to D such that $|u - v| < \delta$ then $|f(u) - f(v)| < 1$.
(b) Let n be a positive integer such that $n > \frac{2M}{\delta}$. Divide the interval $[-M, M]$ into n equal subintervals:$[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$. Show that $x_k - x_{k-1} < \delta$ for all $k = 1, 2, \cdots, n$
(c) Let $[a_1, b_1], [a_2, b_2], \cdots, [a_k, b_k]$ be those intervals in (b) that intersect D. That is, $D \subseteq [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k]$. For $1 \leq i \leq k$ let $u_i \in [a_i, b_i] \cap D$. Show that if v is in D then there is an $1 \leq i \leq k$ such that $|v - u_i| < \delta$ and $|f(v)| < 1 + |f(u_i)|$.
(d) Show that $|f(v)| \leq M$ for all v in D. That is, $f(D)$ is bounded.

Solution.
(a) Let $\epsilon = 1$. By uniform continuity, there is a $\delta > 0$ such that if u and v belong to D such that $|u - v| < \delta$ then $|f(u) - f(v)| < 1$
(b) We have $x_k - x_{k-1} = \frac{2M}{n} < \delta$.
(c) Since $D \subseteq [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k]$, if v is in D then v belongs to $[a_i, b_i]$ for some $1 \leq i \leq k$. But then $|v - u_i| < |b_i - a_i| < \delta$ (by (b)). By (a) we must have $|f(v) - f(u_i)| < 1$. Using the triangle inequality, we conclude that $|f(v)| < 1 + |f(u_i)|$.
(d) Let $M = 1 + |f(u_1)| + |f(u_2)| + \cdots + |f(u_k)|$. By part (c), for any v in D we have $|f(v)| < 1 + |f(u_i)| < M$. Thus, the range of f is bounded ∎

Exercise 15.3
Show that if $f : [a, b] \to \mathbb{R}$ is continuous then $f([a, b])$ is bounded. Hint: Exercises 14.5 and 15.2.

Solution.
Since f is continuous, by Exercise 14.5 it is uniformly continuous. Since $[a, b]$ is bounded, by Exercise 15.2, $f([a, b])$ is bounded ■

Exercise 15.4
Show that if $f : [a, b] \to \mathbb{R}$ is continuous then $\inf\{f(x) : a \leq x \leq b\}$ and $\sup\{f(x) : a \leq x \leq b\}$ exist.

Solution.
By Exercise 15.3, $f([a, b])$ is bounded. Now, the result follows from the Completenes Axiom of real numbers ■

Exercise 15.5
Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $I = \inf\{f(x) : a \leq x \leq b\}$. Note that I exists by Exercise 15.4. Suppose that $I < f(x)$ for all $x \in [a, b]$. That is, the infimum can not be attained in $[a, b]$. Define the function $g : [a, b] \to \mathbb{R}$ by

$$g(x) = \frac{1}{f(x) - I}.$$

(a) Show that g is continuous on $[a, b]$.
(b) Show that there is a positive number M such that $|g(x)| \leq M$ for all $x \in [a, b]$.
(c) Show that $I + \frac{1}{M}$ is a lower bound of $f([a, b])$ and this leads to a contradiction.
Conclusion: There must be a number $x_1 \in [a, b]$ such that $f(x_1) = \inf\{f(x) : a \leq x \leq b\}$.

Solution.
(a) Since the constant function $w_1(x) = 1$ and $w_2(x) = f(x) - I$ are continuous, the function g being the ratio of two continuous functions is also continuous.
(b) By Exercise 15.3, $g([a, b])$ is bounded, that is $|g(x)| \leq M$ for all $x \in [a, b]$.
(c) From (b), we have $\frac{1}{f(x)-I} \leq M$ which implies that $I + \frac{1}{m} \leq f(x)$ for all $x \in [a, b]$. But this says that $I + \frac{1}{M}$ is a lower bound of $f([a, b])$. Since I is the largest lower bound, we must have $I + \frac{1}{M} \leq I$, a contradiction. Hence, there must be a number $x_1 \in [a, b]$ such that $f(x_1) = \inf\{f(x) : a \leq x \leq b\}$ ■

Exercise 15.6
Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $S = \sup\{f(x) : a \leq x \leq b\}$. Note that S exists by Exercise 15.4. Show that there exists $x_2 \in [a, b]$ such that $f(x_2) = S$. Hint: Mimic Exercise 15.5.

Solution.
Suppose that $f(x) < S$ for all $x \in [a, b]$. Define the function $g(x) = \frac{1}{S-f(x)}$. Since the constant function $w_1(x) = 1$ and $w_2(x) = S - f(x)$ are continuous, the function g being the ratio of two continuous functions is also continuous. By Exercise 15.3, $g([a, b])$ is bounded, that is $|g(x)| \leq M$ for all $x \in [a, b]$. Thus, we have $\frac{1}{S-f(x)} \leq M$ which implies that $f(x) \leq S - \frac{1}{M}$ for all $x \in [a, b]$. But this says that $S - \frac{1}{M}$ is an upper bound of $f([a, b])$. Since S is the smallest upper bound, we must have $S < S - \frac{1}{M}$, a contradiction. Hence, there must be a number $x_2 \in [a, b]$ such that $f(x_2) = \sup\{f(x) : a \leq x \leq b\}$ ■

Exercise 15.7
Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $f(a) \leq c \leq f(b)$.
(a) Let $D = \{x \in [a, b] : f(x) \leq c\}$. Show that D is non-empty and that D is bounded from above. By the Completeness Axiom of real numbers there is a number d such that $d = \sup\{x \in D\}$.
(b) Show that $d \in [a, b]$.
(c) Suppose that $f(d) > c$. Show that there is a $\delta > 0$ such that if $|x - d| < \delta$ then $|f(x) - f(d)| < f(d) - c$.
(d) Show that for $x \in [a, b]$ and $|x - d| < \delta$ we must have $f(x) > c$. Hint: Exercise 1.14.
(e) Using (d), show that $d - \delta$ is an upper bound of D. Thus, $f(d) > c$ leads to a contradiction.
(f) Suppose that $f(d) < c$. Show that there is a $\delta > 0$ such that if $d - \delta < x < d + \delta$ and $x \in [a, b]$ we must have $f(x) < c$.
(g) Show that $f(d + \frac{\delta}{2}) < c$. Why this leads to a contradiction?
Conclusion: We must have $f(d) = c$.

Solution.
(a) Since $a \in D$, D is non-empty. Moreover, since $x \leq b$ for all $x \in D$, we see that D is bounded from above. By the Completeness Axiom of real numbers there is a number d such that $d = \sup\{x \in D\}$.
(b) By the definition of d, we have $a \leq x \leq d$ for all $x \in D$. Also, since b is an upper bound of D and d is the smallest upper bound, we must have

$d \leq b$. Hence, $a \leq d \leq b$.
(c) Let $\epsilon = f(d) - c > 0$. Since f is continuous at d, there is a $\delta > 0$ such that if $|x - d| < \delta$ then $|f(x) - f(d)| < \epsilon = f(d) - c$.
(d) If $|x - d| < \delta$ then $|f(x) - f(d)| < f(d) - c$ which is equivalent to $c - f(d) < f(x) - f(d)$ or $f(x) > c$.
(e) If $x \in D$ and $x > d - \delta$ then $x - d > -\delta$ which implies that $f(x) > c$ a contradiction. Hence, $x \leq d - \delta$ so that $d - \delta$ is an upper bound of D. But then $d < d - \delta$ which is impossible. Hence, $f(d) > c$ cannot happen.
(f) If $f(d) < c$. Letting $\epsilon = c - f(d)$, we can find $\delta > 0$ such that if $|x - d| < \delta$ we have $|f(x) - f(d)| < c - f(d)$ or $f(x) < c$.
(g) Let $x = d + \frac{\delta}{2}$. Then $d - \delta < x < d + \delta$ and therefore $f(x) < c$, that is $d + \frac{\delta}{2} \in D$. But the definition of d implies that $d + \frac{\delta}{2} < d$, a contradiction ■

Exercise 15.8
Let $f : [a, b] \to \mathbb{R}$ be continuous. By Exercise 15.5, there exist $x_1 \in [a, b]$ and $x_2 \in [a, b]$ such that $m = f(x_1) = \inf\{f(x) : x \in [a, b]\}$ and $M = \sup\{f(x) : x \in [a, b]\}$.
(a) Show that $f([a, b]) \subseteq [m, M]$.
(b) Use Exercise 15.7 (restricted to $[x_1, x_2]$) to show that $[m, M] \subseteq f([a, b])$.
Conclusion: $f([a, b]) = [m, M]$.

Solution.
(a) Let $y \in f([a, b])$. Then there is $x \in [a, b]$ such that $y = f(x)$. But $m \leq f(x) \leq M$, that is, $m \leq y \leq M$. This shows that $y \in [m, M]$ and therefore $f([a, b]) \subseteq [m, M]$.
(b) Let $y \in [m, M]$. By Exercise 15.7 restricted to the interval $[x_1, x_2]$, there exists $a \leq x_1 \leq x \leq x_2 \leq b$ such that $f(x) = y$. That is, $y \in f([a, b])$. This shows that $[m, M] \subseteq f([a, b])$ ■

Exercise 15.9
Prove that there exists a number $c \in \left(0, \frac{\pi}{2}\right)$ such that $2c - 1 = \sin\left(c^2 + \frac{\pi}{4}\right)$.

Solution.
Let $f(x) = 2x - 1 - \sin\left(x^2 + \frac{\pi}{4}\right)$. Then $f(0) = -1 - \frac{1}{\sqrt{2}} < 0$ and $f\left(\frac{\pi}{2}\right) = \pi - 1 - \sin\left(\frac{\pi^2}{2} + \frac{\pi}{4}\right) > 0$. By the Intermediate value theorem, there is a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f(c) = 0$ or $2c - 1 = \sin\left(c^2 + \frac{\pi}{4}\right)$ ■

Exercise 15.10
Let $f : [a, b] \to [a, b]$ be a continuous function. Prove that there is $c \in [a, b]$ such that $f(c) = c$. We call c a **fixed point** of f. Hint: Intermediate Value Theorem applied to a specific function F (to be found) defined on $[a, b]$.

Solution.
Define $F : [a, b] \to \mathbb{R}$ by $F(x) = x - f(x)$. Then F is continuous on $[a, b]$. Since $a \leq f(a) \leq b$ and $a \leq f(b) \leq b$ we find $F(a) = a - f(a) \leq 0$ and $F(b) = b - f(b) \geq 0$. By the Intermediate Value Theorem, there is a $c \in [a, b]$ such that $F(c) = 0$ or $c - f(c) = 0$. Thus, $f(c) = c$ ■

Exercise 15.11
Using the Intermediate Value Theorem, show that
(a) the equation $3 \tan x = 2 + \sin x$ has a solution in the interval $(0, \frac{\pi}{4})$.
(b) the polynomial $p(x) = -x^4 + 2x^3 + 2$ has at least two real roots.

Solution.
(a) Let $f(x) = 3 \tan x - \sin x - 2$. Then f is continuous on $[0, \frac{\pi}{4}]$ and $f(0) = -2 < 0,\ f\left(\frac{\pi}{4}\right) = 1 - \frac{1}{\sqrt{2}} > 0$. By IVT, there is a $c \in (0, \frac{\pi}{4})$ such that $f(c) = 0$. This means, the given equation has at least one solution in the interval $(0, \frac{\pi}{4})$.
(b) Since $p(-1) = -1 < 0,\ p(0) = 2 > 0$, and $p(3) = -25 < 0$ there exist at least two numbers $-1 < c_1 < 0 < 3$ such that $f(c_1) = f(c_2) = 0$ ■

Exercise 15.12
Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions such that $f(a) \leq g(a)$ and $f(b) \geq g(b)$. Show that there is a $c \in [a, b]$ such that $f(c) = g(c)$.

Solution.
Let $h(x) = f(x) - g(x)$. Then h is continuous on $[a, b]$ with $h(a) \leq 0 \leq h(b)$. By the IVT, there is a $c \in [a, b]$ such that $h(c) = 0$ or $f(c) = g(c)$ ■

Exercise 15.13
Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f(a) \leq a$ and $f(b) \geq b$. Prove that there is a $c \in [a, b]$ such that $f(c) = c$. We call c a **fixed point** of f.

Solution.
Let $g(x) = f(x) - x$. Then g is continuous on $[a, b]$ with $g(a) \leq 0$ and $g(b) \geq 0$. By IVT, there is a $c \in [a, b]$ such that $g(c) = 0$. That is, $f(c) = c$ ■

Exercise 15.14
Let $f : [a,b] \to \mathbb{R}\backslash\mathbb{Q}$ be continuous. Prove that f must be a constant function. Hint: Exercise 2.6(c).

Solution.
Suppose for a contradiction that f is not constant. Then, we can find $x, y \in [a,b]$ with $x < y$ and such that $f(x) \neq f(y)$. Choose an rational number m lying between $f(x)$ and $f(y)$. Then, by the intermediate value theorem, there exists $z \in [x,y]$ with $f(z) = m$. Hence, f takes a rational value, contradicting the hypotheses ∎

Exercise 15.15
Prove that a polynomial of odd degree considered as a function from the reals to the reals has at least one real root.

Solution.
Let $f(x)$ be a polynomial of odd degree. Then $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$ (or $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -\infty$ depending on whether the leading coefficient is positive or negative, respectively). Hence, there exist $a < b$ in $\mathbb{R}$ such that $f(a) < 0$ and $f(b) > 0$: Now the Intermediate Value Theorem applies to give an $x \in (a,b)$ such that $f(x) = 0$ ∎

Exercise 15.16
Suppose $f(x)$ is continuous on the interval $[0,2]$ and $f(0) = f(2)$: Prove there must be a number c between 0 and 1 so that $f(c+1) = f(c)$. Hint: Consider the function $g(x) = f(x+1) - f(x)$ on $[0,1]$.

Solution.
We let $g(x) = f(x+1) - f(x)$. $f(x)$ is continuous on $[0,1]$. Furthermore,

$$g(0) = f(1) - f(0)$$

and

$$g(1) = f(2) - f(1) = f(0) - f(1) = -(f(1) - f(0)).$$

If $f(1) = f(0)$ we have obtained the desired conclusion upon taking $c = 0$. We therefore assume $f(0) \neq f(1)$. But then $g(0)$ and $g(1)$ have opposite signs. The Intermediate Value Theorem therefore guarantees the existence of a number c in the open interval $(0,1)$ satisfying $g(c) = 0$. But by definition of $g(x)$, this means $f(c+1) = f(c)$ ∎

Solutions to Section 16

Exercise 16.1
Consider the function

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is not differentiable at $a = 0$.

Solution.
We have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \sin\left(\frac{1}{h}\right).$$

By Exercise 8.3, the limit does not exist. Hence, f is not differentiable at 0 ■

Exercise 16.2
Consider the function

$$f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is differentiable at $a = 0$. What is $f'(0)$?

Solution.
We have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right).$$

By Exercise 9.4(b), the limit does exist and is equal to 0 ■

Exercise 16.3
Show that $f(x) = |x|$ is not differentiable at 0.

Solution.
We have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

By Exercise 8.2, the limit does not exist. Hence f is not differentiable at 0 ■

Exercise 16.4
Find the derivative of $f(x) = \sin x$. Hint: Recall the trigonometric identity $\sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$ and use Exercise 9.11.

Solution.
Using Exercise 9.11 along with the continuity of the cosine function we have

$$\begin{aligned} f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\ =\lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \\ =\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right] = \cos x \cdot 1 = \cos x \blacksquare \end{aligned}$$

Exercise 16.5
Let $f : D \to \mathbb{R}$ be differentiable at a.
(a) Show that

$$\lim_{x\to a}[f(x) - f(a)] = \lim_{h\to 0}[f(h+a) - f(a)].$$

(b) Show that f is continuous at a. That is,

$$\lim_{x\to a}[f(x) - f(a)] = 0.$$

Solution.
(a) Letting $h = x - a$ and noting that $\lim_{x\to a} f(x) = \lim_{h\to 0} f(h+a)$, We have

$$\lim_{x\to a}[f(x) - f(a)] = \lim_{h\to 0}[f(h+a) - f(a)].$$

(b) We have

$$\begin{aligned} \lim_{x\to a}[f(x) - f(a)] &= \lim_{h\to 0}[f(h+a) - f(a)] \\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot h \\ &= \left[\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right][\lim_{h\to 0} h] = f'(a)\cdot 0 = 0 \end{aligned}$$

Hence, f is continuous at a $\blacksquare$

Exercise 16.6
Give an example of a function $f : D \to \mathbb{R}$ that is continuous at a but not differentiable there.

Solution.
The function $f(x) = |x|$ is continuous at 0 but not differentiable there (Exercise 16.3) ■

Exercise 16.7
Suppose that $f, g : D \to \mathbb{R}$ are differentiable at a. Show that the functions $f \pm g$ are also differentiable at a.

Solution.
We have

$$\begin{aligned}(f \pm g)(a) =& \lim_{h\to 0} \frac{(f \pm g)(a+h) - (f \pm g)(a)}{h} \\ =& \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h} \pm \frac{g(a+h)-g(a)}{h}\right] \\ =& \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right] \pm \lim_{h\to 0}\left[\frac{g(a+h)-g(a)}{h}\right] \\ =& f'(a) \pm g'(a) \blacksquare\end{aligned}$$

Exercise 16.8 *(Product Rule)*
Suppose that $f, g : D \to \mathbb{R}$ are differentiable at a.
(a) Show that $(fg)(a+h) - (fg)(a) = [f(a+h) - f(a)]g(a+h) + f(a)[g(a+h) - g(a)]$.
(b) Show that the function $f \cdot g$ is also differentiable at a.

Solution.
(a) We have

$$\begin{aligned}[f(a+h) - f(a)]g(a+h) + f(a)[g(a+h) - g(a)] =& f(a+h)g(a+h) - f(a)g(a+h) \\ &+ f(a)g(a+h) - f(a)g(a) \\ =& (fg)(a+h) - (fg)(a).\end{aligned}$$

(b) Using (a) and the continuity of g We have

$$\begin{aligned}(f\cdot g)'(a)=&\lim_{h\to 0}\frac{(f\cdot g)(a+h)-(f\cdot g)(a)}{h}\\=&\lim_{h\to 0}\frac{[f(a+h)-f(a)]g(a+h)+f(a)[g(a+h)-g(a)]}{h}\\=&\left[\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right][\lim_{h\to 0}g(a+h)]+f(a)\left[\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}\right]\\=&f'(a)g(a)+f(a)g'(a)\ \blacksquare\end{aligned}$$

Exercise 16.9 *(Quotient Rule)*
Suppose that $f,g:D\to\mathbb{R}$ are differentiable at a with $g(a)\neq 0$.
(a) Show that

$$\frac{\left(\frac{f}{g}\right)(a+h)-\left(\frac{f}{g}\right)(a)}{h}=\frac{f(a+h)-f(a)}{h}\cdot\frac{1}{g(a+h)}-\frac{g(a+h)-g(a)}{h}\cdot\frac{f(a)}{g(a)}\cdot\frac{1}{g(a+h)}.$$

(b) Show that

$$\left(\frac{f}{g}\right)'(a)=\frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}.$$

Solution.
(a) We have

$$\begin{aligned}\frac{\left(\frac{f}{g}\right)(a+h)-\left(\frac{f}{g}\right)(a)}{h}=&\frac{f(a+h)g(a)-f(a)g(a+h)}{hg(a+h)g(a)}\\=&\frac{[f(a+h)-f(a)]g(a)-f(a)[g(a+h)-g(a)]}{hg(a+h)g(a)}\\=&\frac{f(a+h)-f(a)}{h}\cdot\frac{1}{g(a+h)}\\&-\frac{g(a+h)-g(a)}{h}\cdot\frac{f(a)}{g(a)}\cdot\frac{1}{g(a+h)}\end{aligned}$$

(b) Since g is differentiable at a, it is continuous there. That is, $\lim_{h\to 0}g(a+$

$h) = g(a)$. Now,

$$
\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \lim_{h\to 0} \frac{\left(\frac{f}{g}\right)(a+h) - \left(\frac{f}{g}\right)(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \cdot \frac{1}{g(a+h)} \\
&- \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \cdot \frac{f(a)}{g(a)} \cdot \frac{1}{g(a+h)} \\
&= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \cdot \frac{1}{\lim_{h\to 0} g(a+h)} \\
&- \frac{f(a)}{g(a)} \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} \cdot \frac{1}{\lim_{h\to 0} g(a+h)} \\
&= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2} \quad \blacksquare
\end{aligned}
$$

Exercise 16.10 *(Chain Rule)*
Let $f : D \to \mathbb{R}$ and $g : D' \to \mathbb{R}$ be two functions with $f(D) \subseteq D'$. Suppose that f is differentiable at a and g is differentiable at $f(a)$.
(a) Define $w : D' \to \mathbb{R}$ by

$$
w(y) = \begin{cases} \frac{g(y)-g(f(a))}{y-f(a)} & \text{if } y \neq f(a) \\ g'(f(a)) & \text{if } y = f(a). \end{cases}
$$

Show that w is continuous at $f(a)$. That is,

$$
\lim_{h\to 0} w(f(a)+h) = w(f(a)).
$$

(b) Show that $(w \circ f)(x)$ is continuous at a.
(c) Show that

$$
\frac{(g\circ f)(a+h) - (g\circ f)(a)}{h} = (w\circ f)(a+h) \cdot \frac{f(a+h)-f(a)}{h}.
$$

(d) Show that

$$
(g\circ f)'(a) = g'(f(a)) \cdot f'(a).
$$

Solution.
(a) This follows from

$$\lim_{h\to 0} w(h+f(a)) = \lim_{h\to 0}\frac{g(h+f(a))-g(f(a))}{h} = g'(f(a)) = w(f(a))$$

(b) The composition of two conitnuous functions is a continuous function according to Exercise 12.3(a).
(c) We have

$$\begin{aligned}\frac{(g\circ f)(a+h)-(g\circ f)(a)}{h} =&\frac{g(f(a+h))-g(f(a))}{f(a+h)-f(a)}\cdot\frac{f(a+h)-f(a)}{h}\\ =&(w\circ f)(a+h)\cdot\frac{f(a+h)-f(a)}{h}.\end{aligned}$$

(d) We have

$$\begin{aligned}(g\circ f)'(a) =& \lim_{h\to 0}\frac{g(f(a+h))-g(f(a))}{f(a+h)-f(a)}\cdot\frac{f(a+h)-f(a)}{h}\\ =&\lim_{h\to 0}(w\circ f)(a+h)\cdot\left[\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right]\\ =&w(f(a))\cdot f'(a) = g'(f(a))\cdot f'(a) \blacksquare\end{aligned}$$

Exercise 16.11 *(Power Rule)*
Let $f(x) = x^n$ where n is a non-negative integer.
(a) By letting $h = ax - x$, show that

$$f'(x) = \lim_{a\to 1}\frac{f(ax)-f(x)}{ax-x}.$$

(b) What is the quotient of the division of $a^n - 1$ by $a - 1$? Hint: use synthetic division.
(c) Use (a) and (b) to show that $f'(x) = nx^{n-1}$.

Solution.
(a) From the definition of $f'(x)$ we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Letting $h = ax - x$ we see that $a \to 1$ as $h \to 0$. Thus, with this substitution we can write

$$f'(x) = \lim_{a\to 1} \frac{f(ax) - f(x)}{ax - x}.$$

(b) Using synthetic division we find

$$a^n - 1 = (a-1)(1 + a + a^2 + \cdots + a^{n-1}).$$

(c) Using (a) and (b) we find

$$\begin{aligned} f'(x) =& \lim_{a\to 1} \frac{(ax)^n - x^n}{ax - x} = x^{n-1} \lim_{a\to 1} \frac{a^n - 1}{a - 1} \\ =& x^{n-1} \lim_{a\to 1}(1 + a + a^2 + \cdots + a^{n-1}) = nx^{n-1} \blacksquare \end{aligned}$$

Exercise 16.12
(a) Show that the derivative of a constant function is zero and that the derivative of $f(x) = x$ is $f'(x) = 1$.
(b) Show that the function $h(x) = x \sin\left(\frac{1}{x}\right)$ is differentiable for all $x \neq 0$.

Solution.
(a) Suppose that $f(x) = C$ for all x. Then

$$\begin{aligned} f'(x) =& \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ =& \lim_{h\to 0} \frac{C - C}{h} = 0 \end{aligned}$$

Now, suppose that $f(x) = x$. Then

$$\begin{aligned} f'(x) =& \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ =& \lim_{h\to 0} \frac{x + h - x}{h} = 1 \end{aligned}$$

(b) Let $f(x) = \frac{1}{x}$. Then f is differentiable for all $x \neq 0$. Let $g(x) = \sin x$. Then $g(x)$ is differentiable for all x. But $h(x) = g(f(x))$ so that by the chain rule h is differentiable for all $x \neq 0$ $\blacksquare$

Exercise 16.13
Let $f(x) = \sqrt{2x - 1}$. Find $f'(2)$ by using only the definition of derivative.

Solution.
We have

$$\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{2h+3}-\sqrt{3}}{h}\\
&= \lim_{h\to 0}\frac{(\sqrt{2h+3}-\sqrt{3})(\sqrt{2h+3}+\sqrt{3})}{h(\sqrt{2h+3}+\sqrt{3})}\\
&= \lim_{h\to 0}\frac{2h}{h(\sqrt{2h+3}+\sqrt{3})}\\
&= \lim_{h\to 0}\frac{2}{\sqrt{2h+3}+\sqrt{3}} = \frac{\sqrt{3}}{3} \ \blacksquare
\end{aligned}$$

Exercise 16.14
Let

$$f(x) = \begin{cases} 2x+5 & \text{if } x \le 1\\ 9x^2-2 & \text{if } x > 1.\end{cases}$$

Show that $f(x)$ is continuous but not differentiable at $x = 1$.

Solution.
Since

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x+5) = 7$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(9x^2-2) = 7$$

we conclude that f is continuous at $x = 1$.
Now, since

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{2(1+h)+5-7}{h} = 2$$

and

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+}\frac{9(1+h)^2-2-7}{h} = \lim_{h\to 0^+}(18+9h) = 18$$

the function is NOT differentiable at $x = 1$ $\blacksquare$

Exercise 16.15
Find constants a and b such that the piecewise defined function

$$f(x) = \begin{cases} ax^2 - 4 & \text{if } x \leq 1 \\ bx + a & \text{if } x > 1 \end{cases}$$

is differentiable at $x = 1$.

Solution.
Since f is differentiable at $x = 1$, it is continuous there. Thus, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$. That is, $a - 4 = b + a$ and this implies that $b = -4$. Now, since f is differentiable at 1 we must have

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h}.$$

That is, $-4 = 2a$ or $a = -2$ ■

Exercise 16.16
Let $f(x) = x^2 \cos\left(\frac{1}{x}\right)$ if $x \neq 0$ and $f(0) = 0$. Show that f is differentiable at $x = 0$ and find $f'(0)$.

Solution.
We have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} h \cos\left(\frac{1}{h}\right) = 0.$$

Note that the cosine function is bounded and $\lim_{h\to 0} h = 0$ ■

Exercise 16.17
(a) Let $f(x) = x^n$ with n a negative integer. Prove that $f'(x) = nx^{n-1}$.
(b) Let $f(x) = x^{\frac{p}{q}}$ where p and q are integers with $q \neq 0$. Prove that $f'(x) = \frac{p}{q}x^{\frac{p}{q}-1}$. Hint: Let $y = x^{\frac{p}{q}}$ so that $y^q = x^p$ and use Exercise 16.10.

Solution.
(a) We have $f(x) = \frac{1}{x^{-n}}$ so that by the quotient rule we obtain $f'(x) = -\frac{(-n)x^{-n-1}}{(x^{-n})^2} = nx^{n-1}$.
(b) Let $y = x^{\frac{p}{q}}$. The $y^q = x^p$. Differentiate both sides and use (a) and Exercise 16.10 we find $qy^{q-1}y' = px^{p-1}$. Thus,

$$f'(x) = y' = \frac{p}{q}x^{p-1}y^{1-q} = \frac{p}{q}x^{p-1}x^{\frac{p}{q}-p} = \frac{p}{q}x^{\frac{p}{q}-1} \ ■$$

Exercise 16.18
We define the number e to be the unique number satisfying

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

It is an irrational number whose value is approximately 2.718281828459045. Define the function $f(x) = e^x$. Find $f'(x)$ using the definition of the derivative.

Solution.
We have

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \blacksquare$$

Exercise 16.19
The **natural logarithmic function** is the function $f(x) = \ln x$ defined as follows: $y = \ln x$ if and only if $x = e^y$. Find the derivative of f. Hint: Differentiate $x = e^y$ with respect to x.

Solution.
Differentiating $x = e^y$ with respect to x we obtain $1 = y'e^y$ and therefore $y' = \frac{1}{e^y} = \frac{1}{x} \blacksquare$

Exercise 16.20
Consider the function $f(x) = x^n$ where n is a real number.
(a) Suppose that $x > 0$ and x in the domain of f. Using the fact that $x^n = e^{n \ln x}$, show that $f'(x) = nx^{n-1}$.
(b) Suppose that $x < 0$ and x in the domain of f. Show that $f'(x) = nx^{n-1}$. Hint: $x^n = (-1)^n(-x)^n$.

Solution.
(a) We have $f'(x) = e^{\ln x}(n \ln x)' = \frac{n}{x}x^n = nx^{n-1}$.
(b) We have $f'(x) = (-1)^n n(-x)^{n-1}(-1) = (-1)^{n+1}n(-x)^{n-1} = (-1)^{n-1}n(-x)^{n-1} = nx^{n-1} \blacksquare$

Solutions to Section 17

Exercise 17.1
(a) Find the local extrema (if they exist) of the function $f(x) = |x|$.
(b) Find the local extrema (if they exist) of the function $f(x) = x^3$.
(c) Find the local extrema (if they exist) of the function $f(x) = x$ on the interval $[0, 1]$.

Solution.
(a) Since $|x| \geq 0$ for all x, we find that f has a local minimum at $x = 0$. However, f has no local maximum since f is always increasing for $x \geq 0$ and always decreasing for $x < 0$.
(b) The graph of $f(x) = x^3$ is always increasing so that f has no local extrema.
(c) The graph is a straight line that is rising to the right. Thus, $f(x)$ has a local minimum at $x = 0$ and a local maximum at $x = 1$ ■

Exercise 17.2
Let $f : [a, b] \to \mathbb{R}$. Suppose that $c \in (a, b)$ is a local maximum (or local minimum) of f such that $f'(c)$ exists. Let $\epsilon > 0$ such that $f(x) \leq f(c)$ for all $x \in (c - \epsilon, c + \epsilon) \subseteq [a, b]$.
(a) Let $h > 0$ be small enough so that $c + h \in (c - \epsilon, c + \epsilon)$. Using Exercise 9.8, show that $f'(c) \leq 0$.
(b) Let$h < 0$ be large enough so that $c + h \in (c - \epsilon, c + \epsilon)$. Using Exercise 9.8, show that $0 \leq f'(c)$ and therefore $f'(c) = 0$.

Solution.
(a) Since $c+h \in (c-\epsilon, c+\epsilon)$, we have $f(c+h) \leq f(c)$. Thus, $\frac{f(c+h)-f(c)}{h} \leq 0$. By Exercise 9.8, we have

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq \lim_{h \to 0^+} 0 = 0.$$

(b) Since $c+h \in (c-\epsilon, c+\epsilon)$, we have $f(c+h) \leq f(c)$. Thus, $\frac{f(c+h)-f(c)}{h} \geq 0$. By Exercise 9.8, we have

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq \lim_{h \to 0^-} 0 = 0.$$

From (a) and (b) we conclude that $f'(c) = 0$ ■

Exercise 17.3
The condition $a < c < b$ is critical in the previous exercise. Give an example of a function $f : [a, b] \to \mathbb{R}$ such that either a or b is a local extremum but with non-zero derivative there.

Solution.
See Exercise 17.1(c). f has a local minimum at 0 with $f'(0) = 1 \neq 0$ and a local maximum at 1 with $f'(1) = 1 \neq 0$ ■

Exercise 17.4
Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then there exists $x_1, x_2 \in [a, b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2)$$

for all $x \in [a, b]$. Show that x_1 and x_2 are either the endpoints of $[a, b]$ or critical points of f in $a < x < b$.

Solution.
Since $x_1 \in [a, b]$, either $x_1 = a, x_1 = b$, or $a < x_1 < b$. If $a < x_1 < b$ then by Exercise 17.2, $f'(x_1) = 0$. That is, x_1 is a critical point of f. Similar argument holds for x_2 ■

Exercise 17.5 *(Rolle's Theorem)*
Suppose $f : [a, b] \to \mathbb{R}$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. By Exercise 15.8 there exist $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$. Suppose that $f(a) = f(b)$.
(a) Show that if $f(x) = C$ for all $a \leq x \leq b$ then there is at least a number $a < c < b$ such that $f'(c) = 0$.
(b) Suppose that f is a non-constant function. Let $d \in [a, b]$ such that $f(d) \neq f(a)$. Show that if $f(d) < f(a)$ then $a < x_1 < b$. What can you say about the value of $f'(x_1)$.
(c) Show that if $f(a) < f(d)$ then $a < x_2 < b$. What can you say about the value of $f'(x_2)$.

Solution.
(a) If $f(x) = C$ for all $a \leq x \leq b$ then $f'(x) = 0$ for all $a \leq x \leq b$. In particular, $f'\left(\frac{a+b}{2}\right) = 0$ with $a < \frac{a+b}{2} < b$. So $c = \frac{a+b}{2}$.
(b) Suppose $f(d) < f(a)$. If $x_1 = a$ then $f(d) < f(a) = f(x_1)$ which is impossible since $f(x_1) \leq f(x)$ for all $x \in [a, b]$. If $x_1 = b$ then $f(d) < f(a) =$

$f(b) = f(x_1)$ which is again impossible. So we must have $a < x_1 < b$. Now, by Exercise 17.2 we must have $f'(x_1) = 0$.
(c) Suppose $f(a) < f(d)$. If $x_2 = a$ then $f(x_2) < f(a)$ which is impossible since $f(x) \leq f(x_2)$ for all $x \in [a, b]$. If $x_2 = b$ then $f(x_2) = f(b) < f(d)$ which is again impossible. So we must have $a < x_2 < b$. Now, by Exercise 17.2 we must have $f'(x_2) = 0$ ■

Exercise 17.6
Find the number c of Rolle's theorem for the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \sqrt{x} - x$.

Solution.
The function f satisfies the conditions of Rolle's theorem. We have $0 = f'(c) = \frac{1}{2\sqrt{c}} - 1$. Solving this equation for c we find $c = \frac{1}{4}$ ■

Exercise 17.7
Assume $a_0, a_1, \cdots, a_n$ are real numbers such that

$$\frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \cdots + \frac{a_1}{2} + a_0 = 0$$

Show that the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

has at least one root in $(0, 1)$.

Solution.
Let

$$F(x) = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \cdots + \frac{a_1}{2} x^2 + a_0 x.$$

Note that F is continuous in $[0, 1]$ and differentiable in $(0, 1)$ with derivative $F'(x) = f(x)$. Moreover, $F(0) = F(1) = 0$. By Rolle's theorem, there is a $c \in (0, 1)$ such that $F'(c) = 0$. Hence, $f(c) = 0$ ■

Exercise 17.8
(a) Show that the function $f(x) = x^3 - 4x^2 - 3x + 1$ has a root in $[0, 2]$.
(b) Use Rolle's theorem to show that there is exactly one root in $[0, 2]$.

Solution.
(a) We have $f(0) = 1 > 0$ and $f(2) = -13 < 0$ so that by IVT there is a root in $[0, 2]$.
(b) Suppose that x_1 and x_2 are two roots of f in $[0, 2]$. Then by Rolle's theorem we must have $c \in (0, 2)$ such that $f'(c) = 0$. But the solutions to $f'(x) = 0 = 3x^2 - 8x - 3$ are $x = 3$ and $x = -\frac{1}{3}$ where neither is in $[0, 2]$. Hence, f has exactly one solution in $[0, 2]$ ■

Exercise 17.9
Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable, and let $a, b \in \mathbb{R}$ be such that $a < b$. Show that there is a $c \in (a, b)$ such that

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Hint: Apply Rolle's theorem to the function $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$.

Solution.
Let $h; [a, b] \to \mathbb{R}$ be the function defined by

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

Then h is continuous on $[a, b]$ and differentiable in $a < x < b$ with derivative

$$h'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x).$$

Moreover

$$h(a) = h(b) = f(b)g(a) - g(b)f(a).$$

By Rolle's theorem there is a $a < c < b$ such that $h'(c) = 0$. That is

$$[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0$$

or

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c) \blacksquare$$

Exercise 17.10
Suppose $f : [a, b] \to \mathbb{R}$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. Show that there is $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Hint: Apply Rolle's theorem to the function $g : [a, b] \to \mathbb{R}$ defined by

$$g(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a).$$

Solution.
The function $g(x)$ is continuous on $[a, b]$ being a combination of continuous functions on $[a, b]$. Furthermore, $g(x)$ is differentiable for $a < x < b$ with derivative

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Also, $g(a) = g(b) = 0$. By Exercise 17.5, there is $a < c < b$ such that $g'(c) = 0$ which is equivalent to

$$f'(c) = \frac{f(b) - f(a)}{b - a} \blacksquare$$

Solutions to Section 18

Exercise 18.1 *(Mean Value Theorem)*
Suppose $f : [a, b] \to \mathbb{R}$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. Show that there is $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Hint: Use Exercise 17.5 with the function $g : [a, b] \to \mathbb{R}$ defined by

$$g(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a).$$

Solution.
The function $g(x)$ is continuous on $[a, b]$ being a combination of continuous functions on $[a, b]$. Furthermore, $g(x)$ is differentiable for $a < x < b$ with derivative

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Also, $g(a) = g(b) = 0$. By Exercise 17.5, there is $a < c < b$ such that $g'(c) = 0$ which is equivalent to

$$f'(c) = \frac{f(b) - f(a)}{b - a} \blacksquare$$

Exercise 18.2 *(Cauchy Mean Value Theorem)*
Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. Show that there is $a < c < b$ such that

$$[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c).$$

Hint: Use Exercise 17.5 with the function $h : [a, b] \to \mathbb{R}$ defined by

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

Solution.
Let $h; [a, b] \to \mathbb{R}$ be the function defined by

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

Then h is continuous on $[a, b]$ and differentiable in $a < x < b$ with derivative

$$h'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x).$$

Moreover

$$h(a) = h(b) = f(b)g(a) - g(b)f(a).$$

By Rolle's theorem there is a $a < c < b$ such that $h'(c) = 0$. That is

$$[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0$$

or

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c) \blacksquare$$

Exercise 18.3
Let $f : [a, b] \to \mathbb{R}$ be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. We say that f is one-to-one if and only if for any $a \leq x_1 \leq b$ and $a \leq x_2 \leq b$ such that $f(x_1) = f(x_2)$ we must have $x_1 = x_2$. Suppose that $f'(x) \neq 0$ for all $a < x < b$.
(a) Let $a \leq x_1 \leq b$ and $a \leq x_2 \leq b$ such that $f(x_1) = f(x_2)$. Show that if $x_1 < x_2$ then there is $a < x_1 < c < x_2 < b$ such that $f'(c) = 0$ which contradicts the assumption that $f'(x) \neq 0$ for all $a < x < b$. Hint: Use the Mean Value Theorem on the interval $[x_1, x_2]$..
(b) Answer the same question for $x_2 < x_1$.
Conclusion: We must have $x_1 = x_2$. This shows that f is 1-1.

Solution.
(a) Applying the MVT on the interval $x_1 \leq x \leq x_2$, we can find a number $x_1 < c < x_2$ such that $f(x_2)-f(x_1) = f'(c)(x_2-x_1)$. Since $f(x_1) = f(x_2) = 0$ we obtain $(x_2 - x_1)f'(c) = 0$. Since $x_1 \neq x_2$ we find that $f'(c) = 0$ which contradicts the assumption on f'
(b) Argument similar to (a) $\blacksquare$

Exercise 18.4
Let $f : [a, b] \to \mathbb{R}$ be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. We say that f is **increasing** in $[a, b]$ if and only if for every x_1 and x_2 in $[a, b]$, if $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$. Show that if $f'(x) \geq 0$ for all $a < x < b$ then $f(x)$ is increasing in $[a, b]$. Hint: Use the MVT restricted to the interval $[x_1, x_2]$.

Solution.
Let $x_1, x_2 \in [a, b]$. Clearly, if $x_1 = x_2$ then $f(x_1) = f(x_2)$. So assume that $x_1 < x_2$. By the MVT there is a $x_1 < c < x_2$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \geq 0$ which implies that $f(x_1) \leq f(x_2)$. Thus, we have shown that $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$. That is, f is increasing in $[a, b]$ ■

Exercise 18.5
Consider Case (i). We have either $f(y) < f(x) < f(z)$ or $f(z) < f(x) < f(y)$.
(a) Suppose that $f(z) < f(x) < f(y)$. Use the Intermediate Value theorem restricted to $[y, z]$ to show that such a double inequality can not occur.
(b) Suppose that $f(x) < f(z) < f(y)$. Use the Intermediate Value theorem restricted to $[x, y]$ to show that such a double inequality can not occur.
We conclude that Case (i) does not hold.

Solution.
(a) If $f(z) < f(x) < f(y)$, we can apply the Intermediate value theorem to $[y, z]$ to find $y < w < z$ such that $f(w) = f(x)$. Since f is one-to-one we must have $w = x < y$ which contradicts the inequality $y < w$.
(b) If $f(x) < f(z) < f(y)$, we can apply the Intermediate value theorem to $[x, y]$ to find $x < w < y$ such that $f(w) = f(z)$. Since f is one-to-one we must have $w = z < y$ which contradicts the inequality $y < z$ ■

Exercise 18.6
Consider Case (ii). We have either $f(y) < f(x) < f(z)$ or $f(y) < f(z) < f(x)$.
(a) Suppose that $f(y) < f(x) < f(z)$. Use the Intermediate Value theorem restricted to $[y, z]$ to show that such a double inequality can not occur.
(b) Suppose that $f(y) < f(z) < f(x)$. Use the Intermediate Value theorem restricted to $[x, y]$ to show that such a double inequality can not occur.
We conclude that Case (ii) does not hold.

Solution.
(a) If $f(y) < f(x) < f(z)$, we can apply the Intermediate value theorem to $[y, z]$ to find $y < w < z$ such that $f(w) = f(x)$. Since f is one-to-one we must have $w = x < y$ which contradicts the inequality $y < w$.
(a) If $f(y) < f(z) < f(x)$, we can apply the Intermediate value theorem to $[x, y]$ to find $x < w < y$ such that $f(w) = f(z)$. Since f is one-to-one we must have $w = z < y$ which contradicts the inequality $y < z$ ■

Exercise 18.7
Suppose that $f : [a,b] \to \mathbb{R}$ is differentiable such that $f'(x) \neq 0$ for all $a < x < b$. We know from the above discussion that f is monotone.
(a) Show that if f is increasing in $[a,b]$ then $f'(x) \geq 0$ for all $a \leq x \leq b$. Hint: Let $x \in [a,b)$ and choose $h > 0$ small enough so that $x + h \in [a,b)$. If $x = b$, choose $h < 0$ so that $b + h < b$. Now use the definition of the derivative.
(b) Show that if f is decreasing in $[a,b]$ then $f'(x) \leq 0$ for all $a \leq x \leq b$.

Solution.
(a) Let $x \in [a,b)$. Choose $h > 0$ so that $x + h \in [a,b)$. Since $x < x + h$ and f is increasing, we find that $\frac{f(x+h)-f(x)}{h} \geq 0$. Thus,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \geq 0.$$

If $x = b$ choose $h < 0$ so that $b + h < b$. In this case, $f(b+h) \leq f(b)$ and

$$f'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \geq 0.$$

(b) Let $x \in [a,b)$. Choose $h > 0$ so that $x + h \in [a,b)$. Since $x < x + h$ and f is decreasing, we find that $\frac{f(x+h)-f(x)}{h} \leq 0$. Thus,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \leq 0 \blacksquare$$

If $x = b$ choose $h < 0$ so that $b + h < b$. In this case, $f(b+h) \geq f(b)$ and

$$f'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \leq 0 \blacksquare$$

Exercise 18.8
Let $f : [a,b] \to \mathbb{R}$ be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. We say that f is a constant function on $[a,b]$ if and only if there is a constant C such that $f(x) = C$ for all $a \leq x \leq b$. Suppose that $f'(x) = 0$ for all $a < x < b$.
Let x_1 and x_2 be any two numbers in the interval $[a,b]$ with $x_1 < x_2$. Suppose that $f(x_1) \neq f(x_2)$. Show that by applying the Mean Value Theorem on the interval $[x_1, x_2]$ we obtain the contradiction $f(x_1) = f(x_2)$. Thus, we must have $f(x_1) = f(x_2) = C$ for any x_1 and x_2 in $[a,b]$. That is, $f(x) = C$ for all $a \leq x \leq b$.

Solution.
Applying the MVT on the interval $x_1 \leq x \leq x_2$, we can find a number $x_1 < c < x_2$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since $f'(c) = 0$ we obtain $f(x_1) = f(x_2)$, a contradiction. Since x_1 and x_2 were arbitrary, we have $f(x) = C$ for all $x \in [a, b]$ ■

Exercise 18.9
Let $f : [a, b] \to \mathbb{R}$ be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. Suppose that $f'(x) = g'(x)$ for all $a < x < b$. Show that $f(x) = g(x) + C$ for all $a \leq x \leq b$, where C is a constant. Hint: Exercise 18.8

Solution.
Let $h(x) = f(x) - g(x)$. Then $h(x)$ is continuous in $[a, b]$ being the difference of two continuous functions and $h'(x) = 0$ for all $a < x < b$. By Exercise 18.8, there is C such that $h(x) = C$ for all $a \leq x \leq b$ or equivalently $f(x) = g(x) + C$ for all $a \leq x \leq b$ ■

Exercise 18.10
Let $f : [a, b] \to \mathbb{R}$ be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$. We say that f is **decreasing** in $[a, b]$ if and only if for every x_1 and x_2 in $[a, b]$, if $x_1 \leq x_2$ then $f(x_1) \geq f(x_2)$. Show that if $f'(x) \leq 0$ for all $a < x < b$ then $f(x)$ is decreasing in $[a, b]$. Hint: Use the MVT restricted to the interval $[x_1, x_2]$.

Solution.
Let $x_1, x_2 \in [a, b]$. Clearly, if $x_1 = x_2$ then $f(x_1) = f(x_2)$. So assume that $x_1 < x_2$. By the MVT there is a $x_1 < c < x_2$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \leq 0$ which implies that $f(x_1) \geq f(x_2)$. Thus, we have shown that $x_1 \leq x_2$ implies $f(x_1) \geq f(x_2)$. That is, f is decreasing in $[a, b]$ ■

Exercise 18.11
Consider the function $f(x) = (1 + x)^p$ where $0 < p < 1$. Let $h > 0$.
(a) Apply the MVT to the interval $[0, h]$ to show that $f(h) = p(1+t)^{p-1}h + 1$ for some $0 < t < h$.
(b) Use (a) to show that $(1 + h)^p < 1 + ph$.
In annuity theory, $(1 + h)^p$ may represent compound interest and $1 + ph$ represent simple interest. A result in annuity theory says that for time p less than a year compound interest formula can be estimated by the simple interest formula.

Solution.
(a) Applying the mean value theorem to the interval $[0, h]$, we can find a $0 < t < h$ such that $f(h) - f(0) = f'(t)h$ or $f(h) - 1 = p(1+t)^{p-1}h$. Hence, $f(h) = (1+h)^p = p(1+t)^{p-1}h + 1$.
(b) Since $t > 0$, we have $1 + t > 1 \to (1+t)^{1-p} > 1 \to (1+t)^{p-1} < 1 \to p(1+t)^{p-1}h < ph \to 1 + p(1+t)^{p-1}h < 1 + ph$. Hence, $(1+h)^p < 1 + ph$ ■

Exercise 18.12
Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable in $[a, b]$. Let λ be a real number such that either $f'(a) < \lambda < f'(b)$ or $f'(b) < \lambda < f'(a)$.
(a) Define $g(x) = f(x) - \lambda x$. Show that if $f'(a) < \lambda < f'(b)$ then $g'(x)$ changes sign between a and b.
(b) Establish the same result for $f'(b) < \lambda < f'(a)$.
(c) Show that the condition $g'(c) \neq 0$ for all $c \in [a, b]$ leads to a contradiction. Hint: Exercise 18.7. Conclude that there must be a $a < c < b$ such that $f'(c) = \lambda$.

Solution.
(a) Note that g is continuous in $[a, b]$ and differentiable there with derivative $g'(x) = f'(x) - \lambda$. Since $f'(a) < \lambda < f'(b)$, we find $g'(a) = f'(a) - \lambda < 0 < g'(b) = f'(b) - \lambda$. So g' changes sign from negative to positive.
(b) Since $f'(b) < \lambda < f'(a)$, we find $g'(b) = f'(b) - \lambda < 0 < g'(a) = f'(a) - \lambda$. So g' changes sign from positive to negative.
(c) If $g'(c) \neq 0$ for all $c \in [a, b]$ then by Exercise 18.7 either g' is always nonnegative in $[a, b]$ or always nonpositive which contradict (a) and (b). We conclude that there must be a $a < c < b$ such that $g'(c) = 0$ which is the same as $f'(c) = \lambda$ ■

Exercise 18.13
Let $f, g : [a, b] \to \mathbb{R}$ be two differentiable functions on $[a, b]$ such that $f(a) = g(a)$. Show that if $f'(x) = g'(x)$ for all $x \in (a, b)$ then $f(x) = f(x)$ for all $x \in [a, b]$. Hint: Exercise 18.8.

Solution.
Let $F : [a, b] \to \mathbb{R}$ be given by $F(x) = f(x) - g(x)$. Then F' is differentiable on $[a, b]$ and $F'(x) = 0$ for all $x \in (a, b)$. By Exercise 18.8, there is a constant C such that $F(x) = C$ for all $x \in [a, b]$. But $F(a) = 0$ so that $C = 0$. Thus, $F(x) = 0$ for all $x \in [a, b]$. This is equivalent to $f(x) = g(x)$ for all $x \in [a, b]$ ■

Exercise 18.14
Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $|f'(x)| < 1$ for all $x \in \mathbb{R}$. Show that f can have at most one fixed point. That is, There is at most one $c \in \mathbb{R}$ such that $f(c) = c$. Hint: Mean Value Theorem.

Solution.
Suppose the contrary. Let $a, b \in \mathbb{R}$ such that $a < b$, $f(a) = a$, and $f(b) = b$. We have that f is continuous in $[a, b]$ and differentiable in (a, b). By the MVT, there is a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1.$$

This is impossible since $|f'(x)| < 1$ for all $x \in \mathbb{R}$. We conclude that f has at most one fixed point ■

Exercise 18.15
Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere and that $f'(a) < 0$ and $f'(b) > 0$ for some $a < b$. Prove that there is a $c \in (a, b)$ such that $f'(c) = 0$.

Solution.
This is just Exercise 18.12 with $\lambda = 0$ ■

Exercise 18.16
Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $|f'(x)| \leq K < 1$ for all $x \in \mathbb{R}$. Let $a_0 \in \mathbb{R}$. Define the numbers $a_n = f(a_{n-1})$.
(a) Show that $|a_{n+1} - a_n| \leq K^n |a_1 - a_0|$ for all $n \in \mathbb{N}$.
(b) Show that for all $m, n \in \mathbb{N}$ such that $m > n$ we have

$$|a_m - a_n| \leq \frac{K^n}{1 - K}.$$

Solution.
(a) By the MVT there is a $c_1 \in (a_1, a_0)$ such that $f(a_1) - f(a_0) = f'(c_1)(a_1 - a_0)$. Thus, $|a_2 - a_1| \leq K|a_1 - a_0|$ since $|f'(c_1)| \leq K$. Likewise, we can write $|a_3 - a_2| \leq K|a_2 - a_1| \leq K^2|a_1 - a_0|$. Now, suppose that $|a_n - a_{n-1}| \leq K^n|a_1 - a_0|$. Then $|a_{n+1} - a_n| \leq K|a_n - a_{n-1}| \leq K^{n+1}|a_1 - a_0|$.
(b) Let $m, n \in \mathbb{N}$ such that $m > n$. Then we have $|a_m - a_n| \leq |a_m - a_{m-1}| + \cdots + |a_m - a_{m-1}| \leq |a_1 - a_0| \sum_{i=n}^{m} K^i = \frac{K^n}{1-K}|a_1 - a_0|$ ■

Exercise 18.17
Show that if $0 < a < b$ then $1 - \frac{a}{b} < \ln\left(\frac{b}{a}\right) < \frac{b}{a} - 1$. Hint: Apply the MVT for the function $f(x) = \ln x$.

Solution.
The function $f(x) = \ln x$ is continuous on $[a, b]$ and differentiable in (a, b). By the Mean value theorem there is a $a < c < b$ such that $f'(c) = \frac{\ln b - \ln a}{b - a}$. Thus,

$$\frac{1}{b} < \frac{1}{c} = \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

or

$$1 - \frac{a}{b} < \ln\left(\frac{b}{a}\right) < \frac{b}{a} - 1 \blacksquare$$

Solutions to Section 19

Exercise 19.1
Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in $a < x < b$ with $g'(x) \neq 0$ for all $a < x < b$. Suppose that $f(c) = g(c) = 0$ for some $a \leq c \leq b$. Also, suppose that

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = A.$$

(a) Let $\{c_n\}_{n=1}^{\infty} \subset [a, b]$ be an arbitrary sequence with the properties $c_n \neq c$ for all $n \geq 1$ and $\lim_{n\to\infty} c_n = c$. Show that there is a d_n between c_n and c such that

$$[f(c_n) - f(c)]g'(d_n) = [g(c_n) - g(c)]f'(d_n).$$

(b) Show that $d_n \neq c$ for all $n \geq 1$ and $\lim_{n\to\infty} d_n = c$.
(c) Show that $g(d_n) \neq g(c)$ for all $n \geq 1$. Hint: Exercise 18.3.
(d) Show that

$$\frac{f'(d_n)}{g'(d_n)} = \frac{f(c_n)}{g(c_n)}.$$

(e) Show that $\lim_{n\to\infty} \frac{f'(d_n)}{g'(d_n)} = A$. Hint: See Exercise 10.1.
(f) Show that $\lim_{n\to\infty} \frac{f(c_n)}{g(c_n)} = A$.
(g) Show that $\lim_{x\to c} \frac{f(x)}{g(x)} = A$.

Solution.
(a) Since both f and g are continuous in the closed interval with endpoints c_n and c and differentiable in the open interval with the same endpoints, we can use Exercise 13.5 to find a point d_n in that interval such that

$$[f(c_n) - f(c)]g'(d_n) = [g(c_n) - g(c)]f'(d_n).$$

(b) Since d_n is in the open interval with endpoints c_n and c we have $d_n \neq c$ for all $n \geq 1$. Also, by the Squeeze rule we have $\lim_{n\to\infty} d_n = c$.
(c) Since $g'(x) \neq 0$ for all $a < x < b$, by Exercise 18.3 we find that g is one-to-one. Since $d_n \neq c$ for all $n \geq 1$, we must have $g(d_n) \neq g(c)$ for all $n \geq 1$.
(d) This follows from (a) and the assumption that $f(c) = g(c) = 0$.

(e) Since $\lim_{x\to c}\frac{f'(x)}{g'(x)} = A,\ d_n \neq c, d_n \to c$, we can apply Exercise 10.1 to write

$$\lim_{n\to\infty}\frac{f'(d_n)}{g'(d_n)} = A.$$

(f) Using (d) and (e) the result follows.
(g) The result follows from Exercise 10.1 ■

Exercise 19.2
Find

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}+\sqrt{x-2}}{\sqrt{x^2-4}}.$$

Solution.
Let $f(x) = \sqrt{x}-\sqrt{2}+\sqrt{x-2}$ and $g(x) = \sqrt{x^2-4}$. Both functions are continuous in $[2,3]$ and differentiable in $2 < x < 3$. Moreover, $f(2) = g(2) = 0$. Applying L'Hôpital's rule we find

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}+\sqrt{x-2}}{\sqrt{x^2-4}} =& \lim_{x\to 2}\frac{\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{x-2}}}{\frac{x}{\sqrt{x^2-4}}}\\ =& \lim_{x\to 2}\frac{(\sqrt{x-2}+\sqrt{x})(\sqrt{x^2-4})}{2x\sqrt{x}\sqrt{x-2}}\\ =& \lim_{x\to 2}\frac{(\sqrt{x-2}+\sqrt{x})(\sqrt{x+2})}{2x\sqrt{x}}\\ =& \frac{2\sqrt{2}}{4\sqrt{2}} = \frac{1}{2}\ ■\end{aligned}$$

Exercise 19.3
Let $f : [a,b] \to \mathbb{R}$ be a one-to-one function.
(a) Define $g : f([a,b]) \to [a,b]$ by $g(y) = x$ if and only if $f(x) = y$. Show that g is indeed a function. That is, if $y_1, y_2 \in f([a,b])$ are such that $y_1 = y_2$ then $g(y_1) = g(y_2)$.
(b) Show that $f(g(y)) = y$ for all $y \in f([a,b])$ and $g(f(x)) = x$ for all $x \in [a,b]$. Thus, conclude that f is invertible.

Solution.
(a) Define $g : f([a,b]) \to [a,b]$ by $g(y) = x$ if and only if $f(x) = y$. Then g is a well-defined function: Suppose that $y_1, y_2 \in f([a,b])$ such that $y_1 = y_2$.

Let $x_1, x_2 \in [a, b]$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then $x_1 = g(y_1)$ and $x_2 = g(y_2)$. Moreover, we have $f(x_1) = f(x_2)$. Since f is one-to-one, we conclude that $x_1 = x_2$ which implies that $g(y_1) = g(y_2)$.
(b) Let $y \in f([a, b])$. Then $f(x) = y$ for some $a \leq x \leq b$. By the definition of g we have $g(y) = x$. Thus, $f(g(y)) = f(x) = y$. Likewise, let $x \in [a, b]$. Then $y = f(x) \in f([a, b])$. Hence, $g(f(x)) = g(y) = x$. We conclude that f^{-1} exists ■

Exercise 19.4
Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and differentiable in $[a, b]$ with $f'(x) \neq 0$ for all $a < x < b$. Let the range of f be denoted by $[m, M]$.
(a) Show that f is one-to-one, monotone, and invertible with inverse $f^{-1} : [m, M] \to [a, b]$.
(b) Assume that f is strictly increasing. That is, if $x_1 < x_2$ then $f(x_1) < f(x_2)$. In this case, $[m, M] = [f(a), f(b)]$. Let $f(a) < y_0 < f(b)$. Show that there is a $a < x_0 < b$ such that $f(x_0) = y_0$.
(c) Let $\epsilon > 0$ be given. Let $\epsilon_1 = \min\{\epsilon, x_0 - a, b - x_0\}$. Show that if x satisfies $|x - x_0| < \epsilon_1$ then $a < x < b$ and $|x - x_0| < \epsilon$.
(d) Let $y_1 = f(x_0 - \epsilon_1)$ and $y_2 = f(x_0 + \epsilon_1)$. Show that $f[(x_0 - \epsilon_1, x_0 + \epsilon_1)] = (y_1, y_2)$.
(e) Choose a $\delta > 0$ so that $(y_0 - \delta, y_0 + \delta) \subset (y_1, y_2)$. Show that if $|y - y_0| < \delta$ then $|f^{-1}(y) - f^{-1}(y_0)| < \epsilon$. This shows that f^{-1} is continuous in $(f(a), f(b))$.
(f) Show that f^{-1} is right continuous at $f(a)$ and left continuous at $f(b)$.
We conclude from this problem that f^{-1} is continuous on the closed interval $[f(a), f(b)]$.

Solution.
(a) By Exercise 18.3, f is one-to-one in $[a, b]$. By Exercises 18.5 and 18.6, f is monotone. Now, by Exercise 19.3, f is invertible with inverse $f^{-1} : [m, M] \to [a, b]$.
(b) Since $f(a) < y_0 < f(b)$, by the IVT, there is a $x_0 \in [a, b]$ such that $f(x_0) = y_0$. Since $f(a) < f(x_0) < f(b)$, we must have $a < x_0 < b$.
(c) Suppose that $|x - x_0| < \epsilon_1$. Since $\epsilon_1 \leq \epsilon$ we can write that $|x - x_0| < \epsilon$. Since $|x - x_0|$ is less than or equal to both $x_0 - a$ and $b - x_0$, we conclude that $a < x < b$.
(d) First we show that $x_0 - \epsilon_1 \in [a, b]$. Indeed, we have $a \leq a + \epsilon_1 \leq x_0 < b \leq b + \epsilon_1$. This show that $x_0 - \epsilon_1 \in [a, b]$. Now, let $y \in f[(x_0 - \epsilon_1, x_0 + \epsilon_1)]$. Then $y = f(x)$ for some $x \in (x_0 - \epsilon_1, x_0 + \epsilon_1)$. Since f is strictly increasing

we find $y_1 < y < y_2$. That is, $y \in (y_1, y_2)$. The converse is similar.
(e) Suppose that $|y - y_0| < \delta$. Then $y \in (y_1, y_2)$ which implies that $f^{-1}(y) \in (x_0 - \epsilon_1, x_0 + \epsilon_1)$. Hence, $|f^{-1}(y) - f^{-1}(y_0)| < \epsilon_1 \leq \epsilon$. Thus, we have shown that for any positive number ϵ, we can find a number $\delta > 0$ such that if $|y - y_0| < \delta$ then $|f^{-1}(y) - f^{-1}(y_0)| < \epsilon$. This says that f^{-1} is continuous in $(f(a), f(b))$.
(f) Let $\epsilon > 0$ be given. There is a $\delta_1 > 0$ such that if $x - a < \delta_1$ then $f(x) - f(a) < \epsilon$. Let $\delta_2 = \min \delta_1, \epsilon$. Define $\delta = f(a + \delta_2) - f(a)$. Suppose that $y - f(a) < \delta$. Then there is $a < x < a + \delta_2$ such that $f(x) = y$. Thus, $|f^{-1}(y) - a| = |x - a| < \delta_2 \leq \epsilon$. This shows that f^{-1} is continuous at $f(a)$. A similar argument holds for the left continuity of f^{-1} at $f(b)$ ■

Exercise 19.5 *(Inverse Function Theorem)*
Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and differentiable in $[a, b]$ with $f'(x) \neq 0$ for all $a < x < b$. Let $c \in f([a, b])$. Then there is a $d \in [a, b]$ such that $f(d) = c$.
(a) Let $\{c_n\}_{n=1}^{\infty} \subseteq f([a, b])$ such that $c_n \neq c$ for all $n \geq 1$ and $\lim_{n\to\infty} c_n = c$. Show that there is a sequence $\{d_n\}_{n=1}^{\infty} \subseteq [a, b]$ such that

$$\lim_{n\to\infty} d_n = d.$$

Hint: Exercise 14.9(b).
(b) Show that $d_n \neq d$ for all $n \geq 1$.
(c) Show that

$$\lim_{n\to\infty} \frac{f(d_n) - f(d)}{d_n - d} = f'(d).$$

Hint: Exercise 10.1.
(d) Show that $\frac{f(d_n)-f(d)}{d_n-d} \neq 0$ for all $n \geq 1$. Hint: Exercise 18.3.
(e) Show that

$$\lim_{n\to\infty} \frac{f^{-1}(c_n) - f^{-1}(c)}{c_n - c} = \frac{1}{f'(d)}.$$

Thus, conclude that

$$(f^{-1})'(f(d)) = \frac{1}{f'(d)}$$

for all $d \in [a, b]$. That is f^{-1} is differentiable in $f([a, b])$. Hint: Exercise 10.2.

Solution.
(a) Since $\{c_n\}_{n=1}^{\infty} \subseteq f([a,b])$, there is a sequence $\{d_n\}_{n=1}^{\infty} \subseteq [a,b]$ such that $f(d_n) = c_n$ for all $n \geq 1$. By the continuity of f^{-1} we have

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} f^{-1}(c_n) = f^{-1}(\lim_{n\to\infty} c_n)) = f^{-1}(c) = d.$$

(b) If $d_n = d$ then $f^{-1}(d_n) = f^{-1}(d)$ which implies $c_n = c$. This contradicts the fact that $c_n \neq c$ for all $n \geq 1$.
(c) We have

$$\lim_{n\to\infty} \frac{f(d_n) - f(d)}{d_n - d} == f'(d)$$

since f is differentiable at d. (Exercise 10.1).
(d) Since f is one-to-one (Exercise 18.3) and $d_n \neq d$, we have $f(d_n) \neq f(d)$. Thus, the ratio

$$\tfrac{f(d_n)-f(d)}{d_n-d} \neq 0 \text{ for all } n \geq 1.$$

(e) We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{f^{-1}(c_n) - f^{-1}(c)}{c_n - c} &= \lim_{n\to\infty} \frac{d_n - d}{f(d_n) - f(d)} \\
&= \lim_{n\to\infty} \frac{1}{\frac{f(d_n)-f(d)}{d_n-d}} \\
&= \frac{1}{f'(d)}
\end{aligned}$$

Applying Exercise 10.2 we find

$$(f^{-1})'(c) = (f^{-1})'(f(d)) = \frac{1}{f'(d)}$$

for all $d \in [a,b]$. That is f^{-1} is differentiable in $f([a,b])$ ■

Exercise 19.6
Find $\lim_{x\to\infty} \left(\frac{\ln x}{x} \cdot \sin\left(\frac{x\pi+2}{2x}\right)\right.$

Solution.
By L'Hôpital's rule we have

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

By the continuity of the sine function we have

$$\lim_{x\to\infty}\sin\left(\frac{x\pi+2}{2x}\right)=\sin\frac{\pi}{2}=1.$$

Thus,

$$\lim_{x\to\infty}\left(\frac{\ln x}{x}\cdot\sin\left(\frac{x\pi+2}{2x}\right)\right)=0\ \blacksquare$$

Exercise 19.7
Let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable in $a<x<b$ with $g'(x)\neq 0$ for all $a<x<b$. Suppose that $\lim_{x\to c}f(x)=\lim_{x\to c}g(x)=\infty$ for some $a\le c\le b$. Also, suppose that

$$\lim_{x\to c}\frac{f'(x)}{g'(x)}=A.$$

Prove that

$$\lim_{x\to c}\frac{f(x)}{gx(x)}=A.$$

Solution.
We have

$$\begin{aligned}\lim_{x\to c}\frac{f(x)}{g(x)}=&\lim_{x\to c}\frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}\\=&\lim_{x\to c}\frac{g'(x)}{f'(x)}\cdot\left(\lim_{x\to c}\frac{f(x)}{g(x)}\right)^2\end{aligned}$$

Thus, we have

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}=A\ \blacksquare$$

Exercise 19.8
Use L'Hôpital's rule to evaluate $\lim_{x\to 0^+}x^x$. Note that 0^0 is an undeterminate form.

Solution.
We have

$$\lim_{x\to 0^+}x^x=\lim_{x\to 0^+}e^{x\ln x}.$$

But
$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} -x = 0.$$
Thus,
$$\lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x\ln x} = e^0 = 1 \blacksquare$$

Exercise 19.9
Let f and g be invertible differentiable functions such that

$$f(1) = 2; g(2) = 1; f'(1) = g'(2) = 3.$$

Find the derivative $(f^{-1}\circ g^{-1})'(1)$.

Solution.
We have

$$\begin{aligned}(f^{-1}\circ g^{-1})'(1) =& [f^{-1}]'(g^{-1}(1))\cdot [g^{-1}]'(1) = \frac{1}{f'(f^{-1}(g^{-1}(1)))}\cdot\frac{1}{g'(g^{-1}(1))}\\ =&\frac{1}{f'(f^{-1}(2))}\frac{1}{g'(2)} = \frac{1}{f'(1)}\frac{1}{g'(2)} = \frac{1}{9} \blacksquare\end{aligned}$$

Exercise 19.10
Let $f(x) = x\tan^2 x$ for $x \in (0, \frac{\pi}{2})$. Calculate $(f^{-1})'(\pi)$. Note that $f(\frac{\pi}{3}) = \pi$.

Solution.
By the inverse function theorem we have

$$(f^{-1})'(\pi) = \frac{1}{f'(f^{-1}(\pi))} = \frac{1}{f'(\frac{\pi}{3})} = (3 + \frac{8}{\sqrt{3}}\pi)^{-1} \blacksquare$$

Solutions to Section 20

Exercise 20.1
(a) Show that $m \leq m_i(f) \leq M_i(f) \leq M$.
(b) Show that $m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a)$.

Solution.
(a) Since m is a lower bound of f in $[a,b]$ we must have $m \leq m_i(f)$. Since $m_i(f)$ is a lower bound of f and $M_i(f)$ is an upper bound of f we must have $m_i(f) \leq M_i(f)$. Finally, since M is an upper bound of f in $[a,b]$, we have $M_i(f) \leq M$.
(b) For all $1 \leq i \leq n$, we have $m(x_i - x_{i-1}) \leq m_i(f)(x_i - x_{i-1}) \leq M_i(f)(x_i - x_{i-1}) \leq M(x_i - x_{i-1})$. Adding these inequalities, we obtain the desired inequality ■

Exercise 20.2
Let Q be a refinement of P. Suppose that $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ and $Q = \{a = x_0 < x_1 < \cdots < x_{i-1} < z < x_i < \cdots < x_n = b\}$.
(a) Show that $U(f,Q) \leq U(f,P)$.
(b) Show that $L(f,P) \leq L(f,Q)$.

Solution.
Suppose first that P is a partition of $[a,b]$ and that Q is the partition obtained from P by adding an additional point $x_{i-1} < z < x_i$. The general case follows by induction, adding one point at at time. In particular, we let

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

and

$$Q = \{a = x_0 < x_1 < \cdots < x_{i-1} < z < x_i < \cdots < x_n = b\}.$$

Observe that

$$U(f,P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$

and

$$U(f,Q) = \sum_{j=1}^{i-1} M_j(f)(x_j - x_{j-1}) + M(z - x_{i-1}) + M'(x_i - z) + \sum_{j=i+1}^{n} M_j(f)(x_j - x_{j-1})$$

where

$$M = \sup\{f(x) : x \in [x_{i-1}, z]\} \text{ and } M' = \sup\{f(x) : x \in [z, x_i]\}.$$

Since $M(z - x_{i-1}) + M'(x_i - z) \leq M_i(x_i - x_{i-1})$ we conclude that $U(f, Q) \leq U(f, P)$.
(b) Similar argument as in (a) ■

Exercise 20.3
Suppose that f is bounded on $[a, b]$. Show that $\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx$. Hint: Exercise 20.2.

Solution.
Let P be a partition of $[a, b]$. For any partition Q of $[a, b]$ we let $R = P \cup Q$. Since $P \subset R$ and $Q \subset R$ we can use Exercise 20.2 to write

$$L(f, P) \leq L(f, R) \leq U(f, R) \leq U(f, Q).$$

Since Q was arbitrary, $L(f, P)$ is a lower bound of S_U. But the Riemann upper integral is the largest lower bound of S_U. We conclude that

$$L(f, P) \leq \overline{\int_a^b} f(x)dx.$$

Since P was arbitrary, the above inequality says $\overline{\int_a^b} f(x)dx$ is an upper bound of S_L. But the lower Riemann integral is the smallest upper bound of S_L. That is,

$$\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \blacksquare$$

Exercise 20.4
Consider the function $f : [a, b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2 & \text{if } a \leq x < b \\ 3 & \text{if } x = 3 \end{cases}$$

(a) Find numbers m and M such that $m \leq f(x) \leq M$ for all $x \in [a, b]$?
(b) Show that for any partition P of $[a, b]$ we have $L(f, P) = 2(b - a)$. Conclude that

$$\underline{\int_a^b} f(x)dx = 2(b - a).$$

(c) Show that $\overline{\int_a^b} f(x)dx \geq 2(b-a)$. (d) Suppose $\overline{\int_a^b} f(x)dx > 2(b-a)$. Let $\epsilon = \overline{\int_a^b} f(x)dx - 2(b-a) > 0$. Let Q be the partition

$$Q = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

such that $b - x_{n-1} < \epsilon$. Show that $U(f, Q) < \overline{\int_a^b} f(x)dx$. Why this is impossible?
(e) Is $f(x)$ Riemann integrable? If so, what is the value of the integral $\int_a^b f(x)dx$?

Solution.
(a) Since $2 \leq f(x) \leq 3$ for all $a \leq x \leq b$, we have $m = 2$ and $M = 3$.
(b) Let P be a parition of $[a, b]$ given by

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}.$$

Note that $M_i(f) = 2$ for all $1 \leq i \leq n-1$, $M_n(f) = 3$, $m_i(f) = 2$ for all $1 \leq i \leq n$. Thus,

$$L(f, P) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) = 2(b-a).$$

Since P was arbitrary, it follows that

$$\underline{\int_a^b} f(x)dx = 2(b-a).$$

(c) This follows from Exercise 20.3.
(d) We have $U(f, Q) = \sum_{i=1}^{n-1} M_i(f)(x_i - x_{i-1}) + M_n(f)(b - x_{n-1}) = 2(x_{n-1} - a) + 3(b - x_{n-1}) = 2(b-a) + (b - x_{n-1}) < 2(b-a) + \overline{\int_a^b} f(x)dx - 2(b-a) = \inf_{S_U}$, which contradicts the definition of infimum. We conclude that $\overline{\int_a^b} f(x)dx = 2(b-a)$.
(e) Since $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx = 2(b-a)$, the function f is Riemnann integrable with

$$\int_a^b f(x)dx = 2(b-a) \blacksquare$$

Exercise 20.5
Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational.
(a) Compute the upper Riemann integral and the lower Riemann integral. Hint: Exercise 2.6(c).
(b) Is f Riemann integrable on $[0,1]$?

Solution.
(a) Let P be a partition of $[0,1]$:

$$P = \{0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\}.$$

Clearly, $M_i(f) = 1$ and $m_i(f) = 0$ for all $1 \le i \le n$. Hence, $U(f,P) = 1$ and $L(f,P) = 0$. It follows that $\overline{\int_0^1} f(x)dx = 1$ and $\underline{\int_0^1} f(x)dx = 0$.
(b) Since the lower Riemann sum is different from the upper Riemann sum, the function is not Riemann integrable ■

Exercise 20.6
Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose that f is Riemann integrable. We want to show that f satisfis the following property:

(P)$\forall \epsilon > 0$, there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.

(a) Let $\epsilon > 0$ be given. Show that there is a partition P of $[a,b]$ such that

$$\underline{\int_a^b} f(x)dx - \frac{\epsilon}{2} < L(f,P).$$

Hint: Assume the contrary and get a contradiction.
(b) Show that there is a partition Q of $[a,b]$ such that

$$U(f,Q) < \overline{\int_a^b} f(x)dx + \frac{\epsilon}{2}.$$

(c) Let $R = P \cup Q$. Use Exercise 20.2 to show that

$$\int_a^b f(x)dx - \frac{\epsilon}{2} < L(f,R) \le U(f,R) < \int_a^b f(x)dx + \frac{\epsilon}{2}.$$

(d) Show that

$$\left|L(f,R) - \int_a^b f(x)dx\right| < \frac{\epsilon}{2} \text{ and } \left|U(f,R) - \int_a^b f(x)dx\right| < \frac{\epsilon}{2}$$

(e) Use the triangle inequality to show that $U(f,R) - L(f,R) < \epsilon$.

Solution.
(a) Suppose the contrary. That is,

$$\underline{\int_a^b} f(x)dx - \frac{\epsilon}{2} \geq L(f,P)$$

for all partition of $[a,b]$. This means that $\underline{\int_a^b} f(x)dx - \frac{\epsilon}{2}$ is an upper bound of S_L. But $\underline{\int_a^b} f(x)dx$ is the smallest upper bound of S_L. Hence, $\underline{\int_a^b} f(x)dx < \underline{\int_a^b} f(x)dx - \frac{\epsilon}{2}$, which is impossible.
(b) Suppose the contrary. That is,

$$U(f,Q) \geq \overline{\int_a^b} f(x)dx + \frac{\epsilon}{2}$$

for all partition of $[a,b]$. This means that $\overline{\int_a^b} f(x)dx + \frac{\epsilon}{2}$ is a lower bound of S_U. But $\overline{\int_a^b} f(x)dx$ is the largest lower bound of S_U. Hence, $\overline{\int_a^b} f(x)dx + \frac{\epsilon}{2} < \overline{\int_a^b} f(x)dx$, which is impossible.
(c) We have

$$\begin{aligned}
\int_a^b f(x)dx - \frac{\epsilon}{2} =& \underline{\int_a^b} f(x)dx - \frac{\epsilon}{2} \\
<& L(f,P) < L(f,R) \leq U(f,R) \leq U(f,Q) \\
<& \overline{\int_a^b} f(x)dx + \frac{\epsilon}{2} \\
=& \int_a^b f(x)dx - \frac{\epsilon}{2}
\end{aligned}$$

(d) We conclude from (c) that

$$\left|L(f,R) - \int_a^b f(x)dx\right| < \frac{\epsilon}{2} \text{ and } \left|U(f,R) - \int_a^b f(x)dx\right| < \frac{\epsilon}{2}$$

(e) By the triangle inequality we have $U(f,R) - L(f,R) = |U(f,R) - L(f,R)| = \left|(U(f,R) - \int_a^b f(x)dx) + (\int_a^b f(x)dx - L(f,R))\right| \leq \left|U(f,R) - \int_a^b f(x)dx\right| + \left|\int_a^b f(x)dx - L(f,R)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ ■

Exercise 20.7
Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose that f satisfies property (P) above.
(a) Show that for each positive integer n, there is a partition P_n such that

$$U(f,P_n) - L(f,P_n) < \frac{1}{n}.$$

(b) Using (a), show that

$$L(f,P_n) \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx < L(f,P_n) + \frac{1}{n}.$$

(c) Show that

$$0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \frac{1}{n}.$$

(d) Show that

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

Hint: Squeeze rule. We conclude that any bounded function that satisfies property (P) is Riemann integrable.

Solution.
(a) This follows from property (P).
(b) We have

$$\begin{aligned} L(f,P_n) \leq & \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \\ \leq & U(f,P_n) < L(f,P_n) + \frac{1}{n} \end{aligned}$$

(c) Using (b) we have

$$0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < L(f,P_n) + \frac{1}{n} - L(f,P_n) = \frac{1}{n}.$$

(d) This follows from the squeeze rule ■

Exercise 20.8
Let $f : [0,1] \to \mathbb{R}$ be the function $f(x) = x^2$. For any $\epsilon > 0$, choose a partition $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ such that

$$x_i - x_{i-1} < \frac{\epsilon}{2} \text{ for all } 1 \leq i \leq n$$

Show that $U(f,P) - L(f,P) < \epsilon$. Hence, f is Riemann integrable.

Solution.
For $1 \leq i \leq n$ we have $M_i(f) = x_i^2$ and $m_i(f) = x_{i-1}^2$. Then $U(f,P) = \sum_{i=1}^n x_i^2(x_i - x_{i-1})$ and $L(f,P) = \sum_{i=1}^n x_{i-1}^2(x_i - x_{i-1})$. Hence,

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=1}^n (x_i^2 - x_{i-1}^2)(x_i - x_{i-1}) \\
&= \sum_{i=1}^n (x_i + x_{i-1})(x_i - x_{i-1})(x_i - x_{i-1}) \\
&< \sum_{i=1}^n 2\left(\frac{\epsilon}{2}\right)(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^n \epsilon(x_i - x_{i-1}) = \epsilon \ \blacksquare
\end{aligned}$$

Exercise 20.9
Suppose that $f(x) = x$ for $x \in [1,2]$.
(a) Find $U(f,P)$ and $L(f,P)$. Hint: Consider a partition with equal subintervals.
(b) Show that f is Riemann integrable. Hint: Exercise 20.7.
(c) Show that $U(f,P) \geq \frac{3}{2}$ and $L(f,P) \leq \frac{3}{2}$.
(d) Find $\int_1^2 x dx$.

Solution.
(a) Let $P = \{1 = x_0 < x_1 < \cdots < x_n = 2\}$ be a partition of $[1,2]$ with $x_i = 1 + \frac{i}{n}$. Then for $1 \leq i \leq n$ we have $m_i(f) = 1 + \frac{i-1}{n}$ and $M_i(f) = 1 + \frac{i}{n}$.

Hence,

$$\begin{aligned}L(f,P) =&\sum_{i=1}^n m_i(f)(x_i-x_{i-1}) = \sum_{i=1}^n \left(1+\frac{i-1}{n}\right)\cdot\frac{1}{n}\\ =&\frac{1}{n}\left(n+\frac{1}{n}\sum_{i=1}^n(i-1)\right)\\ =&\frac{1}{n}\left(n+\frac{1}{n}\left(\frac{n(n+1)}{2}-n\right)\right)\\ =&1+\frac{n-1}{2n}\end{aligned}$$

and

$$\begin{aligned}U(f,P) =&\sum_{i=1}^n M_i(f)(x_i-x_{i-1}) = \sum_{i=1}^n \left(1+\frac{i}{n}\right)\cdot\frac{1}{n}\\ =&\frac{1}{n}\left(n+\frac{1}{n}\sum_{i=1}^n i\right)\\ =&\frac{1}{n}\left(n+\frac{1}{n}\cdot\frac{n(n+1)}{2}\right)\\ =&1+\frac{n+1}{2n}\end{aligned}$$

(b) Let $\epsilon > 0$ be given. Choose n large enough so that $n > \frac{1}{\epsilon}$. Let $P = \{1 = x_0 < x_1 < \cdots < x_n = 2\}$ be a partition of $[1,2]$ with $x_i = 1 + \frac{i}{n}$. Then

$$U(f,P) - L(f,P) = \frac{1}{n} < \epsilon.$$

By Exercise 20.7, f is Riemann integrable.
(c) Since $\frac{n+1}{2n}$ is a decreasing function of n and converges to $\frac{1}{2}$, we must have $U(f,P) \geq \frac{3}{2}$. Likewise, since $\frac{n-1}{2n}$ is increasing and converges to $\frac{1}{2}$ we must have $L(f,P) \leq \frac{3}{2}$. Hence

$$\underline{\int_a^b} xdx \leq \frac{3}{2} \leq \overline{\int_a^b} xdx.$$

(c) From (c) we have

$$\int_1^2 xdx = \underline{\int_1^2} xdx = \overline{\int_1^2} xdx = \frac{3}{2} \blacksquare$$

Exercise 20.10
Let $f : [a, b] \to \mathbb{R}$ be bounded. Let P and Q be any two partitions of $[a, b]$. Prove that $L(f, P) \leq U(f, Q)$.

Solution.
Let $R = P \cup Q$. Then $L(f, R) \leq L(f, P) \leq U(f, P) \leq U(f, R) \leq U(f, Q)$ ■

Exercise 20.11
Let $f : [a, b] \to \mathbb{R}$ be such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Prove that

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq (M - m)(b - a).$$

Solution.
Let P be a paritition of $[a, b]$. Then $L(f, p) \leq M(b-a)$ and $U(f, P) \geq m(b-a)$. Thus, $M(b - a)$ is an upper bound of S_L so that $\overline{\int_a^b} f(x)dx \leq M(b - a)$. Likewise, $m(b - a)$ is a lower bound of S_U so that $m(b - a) \leq \underline{\int_a^b} f(x)dx$. Hence, $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq (M - m)(b - a)$ ■

Exercise 20.12
Let $f : [a, b] \to \mathbb{R}$ be bounded functions such that $f(x) \leq g(x)$ for all $x \in [a, b]$. Prove the following:
(a) $\overline{\int_a^b} f(x)dx \leq \overline{\int_a^b} g(x)dx$
(b) $\underline{\int_a^b} f(x)dx \leq \underline{\int_a^b} g(x)dx$

Solution.
(a) We prove (a) since (b) is similar. Let P be any partition of $[a, b]$. Then $L(f, P) \leq L(g, P) \leq \overline{\int_a^b} g(x)dx$. That is, $\overline{\int_a^b} g(x)dx$ is an upper bound of $S_L(f)$. This implies that $\overline{\int_a^b} f(x)dx \leq \overline{\int_a^b} g(x)dx$ ■

Exercise 20.13
Let $f : [a, b] \to \mathbb{R}$ be bounded functions. Let P be any partition of $[a, b]$. Prove

$$U(f + g, P) \leq U(f, P) + U(g, P).$$

Solution.
Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Then for all $x \in [x_{i-1}, x_i]$ we have

$$f(x) + g(x) \leq M_i(f) + M_i(g).$$

This implies that

$$M_i(f + g) \leq M_i(f) + M_i(g).$$

Hence,

$$U(f + g, P) \leq U(f, P) + U(g, P) \blacksquare$$

Exercise 20.14
Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Prove that there is a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = \int_a^b f(x)dx$.

Solution.
This follows from Exercise 20.7 $\blacksquare$

Exercise 20.15
Consider the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = ax+b$ where $a > 0$ and $b > 0$. Assume that this function is Riemann integrable. For each positive integer n consider the partition $P_n = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ with equal subintervals.
(a) Compute $L(f, P_n)$ and $U(f, P_n)$.
(b) Show that $\int_0^1 f(x)dx = \frac{a}{2} + b$.

Solution.
(a) We have

$$L(f, P_n) = \sum_{i=1}^{n}(ax_{i-1} + b)\frac{1}{n} = \frac{a}{n}\sum_{i=1}^{n}\frac{i-1}{n} + b = \frac{a}{2}\frac{n-1}{n} + b$$

and

$$U(f, P_n) = \sum_{i=1}^{n}(ax_i + b)\frac{1}{n} = \frac{a}{n}\sum_{i=1}^{n}\frac{i}{n} + b = \frac{a}{2}\frac{n+1}{n} + b$$

(b) We have

$$\int_0^1 f(x)dx = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \frac{a}{2} + b \blacksquare$$

Solutions to Section 21

Exercise 21.1
Let $f : [a, b] \to \mathbb{R}$ be an increasing function on $[a, b]$.
(a) Show that f is bounded on $[a, b]$.
(b) Let $\epsilon > 0$ be given. Choose a positive integer N such that $\frac{f(b)-f(a)}{N} < \epsilon$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ such that $x_i - x_{i-1} < \frac{1}{N}$ for all $1 \le i \le n$. For each $1 \le i \le n$, express $M_i(f)$ and $m_i(f)$ in terms of $f(x)$.
(c) Show that $U(f, P) - L(f, P) < \epsilon$. Thus, conclude that f is Riemann integrable.

Solution.
(a) Since f is increasing, we must have $f(a) \le x \le f(b)$ for all $x \in [a, b]$. Let $m = f(a)$ and $M = f(b)$. Then f is bounded.
(b) Since f is increasing, we must have $M_i(f) = f(x_i)$ and $m_i(f) = f(x_{i-1})$ for $1 \le i \le n$.
(c) We have

$$\begin{aligned} U(f, P) - L(f, P) =& \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})](x_i - x_{i-1}) \\ <& \sum_{i=1}^{n} \frac{[f(x_i) - f(x_{i-1})]}{N} \\ =& \frac{f(b) - f(a)}{N} < \epsilon \end{aligned}$$

It follows from Exercise 20.7 that f is Riemann integrable ∎

Exercise 21.2
Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$.
(a) Show that there exist numbers m and M such that $m \le f(x) \le M$ for all $a \le x \le b$. That is, f is bounded on $[a, b]$.
(b) Show that f is uniformly continuous on $[a, b]$.
(c) Let $\epsilon > 0$. Show that there is a positive number $\delta > 0$ such that if $|u - v| < \delta$ then $|f(u) - f(v)| < \frac{\epsilon}{b-a}$.
(d) Choose a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ such that $x_i - x_{i-1} < \delta$ for all $1 \le i \le n$. Show that for each interval $[x_i, x_{i-1}]$ there exist $s_i, t_i \in [x_i, x_{i-1}]$ such that $M_i(f) = f(t_i)$ and $m_i(f) = f(s_i)$. Hint:

Exercise 17.4.
(e) Show that $M_i(f) - m_i(f) < \frac{\epsilon}{b-a}$ for each $1 \le i \le n$.
(f) Using (e), show that $U(f,P) - L(f,P) < \epsilon$. Hence, conclude that f is Riemann integrable.

Solution.
(a) By Exercise 17.4 there exist $x_1, x_2 \in [a,b]$ such that $f(x_1) \le f(x) \le f(x_2)$ for all $a \le x \le b$. Let $m = f(x_1)$ and $M = f(x_2)$.
(b) This follows from Exercise 14.5.
(c) This follows from the fact that f is uniformly continuous.
(d) This follows from Exercise 17.4.
(e) Since $|t_i - s_i| < \delta$, we have $|f(t_i) - f(s_i)| < \frac{\epsilon}{b-a}$ which implies $-\frac{\epsilon}{b-a} < f(t_i) - f(s_i) < \frac{\epsilon}{b-a}$. Thus, $M_i(f) - m_i(f) = f(t_i) - f(s_i) < \frac{\epsilon}{b-a}$.
(f) We have $U(f,P) - L(f,P) = \sum_{i=1}^n [f(t_i) - f(s_i)](x_i - x_{i-1}) < \sum_{i=1}^n \frac{\epsilon}{b-a}(x_i - x_{i-1}) = \epsilon$. We conclude that f is Riemann integrable ■

Exercise 21.3
Suppose f is continuous except at a point c in $[a,b]$. Let $\epsilon > 0$ be given and consider a partition $Q = \{a = x_0 < x_1 < \cdots < x_{k-1} < c < x_{k+1} < \cdots < x_n = b\}$ such that $\mu(Q) < \frac{\epsilon}{12M}$.
(a) Prove that $|x_{k-1} - xk + 1| < \frac{\epsilon}{6M}$.
(b) Show that there exist $\delta' > 0$ and $\delta'' > 0$ such that for all $x, y \in [a, x_{k-1}]$ with $|x - y| < \delta'$ we have $|f(x) - f(y)| < \frac{\epsilon}{3(b-a)}$ and for all $x, y \in [x_{k+1}, b]$ with $|x - y| < \delta''$ we have $|f(x) - f(y)| < \frac{\epsilon}{3(b-a)}$.
(c) Let P_1 be a refinement of Q on $[a, x_{k-1}]$ such that $\mu(P_1) < \delta'$ and P_2 be a refinement of P on $[x_{k+1}, b]$ such that $\mu(P_2) < \delta''$. Let $P = P_1 \cup P_2$. Then we have

$$U(f,P) - L(f,P) = \sum_{i=1}^{k-1} (M_i - m_i)(x_i - x_{i-1}) + (M_k - m_k)(c - x_{k-1})$$
$$+(M_{k+1} - m_{k+1}(x_{k+1} - c) + \sum_{i=k+2}^{n} (M_i - m_i)(x_i - x_{i-1})$$

Show that

$$\sum_{i=1}^{k-1} (M_i - m_i)(x_i - x_{i-1}) < \frac{\epsilon}{3}$$

$$(M_k - m_k)(c - x_{k-1}) + (M_{k+1} - m_{k+1}(x_{k+1} - c) < \frac{\epsilon}{3}$$

and
$$\sum_{i=k+2}^{n}(M_i - m_i)(x_i - x_{i-1}) < \frac{\epsilon}{3}$$
(d) Conclude that $U(f,P) - L(f,P) < \epsilon$ and therefore f is Riemann integrable.

Solution.
(a) We have $|x_{k-1} - x_{k+}| \leq |c - x_{k-1}| + |x_{k+1} - c| < \frac{\epsilon}{12M} + \frac{\epsilon}{12M} = \frac{\epsilon}{6M}$.
(b) This follows from the fact that f is uniformly continuous on $[a, x_{k-1}]$ and $[x_{k+1}, b]$.
(c) We have
$$\sum_{i=1}^{k-1}(M_i - m_i)(x_i - x_{i-1}) < \frac{\epsilon}{3(b-a)}(x_{k-1} - a) < \frac{\epsilon}{3}$$
$$(M_k - m_k)(c - x_{k-1}) + (M_{k+1} - m_{k+1}(x_{k+1} - c) < 2M(x_{k+1} - x_k) < 2M\frac{\epsilon}{6M} = \frac{\epsilon}{3}$$
$$\sum_{i=k+2}^{n}(M_i - m_i)(x_i - x_{i-1}) < \frac{\epsilon}{3(b-a)}(b - x_{k+1}) < \frac{\epsilon}{3}$$
(d) Taking everything together we have:
$$U(f,P) - L(f,P) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$
by Exercise 20.7, we conclude that f is Riemann integrable ■

Exercise 21.4
Suppose f is continuous except at points $c_1, c_2, \cdots, c_n$ in $[a,b]$. We want to show that f is Riemann integrable on $[a,b]$. The proof is by induction on n. For $n = 1$ the result holds by the previous exercise. Suppose that the result holds for $c_1, c_2, \cdots, c_n$. Suppose that f is continuous except at $c_1 < c_2 < \cdots < c_n < c_{n+1}$. Let $\epsilon > 0$. Choose $\delta > 0$ small enough so that $\delta < \frac{\epsilon}{8M}$ and $(c_{n+1} - \delta, c_{n+1} + \delta) \subset [c_n, b]$.
(a) Show that there is a partition P_1 of $[a, c_{n+1} - \delta]$ such that $U(f,P_1) - L(f,P_1) < \frac{\epsilon}{4}$ and a partition P_2 of $[c_{n+1}, b]$ such that $U(f,P_2) - L(f,P_2) < \frac{\epsilon}{4}$.
(b) Let $P = P_1 \cup P_2$. Show that $U(f,P) - L(f,P) < \epsilon$. Hence, f is Riemann integrable on $[a,b]$.

Solution.
(a) By the induction hypothesis f is Riemann integrable on $[a, c_{n+1}-\delta]$. Since f is continuous on $[c_{n+1}+\delta, b]$, it is integrable there. Then by Exercise 20.6 we can find a partition P_1 of $[a, c_{n+1}-\delta]$ such that $U(f, P_1) - L(f, P_1) < \frac{\epsilon}{4}$ and a partition P_2 of $[c_{n+1}, b]$ such that $U(f, P_2) - L(f, P_2) < \frac{\epsilon}{4}$.
(b) Let M' be the supremum of f and m' the infimum of f on $[c_{n+1}-\delta, c_{n+1}+\delta]$. We have

$$U(f,P) - L(f,P) = U(f,P_1) - L(f,P_1) + (M'-m')(2\delta) + U(f,P_2) - L(f,P_2)$$
$$< \frac{\epsilon}{4} + 2M(2)\frac{\epsilon}{8M} + \frac{\epsilon}{4} = \epsilon$$

Thus, f is Riemann integrable on $[a, b]$ ■

Exercise 21.5
Let $f : [a, b] \to \mathbb{R}$ be a increasing function on $[a, b]$.
(a) Show that f is bounded on $[a, b]$.
(b) Let $\epsilon > 0$ be given. Choose a positive integer N such that $\frac{f(a)-f(b)}{N} < \epsilon$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ such that $x_i - x_{i-1} < \frac{1}{N}$ for all $1 \le i \le n$. For each $1 \le i \le n$, express $M_i(f)$ and $m_i(f)$ in terms of $f(x)$.
(c) Show that $U(f, P) - L(f, P) < \epsilon$. Thus, conclude that f is Riemann integrable.

Solution.
(a) Since f is decreasing, we must have $f(b) \le x \le f(a)$ for all $x \in [a, b]$. Let $m = f(b)$ and $M = f(a)$. Then f is bounded.
(b) Since f is decreasing, we must have $M_i(f) = f(x_{i-1})$ and $m_i(f) = f(x_i)$ for $1 \le i \le n$.
(c) We have

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} [f(x_{i-1}) - f(x_i)](x_i - x_{i-1}) \\ &< \sum_{i=1}^{n} \frac{[f(x_{i-1}) - f(x_i)]}{N} \\ &= \frac{f(a) - f(b)}{N} < \epsilon \end{aligned}$$

It follows from Exercise 20.7 that f is Riemann integrable ■

Exercise 21.6
Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f \geq 0$ on $[a, b]$. Let $[c, d] \subset [a, b]$. Prove that $\int_a^b f(x)dx \geq \int_c^d f(x)dx$.

Solution.
Let $P = \{c = x_0 < x_1 < \cdots < x_n = d\}$ be a partition of $[c, d]$. Since $f \geq 0$ We have $L(f, P) \leq \underline{\int}_a^b f(x)dx = \int_a^b f(x)dx$. Since P was arbitrary, we can say that $\int_a^b f(x)dx$ is an upper bound of $S_L([c, d])$. Hence, $\int_c^d f(x)dx = \underline{\int}_c^d f(x)dx \leq \int_a^b f(x)dx$ ■

Exercise 21.7
(a) Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and $f \geq 0$ on $[0, 1]$. Let $a \in [0, 1]$ be such that $f(a) > 0$. Show that $\int_0^1 f(x)dx > 0$.
(b) Construct a nonnegative function f on $[0, 1]$ such that $f(0.5) > 0$ but $\int_0^1 f(x)dx = 0$.

Solution.
(a) Let $\epsilon = \frac{f(a)}{2} > 0$. Since f is continuous at a we can find a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon = \frac{f(a)}{2}$. Using the triangle inequality we end up with $|f(x) > \frac{f(a)}{4} > 0$ for all $x \in (a - \delta, a + \delta)$. Thus, $\int_0^1 f(x)dx > \int_{a-\delta}^{a+\delta} f(x)dx > \int_{a-\delta}^{a+\delta} \frac{f(a)}{4}dx = \frac{f(a)\delta}{4} > 0$.
(b) Let

$$f(x) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{2}. \end{cases}$$

Then $f(0.5) = 1 > 0$ and $\int_0^1 f(x)dx = 0$ ■

Exercise 21.8
Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$. Prove that f is Riemann integrable on $[a, b]$.

Solution.
Since f is differentiable, f is continuous; this implies f is integrable ■

Exercise 21.9
Let $f : [a, b] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

(a) Prove that f is not Riemann integrable on $[a, b]$. Hint Show that the lower Riemann integral is different from the upper Riemann integral.
(b) Prove that $|f|$ is Riemann integrable.

Solution.
(a) Since between any two real numbers we can find a rational number and an irrational number, we can write $U(f, P) = b - a$ and $L(f, P) = a - b$ for all P. Thus

$$\overline{\int}_a^b f(x)dx = b - a \text{ and } \underline{\int}_a^b f(x)dx = a - b$$

It follows that f is not integrable.
(b) Since $|f(x)| = 1$ for all $x \in [a, b]$, $|f|$ is continuous on $[a, b]$ and therefore integrable ■

Exercise 21.10
Suppose f is a continuous function on $[a, b]$ and that $f(x) \geq 0$ for all $x \in [a, b]$. Show that if $\int_a^b f(x)dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$. Hint: Assume the contrary and get a contradiction.

Solution.
Suppose that there is an $c \in [a, b]$ such that $f(c) > 0$. Let $\epsilon = \frac{f(c)}{2} > 0$. Since f is continuous at c we can find a $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon = \frac{f(c)}{2}$. Using the triangle inequality we end up with $|f(x) > \frac{f(c)}{4} > 0$ for all $x \in (c - \delta, c + \delta)$. Thus, $\int_0^1 f(x)dx > \int_{c-\delta}^{c+\delta} f(x)dx > \int_{c-\delta}^{c+\delta} \frac{f(c)}{4}dx = \frac{f(c)\delta}{4} > 0$ which is a contradiction. Hence, $f(x) = 0$ for all $x \in [a, b]$ ■

Solutions to Section 22

Exercise 22.1
Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose that $\lim_{\mu(P)\to 0} S(f, P) = A$.
(a) Let $\epsilon > 0$. Show that there is a $\delta > 0$ such that for any partition P of $[a, b]$ such that $\mu(P) < \delta$ we must have $|S(f, P) - A| < \frac{\epsilon}{4}$.
(b) Let $Q = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ such that $\mu(Q) < \delta$, that is, $x_i - x_{i-1} < \delta$ for all $1 \le i \le n$. Fix $1 \le i \le n$. Show that if $f(u_i) \ge m_i(f) + \frac{\epsilon}{4(b-a)}$ for all $x_{i-1} \le u_i \le x_i$ then this contradicts the definition of $m_i(f)$.
(c) With Q as above, show that if $f(v_i) \le M_i(f) - \frac{\epsilon}{4(b-a)}$ for all $x_{i-1} \le v_i \le x_i$ then this contradicts the definition of $M_i(f)$.
(d) Show that for every $1 \le i \le n$, there exists $u_i, v_i \in [x_{i-1}, x_i]$ such that $f(u_i) < m_i(f) + \frac{\epsilon}{4(b-a)}$ and $f(v_i) > M_i(f) - \frac{\epsilon}{4(b-a)}$
(e) Show that $\sum_{i=1}^n f(u_i)(x_i - x_{i-1}) < L(f, Q) + \frac{\epsilon}{4}$ and $\sum_{i=1}^n f(v_i)(x_i - x_{i-1}) > U(f, Q) - \frac{\epsilon}{4}$
(f) Show that

$$\begin{array}{c} A - \frac{\epsilon}{4} < \sum_{i=1}^n f(u_i)(x_i - x_{i-1}) < A + \frac{\epsilon}{4} \text{ and} \\ A - \frac{\epsilon}{4} < \sum_{i=1}^n f(v_i)(x_i - x_{i-1}) < A + \frac{\epsilon}{4} \end{array}$$

(g) Use (f) to show that

$$A - \frac{\epsilon}{2} < L(f, Q) \le U(f, Q) < A + \frac{\epsilon}{2}.$$

(h) Show that $U(f, Q) - L(f, Q) < \epsilon$. That is, f is Riemann integrable.
(i) Show that

$$\left| \int_a^b f(x)dx - A \right| < \epsilon.$$

(k) Use the Squeeze rule to show that $\int_a^b f(x)dx = A$.
Conclusion: Suppose that there is a number A such that $\lim_{\mu(P)\to 0} S(f, P) = A$. Then f is Riemann integrable with $\int_a^b f(x)dx = A$.

Solution.
(a) This follows from the hypothesis that $\lim_{\mu(P)\to 0} S(f, P) = A$.
(b) Let $Q = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ such that $\mu(Q) < \delta$, that is, $x_i - x_{i-1} < \delta$ for all $1 \le i \le n$. If $f(u_i) \ge m_i(f) + \frac{\epsilon}{4(b-a)}$

for all $x_{i-1} \leq u_i \leq x_i$ then $m_i(f) + \frac{\epsilon}{4(b-a)}$ is a lower bound of f in $[x_{i-1}, x_i]$ so by the definition of $m_i(f)$ we must have $m_i(f) + \frac{\epsilon}{4(b-a)} < m_i(f)$, which is impossible.
(c) If $f(v_i) \leq M_i(f) - \frac{\epsilon}{4(b-a)}$ for all $x_{i-1} \leq v_i \leq x_i$ then $M_i(f) - \frac{\epsilon}{4(b-a)}$ is an upper bound of f on $[x_{i-1}, x_i]$. By the definition of $M_i(f)$, we must have $M_i(f) \leq M_i(f) - \frac{\epsilon}{4(b-a)}$, which is impossible.
(d) This follows from (b) and (c).
(e) We have

$$\sum_{i=1}^{n} f(u_i)(x_i - x_{i-1}) < \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) + \frac{\epsilon}{4}\sum_{i=1}^{n} \frac{x_i - x_{i-1}}{b-a} = L(f, Q) + \frac{\epsilon}{4}.$$

Similar argument for the second part of the question.
(f) Since $\sum_{i=1}^{n} f(u_i)(x_i - x_{i-1})$ and $\sum_{i=1}^{n} f(v_i)(x_i - x_{i-1})$ are Riemann sums with $\mu(Q) < \delta$ they are within $\frac{\epsilon}{4}$ of A.
(g) we have $A - \frac{\epsilon}{2} < \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1}) - \frac{\epsilon}{4} \leq L(f, Q) \leq U(f, Q) \leq \sum_{i=1}^{n} f(v_i)(x_i - x_{i-1}) + \frac{\epsilon}{4} < A + \frac{\epsilon}{2}$.
(h) Since both $U(f, Q)$ and $L(f, Q)$ are inside the interval centered at A and of length at most ϵ, we must have $U(f, Q) - L(f, Q) < \epsilon$. By Exercise **??**, f is Riemann integrable.
(i) Since $L(f, Q) \leq \int_a^b f(x)dx \leq U(f, Q)$, we must have $\left|\int_a^b f(x)dx - A\right| < \epsilon$.
(k) Let $\epsilon \to 0$ and apply the Squeeze rule ■

Exercise 22.2
Prove that $A - L(f, R) < \frac{\epsilon}{2}$ and $U(f, R) - A < \frac{\epsilon}{2}$.

Solution.
by Exercise **??** we have $L(f, R) \geq L(f, P)$ and $U(f, R) \leq U(f, P)$. Thus, $A - L(f, R) \leq A - L(f, P) < \frac{\epsilon}{2}$ and $U(f, R) - A \leq U(f, P) - A < \frac{\epsilon}{2}$ ■

Exercise 22.3
(a) For $1 \leq i \leq m$ such that $L(f, R_i) - m_i(z_i - z_{i-1}) \neq 0$ and $M_i(z_i - z_{i-1}) - U(f, R_i)$ prove that

$$L(f, R_i) - m_i(z_i - z_{i-1}) < 2M\delta \text{ and } M_i(z_i - z_{i-1}) - U(f, R_i) < 2M\delta.$$

(b) Use (a) and the sums above to show that

$$L(f, R) - L(f, Q) < \frac{\epsilon}{2} \text{ and } U(f, Q) - U(f, R) < \frac{\epsilon}{2} >$$

Solution.
(a) We have

$$L(f, R_i) - m_i(z_i - z_{i-1}) \leq 2M(z_i - z_{i-1}) < 2M\delta$$

and

$$M_i(z_i - z_{i-1}) - U(f, R_i) \leq 2M(z_i - z_{i-1}) < 2M\delta$$

(b) Because there are at most $n - 1$ such terms, we obtain the bounds

$$L(f, R) - L(f, Q) < 2nM\delta = \frac{\epsilon}{2}$$

and

$$U(f, Q) - U(f, R) < 2nM\delta = \frac{\epsilon}{2} \blacksquare$$

Exercise 22.4
Use Exercise 22.2 and 22.3 to prove that

$$U(f, Q) < A + \epsilon \text{ and } L(f, Q) > A - \epsilon.$$

Solution.
We have

$$U(f, Q) - A = [U(f, Q) - U(f, R)] + [U(f, R) - A] < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

and

$$A - L(f, Q) = [A - L(f, R)] + [L(f, R) - L(f, Q)] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \blacksquare$$

Exercise 22.5
Using the previous problem, show that

$$A - \epsilon < S(f, Q) < A + \epsilon.$$

That is,

$$|S(f, Q) - A| < \epsilon.$$

Solution.
We have $A - \epsilon \leq L(f, Q) \leq S(f, Q) \leq U(f, Q) < A + \epsilon$ ■

Exercise 22.6
Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and Riemann integrable. The goal of this problem is to show that for any sequence $\{P_n\}_{n=1}^{\infty}$ of partitions of $[a, b]$ such that $\lim_{n\to\infty} \mu(P_n) = 0$ we have $\lim_{n\to\infty} S(f, P_n) = \int_a^b f(x)dx$.
(a) Let $\epsilon > 0$. Show that there is a $\delta > 0$ such that if P is a partition of $[a, b]$ with $\mu(P) < \delta$ we have

$$\left| S(f, P) - \int_a^b f(x)dx \right| < \epsilon.$$

(b) Show that there is a positive integer N such that if $n \geq N$ then $\mu(P_n) < \delta$.
(c) Use (a) and (b) to conclude that for $n \geq N$ we have

$$\left| S(f, P_n) - \int_a^b f(x)dx \right| < \epsilon.$$

Hence,

$$\lim_{n\to\infty} S(f, P_n) = \int_a^b f(x)dx.$$

Solution.
(a) This follows from Exercise 16.6.
(b) This follows from the definition of convergence of a sequence.
(c) If $n \geq N$ then $\mu(P_n) < \delta$ and this implies by (a) that

$$\left| S(f, P_n) - \int_a^b f(x)dx \right| < \epsilon \blacksquare$$

Exercise 22.7
Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable. Let $\epsilon > 0$ be given. Show that there is a $\delta > 0$ such that for any partition P of $[a, b]$ with $\mu(P) < \delta$ we have

$$U(f, P) - L(f, P) < \epsilon.$$

Solution.
This follows from Exercise 22.4 $\blacksquare$

Exercise 22.8
Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable in $[a, b]$ and that $f' : [a, b] \to \mathbb{R}$ is Riemann integrable. Let $P_n = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition

of $[a,b]$ such that $x_i - x_{i-1} = \frac{b-a}{n}$.
(a) For each $1 \le i \le n$, show that there exists $x_{i-1} < t_i < x_i$ such that $f(x_i) - f(x_{i-1}) = f'(t_i)(x_i - x_{i-1})$.
(b) Show that $S(f', P_n) = \sum_{i=1}^n f'(t_i)(x_i - x_{i-1}) = f(b) - f(a)$.
(c) Show that $\lim_{n\to\infty} \mu(P_n) = 0$.
(d) Show that $\lim_{n\to\infty} S(f', P_n) = \int_a^b f'(x)dx$.
(e) Show that

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Solution.
(a) This follows from the Mean Value Theorem.
(b) We $\sum_{i=1}^n f'(t_i)(x_i - x_{i-1}) = (f(x_1) - f(a)) + (f(x_2) - f(x_1)) + \cdots + (f(x_{n-1}) - f(x_{n-2}) + (f(b) - f(x_{n-1})) = f(b) - f(a)$.
(c) We have $\lim_{n\to\infty} \mu(P_n) = \lim_{n\to\infty} \frac{b-a}{n} = 0$.
(d) This follows from the previous exercise.
(e) We have

$$\int_a^b f'(x)dx = \lim_{n\to\infty} S(f', P_n) = \lim_{n\to\infty} (f(b) - f(a)) = f(b) - f(a) \blacksquare$$

Exercise 22.9 *(Fundamental Theorem of Calculus)*
Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and let $F : [a,b] \to \mathbb{R}$ be a differentiable function such that $F'(x) = f(x)$ for all $a \le x \le b$. Show that

$$\int_a^b f(x)dx = F(b) - F(a).$$

The function $F(x)$ is called an **antiderivative** of f.

Solution.
Using the previous problem we see that

$$\int_a^b f(x)dx = \int_a^b F'(x)dx = F(b) - F(a) \blacksquare$$

Solutions to Section 23

Exercise 23.1
Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions and α, β be real numbers. Let $\epsilon > 0$.
(a) Show that there is a $\delta_1 > 0$ such that if $\mu(P) < \delta_1$ then $\left|S(f, P) - \int_a^b f(x)dx\right| < \frac{\epsilon}{|\alpha|+|\beta|}$.
(b) Show that there is a $\delta_2 > 0$ such that if $\mu(P) < \delta_2$ then $\left|S(g, P) - \int_a^b g(x)dx\right| < \frac{\epsilon}{|\alpha|+|\beta|}$.
(c) Show that there is a $\delta > 0$ such that if $\mu(P) < \delta$ then

$$\left|S(\alpha f + \beta g, P) - \left[\alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx\right]\right| < \epsilon.$$

We conclude that $\alpha f + \beta g$ is Riemann integrable and

$$\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

Solution.
(a) Follows from Exercises 22.2 - 22.5.
(b) Follows from Exercises 22.2 - 22.5.
(c) Let $\delta = \min\{\delta_1, \delta_2\}$. Suppose that $\mu(P) < \delta$. Then $\mu(P) < \delta_1$ and $\mu(P) < \delta_2$. Moreover, we have

$$\begin{aligned}
&\left|S(\alpha f + \beta g, P) - \left[\alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx\right]\right| \\
=&\left|\alpha S(f, P) + \beta S(g, P) - \left[\alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx\right]\right| \\
\leq&|\alpha| \left|S(f, P) - \int_a^b f(x)dx\right| + |\beta| \left|S(g, P) - \int_a^b g(x)dx\right| \\
<&|\alpha| \frac{\epsilon}{|\alpha| + |\beta|} + |\beta| \frac{\epsilon}{|\alpha| + |\beta|} = \epsilon.
\end{aligned}$$

By Exercise 22.1, $\alpha f + \beta g$ is Riemann integrable and

$$\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx \blacksquare$$

Exercise 23.2
Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions such that $f(x) \le g(x)$ for all $x \in [a, b]$.
(a) Show that for any partition P of $[a, b]$ we have $L(f, P) \le L(g, P)$.
(b) Show that $\underline{\int_a^b} f(x)dx \le \underline{\int_a^b} g(x)dx$.
(c) Show that $\int_a^b f(x)dx \le \int_a^b g(x)dx$.

Solution.
(a) Since $f(x) \le g(x)$ for all $x \in [a, b]$, we have $m_i(f) \le m_i(g)$ for all $1 \le i \le n$. Thus, $\sum_{i=1}^n m_i(f)(x_i - x_{i-1}) \le \sum_{i=1}^n m_i(g)(x_i - x_{i-1})$, that is, $L(f, P) \le L(f, g)$.
(b) From (b), we have

$$L(f, P) \le L(g, P) \le \underline{\int_a^b} g(x)dx$$

for all partitions P of $[a, b]$. Thus, $\underline{\int_a^b} g(x)dx$ is an upper bound for $S_L = \{L(f, P) : P \text{ a partition of } [a, b\}$. But $\underline{\int_a^b} f(x)dx$ is the smallest upper bound so that

$$\underline{\int_a^b} f(x)dx \le \underline{\int_a^b} g(x)dx.$$

(c) This follows from the fact that

$$\underline{\int_a^b} f(x)dx = \int_a^b f(x)dx \text{ and } \underline{\int_a^b} g(x)dx = \int_a^b g(x)dx \ \blacksquare$$

Exercise 23.3
Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function such that $m \le f(x) \le M$ for all $x \in [a, b]$.
(a) Show that $m(b - a) \le L(f, P) \le U(f, P) \le M(b - a)$ for any partition P of $[a, b]$.
(b) Show that $\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx \le M(b - a)$.
(c) Show that $m(b - a) \le \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx$.
Conclusion: $m(b - a) \le \int_a^b f(x)dx \le M(b - a)$.

Solution.
(a) Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Then

$$m(x_i - x_{i-1}) \le m_i(f)(x_i - x_{i-1}) \le M_i(f)(x_i - x_{i-1}) \le M(x_i - x_{i-1}).$$

Summing from $i = 1$ to $i = n$ to obtain

$$m(b-a) \le L(f,P) \le U(f,P) \le M(b-a)$$

(b) $M(b-a)$ is an upper bound of S_L. But $\underline{\int_a^b} f(x)dx$ is the smallest upper bound so that

$$\underline{\int_a^b} f(x)dx \le M(b-a).$$

Since f is Riemann integrable, we have

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx \le M(b-a).$$

(c) $m(b-a)$ is a lower bound of S_U. But $\overline{\int_a^b} f(x)dx$ is the largest lower bound so that

$$m(b-a) \le \overline{\int_a^b} f(x)dx.$$

Since f is Riemann integrable, we have

$$m(b-a) \le \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx \blacksquare$$

Exercise 23.4
Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function and $a < c < b$.
(a) Let $\epsilon > 0$. Show that there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.
(b) Let $Q = P \cup \{c\}$, $Q_1 = Q \cap [a,c]$, and $Q_2 = Q \cap [c,b]$. That is, Q is partition of $[a,b]$, Q_1 is a partition of $[a,c]$, and Q_2 is a partition of $[c,b]$. Show that

$$[U(f,Q_1) - L(f,Q_1)] + [U(f,Q_2) - L(f,Q_2)] < \epsilon.$$

(c) Show that $U(f,Q_1) - L(f,Q_1) < \epsilon$. Thus, by Exercise 21.1, $\int_a^c f(x)dx$ exists and is finite.
(d) Show that $U(f,Q_2) - L(f,Q_2) < \epsilon$. Thus, by Exercise 21.1, $\int_c^b f(x)dx$ exists and is finite.

Solution.
(a) This follows from Exercise 21.1.
(b) We have

$$\begin{aligned}[U(f,Q_1)-L(f,Q_1)]+[U(f,Q_2)-L(f,Q_2)] =&[U(f,Q_1)+U(f,Q_2]\\ &-[L(f,Q_1)]+L(f,Q_2)]\\ =&U(f,Q)-L(f,Q)\\ \le&U(f,P)-L(f,P)<\epsilon\end{aligned}$$

(c) Since $U(f,Q_1)-L(f,Q_1)>0$ we must have $U(f,Q_1)-L(f,Q_1)<\epsilon$. By Exercise 21.1, f is Riemann integrable in $[a,c]$.
(d) Since $U(f,Q_2)-L(f,Q_2)>0$ we must have $U(f,Q_2)-L(f,Q_2)<\epsilon$. By Exercise 21.1, f is Riemann integrable in $[c,b]$ ■

Exercise 23.5
Let $f:[a,b]\to\mathbb{R}$ be a Riemann integrable function and $a<c<b$. Let $\epsilon>0$.
(a) Show that there is a $\delta_1>0$ such that if P_1 is a partition of $[a,c]$ such that $\mu(P_1)<\delta_1$ then $\left|S(f,P_1)-\int_a^c f(x)dx\right|<\frac{\epsilon}{2}$.
(b) Show that there is a $\delta_2>0$ such that if P_2 is a partition of $[c,b]$ such that $\mu(P_2)<\delta_2$ then $\left|S(f,P_2)-\int_c^b f(x)dx\right|<\frac{\epsilon}{2}$.
(c) Let $P=P_1\cup P_2$. Then P is a partition of $[a,b]$. Show that there is $\delta>0$ such that $\mu(P)<\delta$ and

$$\left|S(f,P)-\left[\int_a^c f(x)dx+\int_c^b f(x)dx\right]\right|<\epsilon.$$

That is,

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx.$$

Solution.
(a) By the previous exercise, f is Riemann integrable in $[a,c]$. Now, the result follows from Exercises 22.2 - 22.5.
(b) Similar to (a).

(c) Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\mu(P) < \delta$. Also, we have

$$\begin{aligned}
&\left|S(f,P) - \left[\int_a^c f(x)dx + \int_c^b f(x)dx\right]\right| \\
\leq &\left|S(f,P_1) - \int_a^c f(x)dx\right| + \left|S(f,P_2) - \int_c^b f(x)dx\right| \\
< &\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Now the result follows from Exercise 22.1 ■

Exercise 23.6
Let $f : [a,b] \to \mathbb{R}$ be continuous. Use the Intermediate Value Theorem to prove the existence of a number $c \in [a,b]$ such that

$$\int_a^b f(x)dx = (b-a)f(c).$$

The number $f(c)$ is called the **average value** of f on $[a,b]$.

Solution.
We know that $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$ (Exercise 23.3). Let $d = \frac{1}{b-a}\int_a^b f(x)dx$. Then $d \in [m,M]$. By the IVT there is a number $c \in [a,b]$ such that $f(c) = d$. This implies that

$$\int_a^b f(x)dx = (b-a)f(c) \blacksquare$$

Exercise 23.7
Suppose that f and g are continuous function on $[a,b]$ such that $\int_a^b f(x)dx = \int_a^b g(x)dx$. Prove there is a $c \in [a,b]$ such that $f(c) = g(c)$.

Solution.
Let $h(x) = f(x) - g(x)$. Then h is continuous on $[a,b]$. By the previous exercise, there is a $c \in [a,b]$ such that $\frac{1}{b-a}\int_a^b h(x)dx = h(c)$. But $\int_a^b h(x)dx = 0$. Thus, $h(c) = 0$ or $f(c) = g(c)$ ■

Exercise 23.8
(a) For any set S, one can see that $M(f,S) - m(f,S) = \sup_{s,t\in S}|f(s) - f(t)|$.

Let f be a function defined on a set S. Show that $M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S)$.
(b) Suppose that $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Show that $|f$ is also Riemann integrable.

Solution.
(a) For any $s, t \in S$ we have

$$||f(s)| - |f(t)|| \leq |f(s) - f(t)|.$$

It follows that

$$M(|f|, S) - m(|f|, S) = \sup_{s,t \in S} ||f(s)| - |f(t)|| \leq \sup_{s,t \in S} |f(s) - f(t)| = M(f, S) - m(f, S).$$

(b) Let $\epsilon > 0$ be given. Then there is a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ such that $U(f, P) - L(f, P) < \epsilon$. From part (a) we have that for $1 \leq i \leq n$

$$M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f).$$

Hence,

$$U(|f|, P) - L(|f|, P) < U(f, P) - L(f, P) < \epsilon.$$

That is, $|f|$ is integrable ■

Exercise 23.9
Let $f : [a, b] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

(a) Compute $\underline{\int_a^b} f(x)dx$ and $\overline{\int_a^b} f(x)dx$.
(b) Is f Riemann integrable?
(c) Show that $|f|$ is Riemann integrable.

Solution.
Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be any partition of $[a, b]$. Then $m_i = -1$ and $M_i = 1$ for all $1 \leq i \leq n$. Hence, $L(f, P) = a - b$ and $U(f, P) = b - a$. It follows that

$$\underline{\int_a^b} f(x)dx = a - b \text{ and } \overline{\int_a^b} f(x)dx = b - a.$$

(b) It follows from (a) that f is not Riemann integrable.
(c) $|f|(x) = 1$ for all $x \in [a,b]$. Since $|f|$ is a continuous function, $|f|$ is integrable on [a,b] ■

Exercise 23.10
Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable with $|f(x)| \le M$ for all $x \in [a,b]$.
(a) Prove that $|f^2(x) - f^2(y)| \le 2M|f(x) - f(y)|$ for all $x, y \in [a,b]$ where $f^2(x) = (f(x))^2$.
(b) Let $\epsilon > 0$. Show that there is a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \frac{\epsilon}{2M}.$$

(c) Prove that $U(f^2,P) - L(f^2,P) < \epsilon$. That is, f^2 is Riemann integrable.

Solution.
(a) We have $|f^2(x) - f^2(y)| = |f(x)+f(y)||f(x)-f(y)| \le (|f(x)|+|f(y)|)|f(x)-f(y)| \le 2M|f(x) - f(y)|$.
(b) This follows from Exercise 20.6.
(c) Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. We have

$$\begin{aligned} U(f^2,P) - L(f^2,P) =& \sum_{i=1}^{n} \sup_{s,t\in[x_{i-1},x_i]} |f^2(s) - f^2(t)|(x_i - x_{i-1}) \\ \le& 2M \sum_{i=1}^{n} \sup_{s,t\in[x_{i-1},x_i]} |f(s) - f(t)|(x_i - x_{i-1}) \\ =& 2M(U(f,P) - L(f,P)) < 2M \cdot \frac{\epsilon}{2M} = \epsilon \; ■ \end{aligned}$$

Exercise 23.11
Let $f, g : [a,b] \to \mathbb{R}$ be two Riemann integrable functions.
(a) Show that

$$f \cdot g = \frac{1}{2}[(f+g)^2 - f^2 - g^2].$$

(b) Prove that $f \cdot g$ is Riemann integrable.

Solution.
(a) Trivial algebra.
(b) Since f and g are integrable so are the functions f^2, g^2, and $(f+g)^2$ according to the previous problem. Now the result follows from Exercise 23.1 ■

Solutions to Section 24

Exercise 24.1
Suppose that $f : [a, b] \to [c, d]$ is a Riemann integrable function on $[a, b]$ and that $g : [c, d] \to \mathbb{R}$ is continuous (and hence integrable by Exercise **??**).
(a) Show that the set $\{|g(x)| : x \in [c, d]\}$ is bounded. Hence, by the Completeness Axiom of $\mathbb{R}$ there exists $K > 0$ such that $K = \sup\{|g(x)| : x \in [c, d]\}$.
(b) Let $\epsilon > 0$. Chosse ϵ' so that $\epsilon' < \frac{\epsilon}{b-a+2K}$. Show that there is a $\delta < \epsilon'$ such that if $|s - t| < \delta$, where $s, t \in [c, d]$, then $|g(s) - g(t)| < \epsilon'$.
(c) Show that there is a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ such that $U(f, P) - L(f, P) < \delta^2$.
(d) Let $A = \{1 \le i \le n : M_i(f) - m_i(f) < \delta\}$. Show that if $i \in A$ then $M_i(g \circ f) - m_i(g \circ f)| < \epsilon'$.
(e) Let $B = \{1 \le i \le n : M_i(f) - m_i(f) \ge \delta\}$. Show that $\delta \sum_{i \in B}(x_i - x_{i-1}) < \delta^2$ and hence $\sum_{i \in B}(x_i - x_{i-1}) < \delta$.
(f) Show that for all $1 \le i \le n$ we have $M_i(g \circ f) - m_i(g \circ f) < 2K$. Hint: Use Exercise 15.8 and the triangle inequality.
(g) Use (d) (e) and (f) to show that $U(g \circ f, P) - L(g \circ f, P) < \epsilon$. Hence, by Exercise 20.7, $g \circ f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Solution.
(a) This follows from Exercise 15.3.
(b) This is just the uniform continuity of g. See Exercise 14.5(d).
(c) This follows from Exercise 20.6 since f is Riemann integrable on $[a, b]$.
(d) Let $i \in A$ and $s, t \in [x_{i-1}, x_i]$. Since $M_i(f) - m_i(f) < \delta$, we must have $|f(s) - f(t)| < \delta$. Now by (b) we have $|g(f(s)) - g(f(t))| < \epsilon'$ and this implies that $M_i(g \circ f) - m_i(g \circ f)| < \epsilon'$.
(e) We have $\delta \sum_{i \in B}(x_i - x_{i-1}) < \sum_{i \in B}[M_i(f) - m_i(f)](x_i - x_{i-1}) \le U(f, P) - L(f, P) < \delta^2$. Divide both sides by δ to obtain the required result.
(f) Note that by Exercise 15.8 we have $M_i(g \circ f) \in g([c, d])$ and $m_i(g \circ f) \in g([c, d])$. Hence, $M_i(g \circ f) - m_i(g \circ f) \le |M_i(g \circ f) - m_i(g \circ f)| \le |M_i(g \circ f)| + |m_i(g \circ f)| < K + K = 2K$.

(g) We have

$$\begin{aligned}
U(g\circ f,P)-L(g\circ f,P) =&\sum_{i=1}^{n}[M_i(g\circ f)-m_i(g\circ f)](x_i-x_{i-1})\\
=&\sum_{i\in A}[M_i(g\circ f)-m_i(g\circ f)](x_i-x_{i-1})\\
+&\sum_{i\in B}[M_i(g\circ f)-m_i(g\circ f)](x_i-x_{i-1})\\
<&\epsilon'\sum_{i\in A}(x_i-x_{i-1})+2K\sum_{i\in B}(x_i-x_{i-1})\\
<&\epsilon'\sum_{i=1}^{n}(x_i-x_{i-1})+2K\delta\\
=&\epsilon'(b-a)+2K\delta\\
<&\epsilon'(b-a)+2K\epsilon'\\
=&\epsilon'(b-a+2K)<\epsilon
\end{aligned}$$

Thus, by Exercise 20.7, $g\circ f$ is Riemann integrable ■

Exercise 24.2
Let $f,g:[a,b]\to\mathbb{R}$ be Riemann integrable and bounded such that $|f(x)|\le M_1$ and $|g(x)|\le M_2$ for all $x\in[a,b]$.
(a) Find a positive constant M such that $|f(x)|\le M$ and $|g(x)|\le M$. Thus, $f([a,b])\subseteq[-M,M]$ and $g([a,b])\subseteq[-M,M]$
(b) Consider the continuous function $h:[-2M,2M]\to\mathbb{R}$ given by $h(x)=x^2$. Show that $(f+g)^2$ and $(f-g)^2$ are Riemann integrable on $[a,b]$. Hint: Note that $h\circ(f+g)=(f+g)^2$ and $h\circ(f-g)=(f-g)^2$.
(c) Show that $f\cdot g$ is Riemann integrable on $[a,b]$.

Solution.
(a) Let $M=M_1+M_2$.
(b) This follows from Exercise 24.4.
(c) Since $f\cdot g=\frac{1}{4}[(f+g)^2-(f-g)^2]$, by Exercise 23.1 we conclude that $f\cdot g$ is Riemann integrable on $[a,b]$ ■

Exercise 24.3
Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable and bounded such that $|f(x)|\le M$

for all $x \in [a,b]$.
(a) consider the continuous function $g : [-M, M] \to \mathbb{R}$ defined by $g(x) = |x|$. Show that $|f|$ is Riemann integrable on $[a,b]$.
(b) Using the fact that $-|f(x)| \le f(x) \le |f(x)|$ for all $x \in [a,b]$, show that

$$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx.$$

Hence, show that

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx.$$

Solution.
(a) This follows from Exercise 24.4.
(b) This follows from Exercise 23.2. The last equality follows from Exercise 1.14 ■

Exercise 24.4 *(Integration by Parts)*
Let $f, g : [a,b] \to \mathbb{R}$ be continuous and $f', g' : [a,b] \to \mathbb{R}$ be Riemann integrable.
(a) Show that f and g are Riemann integrable on $[a,b]$.
(b) Show that $f' \cdot g$ and $f \cdot g'$ are Riemann integrable on $[a,b]$.
(c) Show that $\int_a^b f'gdx + \int_a^b fg'dx = (fg)(b) - (fg)(a)$. Hint: Use product rule and Exercise 22.8.

Solution.
(a) This follows from Exercise 21.2.
(b) This follows from Exercise 24.5.
(c) By the product rule we have $(fg)' = f'g + fg'$. Hence, using Exercise 22.8 we can write

$$\int_a^b f'gdx + \int_a^b fg'dx = \int_a^b (fg)'dx = (fg)(b) - (fg)(a)$$

or

$$\int_a^b f'gdx = (fg)(b) - (fg)(a) - \int_a^b fg'dx \blacksquare$$

Exercise 24.5
Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is Riemann integrable on $[0,1]$. What is the value of $\int_0^1 f(x)dx$?

Solution.
Let P be a parition of $[0,1]$ given by

$$P = \{0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\}.$$

Note that $M_i(f) = 1$ for all $1 \leq i \leq n$, $m_1(f) =$ and $m_i(f) = 1$ for all $1 \leq i \leq n-1$. Thus,

$$U(f,P) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) = 1.$$

Since P was arbitrary, it follows that

$$\overline{\int_0^1} f(x)dx = 1.$$

We know from Exercise 20.3 that $\underline{\int_0^1} f(x)dx \leq 1$. Suppose that $\underline{\int_0^1} f(x)dx < 1$. Let $\epsilon = 1 - \underline{\int_0^1} f(x)dx > 0$. Let $Q = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be a partition of $[0,1]$ such that $x_1 > \epsilon$. Then $L(f,Q) = m_1(f)(x_1 - x_0) + \sum_{i=2}^{n-1} m_i(f)(x_i - x_{i-1}) = 1 - x_1 < \underline{\int_0^1} f(x)dx$ which contradicts the definition of supremum. We conclude that $\underline{\int_0^1} f(x)dx = 1$.
Since $\overline{\int_0^1} f(x)dx = \underline{\int_0^1} f(x)dx = 1$, the function f is Riemnann integrable with

$$\int_0^1 f(x)dx = 1 \blacksquare$$

Exercise 24.6
Consider the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } x = 1 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is rational with } p \text{ and } q > 0 \text{ in lowest terms} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

(a) Let $\epsilon > 0$ and $\epsilon' = \min\{0.5, \epsilon\}$. Thus, $0 < \epsilon' \leq 0.5$ and $0 < \epsilon' \leq \epsilon$. Show that there is a finite number of rationals in $[0, 1]$ such that $g(x) \geq \frac{\epsilon'}{2}$. Denote the rationals by $\{r_0, r_1, \cdots, r_n\}$ where $r_0 = 0$ and $r_n = 1$.
(b) Define the partition $Q = \{0 = x_0 < x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = 1\}$ where $x_0 = 0; x_1 < r_1$ with $x_1 < \frac{\epsilon'}{2(n+1)}; x_1 < x_2 < r_1 < x_3$ with $x_3 - x_2 < \frac{\epsilon'}{2(n+1)}; \cdots; x_{2n-2} < r_{n-1} < x_{2n-1}$ with $x_{2n-1} - x_{2n-2} < \frac{\epsilon'}{2(n+1)}; x_{2n-1} < x_{2n} < 1$ with $1 - x_{2n} < \frac{\epsilon'}{2(n+1)}$ and $x_{2n+1} = 1$. Show that $U(g, Q) < \epsilon'$. Hint: Note that the sum involves intervals containing $r_i's$ and those that do not.
(c) Show that $L(g, Q) = 0$. Hint: Exercise 2.6.
(d) Using (b) and (c) show that $U(g, Q) - L(g, Q) < \epsilon$. Thus, g is Riemann integrable.
(e) What is the value of the integral $\int_0^1 g(x)dx$?

Solution.
(a) Let $x = \frac{p}{q} \in (0, 1)$ such that $g(x) \geq \frac{\epsilon'}{2}$. Then $\frac{1}{q} \geq \frac{\epsilon'}{2}$ which implies that $0 < q \leq \frac{2}{\epsilon'}$. This shows that there are only a finite number of positive integers q that satisfy this inequality. Moreover, $g(0) = g(1) > \frac{\epsilon'}{2}$. Let's denote these rationals by $\{r_0, r_1, \cdots, r_n\}$ where $r_0 = 0$ and $r_n = 1$.
(b) The terms of $U(g, Q)$ consists of two types of intervals: The intervals $[x_{i-1}, x_i]$ not containing r_is and in this case we have $M_i(g)(x_i - x_{i-1}) \leq \frac{\epsilon'}{2}(x_i - x_{i-1})$ and there are n such intervals. The second type consists of those intervals containing r_is and in this case $M_i(g)(x_i - x_{i-1}) < \frac{\epsilon'}{2(n+1)}$ and there are $n + 1$ such intervals. Thus,

$$\begin{aligned}
U(g, Q) =& \sum_{i=0}^{n} M_i(g)(x_{2i+1} - x_{2i}) + \sum_{i=1}^{n} M_i(g)(x_{2i} - x_{2i-1}) \\
<& \sum_{i=0}^{n} (x_{2i+1} - x_{2i}) + \frac{\epsilon'}{2} \sum_{i=1}^{n} (x_{2i} - x_{2i-1}) \\
<& (n + 1)\frac{\epsilon'}{2(n + 1)} + \frac{\epsilon'}{2} = \epsilon' \leq \epsilon.
\end{aligned}$$

(c) By Exercise 2.6, we have $m_i(g) = 0$ for all i. Hence, $L(g, Q) = 0$. Therefore, $U(g, Q) - L(g, Q) < \epsilon$.
(d) By Exercise 20.7 and (c) , we conslude that g is Riemann integrable.
(e) Since $L(g, P) = 0$ for all partitions P of $[0, 1]$ we find $\underline{\int_0^1} g(x)dx = 0 = \int_0^1 g(x)dx$ ■

Exercise 24.7
Consider the functions f and g introduced in the previous two exercises. Let $h(x) = (f \circ g)(x)$.
(a) Write explicitly the formula of $h(x)$ as a piecewise defined function.
(b) Show that h is not Riemann integrable on $[0, 1]$.

Solution.
(a) We have

$$h(x) = \begin{cases} 1 & \text{if } 0 < g(x) \leq 1 \\ 0 & \text{if } g(x) = 0 \end{cases} = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

(b) This follows from Exercise 20.5 ■

Exercise 24.8
Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable.
(a) Show that $\max\{f(x), g(x)\} = \frac{|f(x)-g(x)|+f(x)+g(x)}{2}$.
(b) Show that the function $\max\{f(x), g(x)\}$ is Riemann integrable.

Solution.
(a) If $f(x) \geq g(x)$ then $\max\{f(x), g(x)\} = f(x)$ and $\frac{|f(x)-g(x)|+f(x)+g(x)}{2} = \frac{f(x)-g(x)+f(x)+g(x)}{2} = f(x)$. Similar argument when $f(x) < g(x)$.
(b) All functions $|f - g|$, f, and g are Riemann integrable so the combination $\max\{f(x), g(x)\}$ is also Riemann integrable ■

Exercise 24.9
Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable.
(a) Show that $\min\{f(x), g(x)\} = \frac{f(x)+g(x)-|f(x)-g(x)|}{2}$.
(b) Show that the function $\min\{f(x), g(x)\}$ is Riemann integrable.

Solution.
Similar to the previous exercise ■

Solutions to Section 25

Exercise 25.1
Let f and F as defined in Definition 24. Let M be such that $|f(x)| \leq M$ for all $x \in [a, b]$. Fix c in $[a, b]$.
(a) Show that for any $x \in [a, b]$ we have

$$-M(x-c) \leq \int_c^x f(t)dt \leq M(x-c).$$

Hence, we can write

$$\left|\int_c^x f(t)dt\right| \leq M|x-c|.$$

Hint: Exercise 23.2.
(b) Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{M}$. Show that for any $x \in [a, b]$ such that $|x-c| < \delta$ we must have $|F(x) - F(c)| < \epsilon$. Hence, F is continuous at c. Since c was arbitrary in $[a, b]$, we conclude that F is continuous on $[a, b]$.

Solution.
(a) Since $-M \leq f(t) \leq M$ for all $t \in [a, b]$ we can use Exercise 23.2 to obtain

$$-M\int_c^x dt \leq \int_c^x f(t)dt \leq M\int_c^x dt$$

which implies

$$-M(x-c) \leq \int_c^x f(t)dt \leq M(x-c)$$

or

$$\left|\int_c^x f(t)dt\right| \leq M|x-c|.$$

(b) We have

$$\begin{aligned}
|F(x) - F(c)| =& \left|\int_a^x f(t)dt - \int_a^c f(t)dt\right| \\
=& \left|\int_c^a f(t)dt + \int_a^x f(t)dt\right| \\
=& \left|\int_c^x f(t)dt\right| \leq M|x-c| \\
<& M\delta = \epsilon
\end{aligned}$$

Hence, F is continuous at c. Since c was arbitrary in $[a, b]$, we conclude that F is continuous on $[a, b]$ ■

Exercise 25.2
Let f and F as above. Suppose furthermore that f is continuous at $c \in [a, b]$.
(a) Show that

$$\frac{F(c+h) - F(c)}{h} - f(c) = \frac{1}{h}\int_c^{c+h}[f(t) - f(c)]dt.$$

(b) Show that $F'(c)$ exists and is equal to $f(c)$.

Solution.
(a) We have

$$\begin{aligned}\frac{F(c+h) - F(c)}{h} - f(c) =& \frac{1}{h}\int_c^{c+h} f(t)dt - \frac{1}{h}\int_c^{c+h} dt\\ =& \frac{1}{h}\int_c^{c+h}[f(t) - f(c)]dt.\end{aligned}$$

(b) Let $\epsilon > 0$. By the continuity of f at c we can find a $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$. Choose h such that $|h| < \delta$. Then for any t between c and $c + h$ we have $|t - c| < \delta$ and therefore $|f(t) - f(c)| < \epsilon$. Thus,

$$\begin{aligned}\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| = \frac{1}{|h|}\left|\int_c^{c+h}[f(t) - f(c)]dt\right|&\\ <\frac{1}{|h|}\epsilon|h| = \epsilon.&\end{aligned}$$

It follows that $F'(c)$ exists and is equal to $f(c)$ ■

Exercise 25.3
Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and f' continuous on $[a, b]$.
(a) Show that f' is Riemann integrable on $[a, b]$.
(b) Define $F : [a, b] \to \mathbb{R}$ by $F(x) = \int_a^x f'(t)dt$. Show that $F'(x) = f'(x)$ for all $x \in [a, b]$.
(c) Show that $F(x) = f(x) - f(a)$ for all $x \in [a, b]$.
(d) Use (c) to show that

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Solution.
(a) This follows from Exercise 21.2.
(b) This follows from Exercise 25.2.
(c) From Exercise 18.9 we have $F(x) = f(x) + C$ for all $x \in [a, b]$, where C is a constant. Letting $x = a$ we find $0 = F(a) = f(a) + C$ or $C = -f(a)$. Thus, $F(x) = f(x) - f(a)$.
(d) The result follows by letting $x = b$ in (c) ■

Exercise 25.4
Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $g : [c, d] \to [a, b]$ is differentiable on $[a, b]$. Define $F : [c, d] \to \mathbb{R}$ by

$$F(x) = \int_a^{g(x)} f(t)dt.$$

(a) Show that f is Riemann integrable on $[a, b]$.
(b) Define $G : [a, b] \to \mathbb{R}$ by $G(x) = \int_a^x f(t)dt$. Show that G is differentiable and $G'(x) = f(x)$ for all $x \in [a, b]$.
(c) Write F in terms of G and g. Show that F is differentiable on $[c, d]$ with

$$F'(x) = f(g(x)) \cdot g'(x).$$

Solution.
(a) This follows from Exercise 21.2.
(b) This follows from Exercise 25.2.
(c) We have $F(x) = (G \circ g)(x)$. Since both G and g are differentiable, by Exercise 16.10, we find that F is differentiable with derivative

$$F'(x) = G'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x) \blacksquare$$

Exercise 25.5 *(Mean Value Theorem for Integrals)*
Let $f : [a, b] \to \mathbb{R}$ be continuous.
(a) Show that f is Riemann integrable on $[a, b]$.
(b) Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)dt.$$

Show that F is differentiable with $F'(x) = f(x)$.
(c) Show that there is $a < c < b$ such that $F(b) - F(a) = F'(c)(b - a)$.
(d) Use (c) to show that

$$\int_a^b f(x)dx = f(c)(b - a).$$

Solution.
(a) This follows from Exercise 21.2.
(b) This follows from Exercise 25.2.
(c) This follows from the Mean Value Theorem (Exercise 18.1).
(d) Since $F(a) = 0$, we have $F(b) = F(b) - F(a) = \int_a^b f(x)dx$. Also, $F'(c) = f(c)$. Hence,

$$\int_a^b f(x)dx = f(c)(b-a) \blacksquare$$

Exercise 25.6 *(Change of Variables Formula)*
Let $\phi : [a, b] \to [c, d]$ be differentiable with continuous derivative and such that $\phi(a) = c, \phi(b) = d$. Let $f : [c, d] \to \mathbb{R}$ be continuous.
(a) Show that the functions f and $(f \circ \phi) \cdot \phi'$ are Riemann integrable.
(b) Define $F(x) = \int_c^x f(t)dt$. Show that F is differentiable with $F'(x) = f(x)$ for all $x \in [c, d]$.
(c) Define $G(x) = \int_a^x f(\phi(t))\phi'(t)dt$. Show that G is differentiable with $G'(x) = f(\phi(x))\phi'(x)$ for all $x \in [a, b]$.
(d) Show that $F \circ \phi$ is differentiable on $[a, b]$ with $(F \circ \phi)'(x) = G'(x)$ for all $x \in [a, b]$. Hint:Exercise **??**.
(e) Use (d) and Exercise 18.9 to show that $(F \circ \phi)(x) = G(x)$ for all $x \in [a, b]$.
(f) Use (e) to show that

$$\int_a^b f(\phi(x))\phi'(x)dx = \int_c^d f(x)dx.$$

Solution.
(a) Since f, ϕ, and ϕ' are continuous, they are Riemann integrable by Exercise 21.2. By Exercise 24.1, $f \circ \phi$ is Riemann integrable. Also, $f \circ \phi$ is continuous being the composition of two continuous functions. Since $f \circ \phi : [a, b] \to \mathbb{R}$, by Exercise 15.3 , $f \circ \phi$ is bounded. Since $\phi' : [a, b] \to \mathbb{R}$, ϕ' is also bounded. Thus, by Exercise 24.2, the product $(f \circ \phi) \cdot \phi'$ is integrable.
(b) Since f is continuous, from Exercise 25.2 we conclude that F is differentiable with $F'(x) = f(x)$ for all $x \in [c, d]$.
(c) Since $(f \circ \phi) \cdot \phi'$ is continuous, by Exercise 25.2, G is differentiable with $G'(x) = f(\phi(x))\phi'(x)$ for all $x \in [a, b]$.
(d) Since both F and ϕ are differentiable, the composition $F \circ \phi$ is also differentiable (Exercise 18.1) with derivative

$$(F \circ \phi)'(x) = F'(\phi(x)) \cdot \phi'(x) = f(\phi(x)) \cdot \phi'(x) = G'(x).$$

(e) Using (d) and Exercise 10.1 we can write $(F \circ \phi)(x) = G(x) + C$ for all $x \in [a, b]$, where C is a constant. In particular, letting $x = c$ we find $0 = F(c) = (F \circ \phi)(a) = G(a) + C = C$. Thus, $(F \circ \phi)(x) = G(x)$ for all $x \in [a, b]$.
(f) This follows by letting $x = b$ in (e) ■

Exercise 25.7
Find the derivative of

$$F(x) = \int_1^{\sqrt{x}} \cos(t^2) dt.$$

Solution.
From Exercise 25.4 with $g(x) = \sqrt{x}$ and $f(x) = \cos x^2$ we find

$$F'(x) = f(g(x)) \cdot g'(x) = \frac{\cos x}{2\sqrt{x}} \blacksquare$$

Exercise 25.8 *(Mean Value Theorem for Monotone Functions)*
Let $f : [a, b] \to \mathbb{R}$ be increasing on $[a, b]$.
(a) Show that f is Riemann integrable on $[a, b]$.
(b) Define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(a)(x - a) + f(b)(b - x)$. Show that g is continuous on $[a, b]$.
(c) Show that $g(b) \leq \int_a^b f(x)dx \leq g(a)$.
(d) Show that there is $c \in [a, b]$ such that

$$\int_a^b f(x)dx = f(a)(c - a) + f(b)(c - b).$$

Solution.
(a) This follows from Exercise 21.1.
(b) This follows from the fact that g is a combination of continuous functions.
(c) Since $a \leq x \leq b$ and f is increasing, we find $f(a) \leq f(x) \leq f(b)$. Integrate each side from a to b we find

$$f(a)(b - a) \leq \int_a^b f(x)dx \leq f(b)(b - a)$$

or

$$g(b) \leq \int_a^b f(x)dx \leq g(a).$$

(d) By the Intermediate value Theorem we can find a $c \in [a, b]$ such that

$$g(c) = \int_a^b f(x)dx$$

or

$$\int_a^b f(x)dx = f(a)(c-a) + f(b)(c-b) \blacksquare$$

Exercise 25.9
Use change of variables to evaluate $\int_1^3 (3x+1)^{100} dx$.

Solution.
Let $f(x) = x^{100}$ and $\phi(x) = 3x + 1$. Then

$$\int_1^3 (3x+1)^{100} dx = 3\int_4^{10} \phi^{100} d\phi = \left.\frac{3}{101}\phi^{101}\right|_3^{10} \blacksquare$$

Exercise 25.10
Find the smallest positive critical point of

$$F(x) = \int_0^x \cos{(t^{\frac{3}{2}})} dt.$$

Solution.
We have $F'(x) = \cos{(x^{\frac{3}{2}})}$, so the smallest positive critical point is $x = \left(\frac{\pi}{2}\right)^{\frac{2}{3}}$ $\blacksquare$

Exercise 25.11
Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$. Find

$$\lim_{x \to a} \frac{1}{x-a} \int_a^x f(t)dt.$$

Solution.
Let $F(x) = \int_a^x f(t)dt$. Then F is differentiable at a with $F'(a) = f(a)$. Thus,

$$\lim_{x \to a} \frac{1}{x-a} \int_a^x f(t)dt = \lim_{x \to a} \frac{F(x) - F(a)}{x-a} = F'(a) = f(a) \blacksquare$$

Exercise 25.12
Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and $A, B:\mathbb{R}\to\mathbb{R}$ be differentiable functions. Define $g:\mathbb{R}\to\mathbb{R}$ by

$$g(x)=\int_{A(x)}^{B(x)} f(t)dt.$$

Prove that g is differentiable and find a formula for $g'(x)$.

Solution.
Define $F(x)=\int_0^x f(t)dt$. Since f is continuous, the First Fundamental Theorem of Calculus shows that F is differentiable everywhere and $F'(x)=f(x)$. Note that

$$g(x)=\int_0^{B(x)} f(t)dt-\int_0^{A(x)} f(t)dt=F(B(x))-F(A(x)).$$

Since A, B, F are differentiable, so is g. By the chain rule,

$$g'(x)=F'(B(x))B'(x)-F'(A(x))A'(x)=f(B(x))B'(x)'f(A(x))A'(x) \blacksquare$$

Exercise 25.13
Suppose $f:\mathbb{R}\to\mathbb{R}$ is continuous at 2 and $f(2)=4$. Find

$$\lim_{x\to 2}\frac{1}{x-2}\int_2^x xf(t)dt.$$

Solution.
We have

$$\lim_{x\to 2}\frac{1}{x-2}\int_2^x xf(t)dt=\lim_{x\to 2} x\cdot\frac{1}{x-2}\int_2^x f(t)dt=2f(2)=8 \blacksquare$$

Exercise 25.14
Use a definite integral to define a function $F(x)$ having derivative $\frac{\cos 2x^3}{\sqrt{1+x^4}}$ for all x and satisfying $F(\sqrt[3]{2})=0$.

Solution.
The answer is

$$F(x)=\int_{\sqrt[3]{2}}^x \frac{\cos 2t^3}{\sqrt{1+t^4}}dt \blacksquare$$

Solutions to Section 26

Exercise 26.1
Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1. Hint: Show that for each $n \geq 1$ we have $S_n = 1 - \frac{1}{n+1}$.

Solution.
Using partial fractions we can write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Thus,

$$\begin{array}{rcl} S_1 & = & 1 - \frac{1}{2} \\ S_2 & = & (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) = 1 - \frac{1}{3} \\ S_3 & = & S_2 + (\frac{1}{3} - \frac{1}{4}) = (1 - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) = 1 - \frac{1}{4} \\ \vdots & & \\ S_n & = & 1 - \frac{1}{n+1}. \end{array}$$

It follows that $\lim_{n\to\infty} S_n = 1$.■

Exercise 26.2
Is the series $\Sigma_{n=1}^{\infty}(-1)^n$ convergent or divergent?

Solution.
The series $\Sigma_{n=1}^{\infty}(-1)^n$ diverges since the sequence of partial sums alternates between the values -1 and 0. ■

Exercise 26.3
Suppose that $\sum_{i=1}^{\infty} a_n = L$. Show that $\lim_{n\to\infty} a_n = 0$. Hint: Note that $S_{n+1} - S_n = a_n$.

Solution.
We know that $S_n = a_1 + a_2 + \cdots + a_n$ and $S_{n+1} = a_1 + a_2 + \cdots + a_n + a_{n+1} = S_n + a_n$ so it follows that $S_{n+1} - S_n = a_n$. Suppose that the series converges to a number L. Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n+1} = L$. Thus, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_{n+1} - S_n) = L - L = 0$. ■

Exercise 26.4
Consider the series $\sum_{i=1}^{n} \log\left(\frac{n+1}{n}\right)$.
(a) Show that $\lim_{n\to\infty} a_n = 0$.
(b) Show that $\lim_{n\to\infty} S_n = \infty$. Hence, the series is divergent.

Solution.
(a) We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \log\left(1+\frac{1}{n}\right) = \log 1 = 0.$
(b) We have that $S_n = \log(n+1)$. Hence, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \log(n+1) = \infty$ so that the given series is divergent ■

Exercise 26.5
Consider the sequence $\{r^n\}_{n=1}^{\infty}$.
(a) Show that if $r = -1$ the sequence is divergent.
(b) Show that if $|r| > 1$, i.e. $r < -1$ or $r > 1$, the sequence is divergent.
(c) Show that if $|r| < 1$, the sequence is convergent.

Solution.
(a) This follows from Exercise .
(b) $|r| > 1$ implies $r > 1$ or $r < -1$. Suppose first that $r > 1$. Let $\epsilon > 0$. Let N be a positive integer greater than $\frac{\epsilon}{r-1}$. Then for $n \geq N$ we have

$$\begin{array}{rcl} r^n &=& (1+(r-1))^n \\ &\geq& 1+n(r-1)(by\ the\ binomial\ formula) \\ &>& 1+N(r-1) \\ &>& 1+\epsilon \\ &>& \epsilon. \end{array}$$

This shows that for any given positive number we can find a term in the sequence $\{r^n\}_{n=1}^{\infty}$ which is greater than the number. This means that $r^n \to \infty$ as $n \to \infty$.
If $r < -1$ then $r^n = (-1)^n(-r)^n$ with $-r > 1$. Thus, as n becomes large, r^n alternates between large positive numbers and negative numbers with large absolute value so that again the limit r^n as $n \to \infty$ does not exist.
(c) If $0 < r < 1$ then $r^n = \frac{1}{(r^{-1})^n}$ with $(r^{-1})^n \to \infty$ as $n \to \infty$. (See (b)). Hence, $r^n \to 0$ as $n \to \infty$. If $-1 < r < 0$ then $0 < -r < 1$. In this case, $r^n = (-1)^n(-r)^n \to 0$ as $n \to \infty$. If $r = 0$ then $r^n = 0$ and $\lim_{n\to\infty} r^n = 0$ ■

Exercise 26.6
The series $\sum_{n=1}^{\infty} ar^{n-1}$ is called a **geometric series** with ratio r.
(a) Show that

$$S_n = a\frac{1-r^n}{1-r} \text{ for } r \neq 1.$$

Hint: Calculate $S_n - rS_n$.
(b) Show that the series converges to $\frac{a}{1-r}$ for $|r| < 1$ and diverges for $|r| \geq 1$.

Solution.
(a) We have $S_n - rS_n = a - a^{r^n}$ so that $S_n = a\frac{1-r^n}{1-r}$ for $r \neq 1$.
(b) If $|r| < 1$, using Exercise 26.5, we find $\lim_{n\to\infty} S_n = \frac{a}{1-r}$. If $r = 1$ then $S_n = na$ and this series diverges to either ∞ (if $a > 0$) or $-\infty$ (if $a < 0$.)
If $r = -1$ the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent and therefore the sequence $\{S_n\}_{n=1}^{\infty}$ is divergent.
The same applies if $|r| > 1$ ■

Exercise 26.7 *(Harmonic Series)*
Consider the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.
(a) Let $n = 2^m$ where m is a positive integer. Then

$$\begin{aligned}S_n =&1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^m}\\ =&1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)\\ &+ \cdots + \left(\frac{1}{2^{m-1}+1} + \cdots + \frac{1}{2^m}\right)\end{aligned}$$

Show that $S_n \geq 1 + \frac{m}{2}$.
(b) Use (a) to show that $\lim_{n\to\infty} S_n = \infty$. Thus, the Harmonic series is divergent.

Solution.
(a) We have $S_n \geq 1+\frac{1}{2}+2\cdot\frac{1}{4}+4\cdot\frac{1}{8}+\cdots+2^{m-1}\cdot\frac{1}{2^m} = 1+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2} = 1+\frac{m}{2}$.
(b) If n goes to ∞ then m goes to ∞ and therefore $\lim_{m\to\infty}\left(1 + \frac{m}{2}\right) = \infty$. By (a), we conclude that $\lim_{n\to\infty} S_n = \infty$ and therefore the Harmonic series is divergent ■

Exercise 26.8
Show that if $\sum_{n=1}^{\infty} a_n = L_1$ and $\sum_{n=1}^{\infty} b_n = L_2$ then $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha L_1 + \beta L_2$ for all $\alpha, \beta \in \mathbb{R}$.

Solution.
Let $S_n^2 = a_1+a_2+\cdots+a_n$ and $S_n^2 = b_1+b_2+\cdots+b_n$. Then $\lim_{\,n} \to \infty(\alpha S_n^1 + \beta S_n^2) = \alpha L_1 + \beta L_2$ ■

Exercise 26.9
Find the value of the infinite sum $\sum_{n=1}^{\infty}\left(\frac{3}{4^n} + \frac{5}{n(n+1)}\right)$.

Solution.
We have $\sum_{n=1}^{\infty}\frac{3}{4^n} = \frac{3}{4}\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1} = \frac{3}{4}\cdot\frac{1}{1-\frac{1}{4}} = 1$ and $\sum_{n=1}^{\infty}\frac{5}{n(n+1)} = 5$. Thus, $\sum_{n=1}^{\infty}\left(\frac{3}{4^n}+\frac{5}{n(n+1)}\right) = 6$ ■

Exercise 26.10
Show that the sequence $\{\sqrt{n^2-1}-n\}_{n=1}^{\infty}$ is convergent and find its limit.

Solution.
We have

$$\begin{aligned}\lim_{n\to\infty}(\sqrt{n^2-1}-n) =& \lim_{n\to\infty}\frac{(\sqrt{n^2-1}-n)(\sqrt{n^2-1}+n)}{(\sqrt{n^2-1}+n)}\\ =& \lim_{n\to\infty}\frac{-1}{\sqrt{n^2-1}+n} = 0 \blacksquare\end{aligned}$$

Exercise 26.11
Let $\sum_{n=1}^{\infty}a_n$ be a conditionally convergent series. Define $b_n = \frac{1}{2}(a_n+|a_n|)$ and $c_n = \frac{1}{2}(a_n-|a_n|)$. Prove that the two series $\sum_{n=1}^{\infty}$ and $\sum_{n=1}^{\infty}$ are divergent.

Solution.
Note first that $|a_n| = b_n - c_n$ for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty}|a_n|$ is divergent, either $\sum_{n=1}^{\infty}b_n$ is divergent or $\sum_{n=1}^{\infty}|a_n|$. Suppose that $\sum_{n=1}^{\infty}b_n$ is convergent and $\sum_{n=1}^{\infty}c_n$ is divergent. Since $b_n = |a_n| + c_n$ we must have that $\sum_{n=1}^{\infty}b_n$ is divergent which is a contradiction. A similar contradiction is obtained if $\sum_{n=1}^{\infty}b_n$ is divergent and $\sum_{n=1}^{\infty}c_n$ is convergent. It follows that both $\sum_{n=1}^{\infty}b_n$ and $\sum_{n=1}^{\infty}c_n$ must be divergent ■

Exercise 26.12
Let S_n be the n-th partial sum of the series $\sum_{n=1}^{\infty}\frac{n-2}{n(n+1)(n+2)}$.
(a) Show that $S_n = \frac{3}{n+1} - \frac{2}{n+1} - \frac{2}{n+2}$. Hint: Partial fractions.
(b) Find the value of the series $\sum_{n=1}^{\infty}\frac{n-2}{n(n+1)(n+2)}$.

Solution.
(a) Using partial fraction decomposition we find that

$$\frac{n-2}{n(n+1)(n+2)} = \frac{3}{n+1} - \frac{2}{n+2} - \frac{1}{n}.$$

Thus,

$$S_n = \frac{3}{2} - 3\sum_{i=3}^{n}\frac{1}{i} - 2\sum_{i=3}^{n}\frac{1}{i} - \sum_{i=3}^{n}\frac{1}{i} - 1 - \frac{1}{2}\frac{3}{n+1} - \frac{2}{n+1} - \frac{2}{n+2}$$
$$= \frac{3}{n+1} - \frac{2}{n+1} - \frac{2}{n+2}$$

Thus, the series is convergent with sum

$$\sum_{n=1}^{\infty}\frac{n-2}{n(n+1)(n+2)} = \lim_{n\to\infty} S_n = 0 \blacksquare$$

Exercise 26.13
Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence such that $\sum_{n=1}^{\infty} a_n$ is convergent.
(a) Show that $a_n \geq 0$ for all $n \in \mathbb{N}$.
(b) Let $\epsilon > 0$. Show that there is a positive integer N such that if $n > m \geq N$ we have

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

(c) Show that $(n-N)a_n < \epsilon$.
(d) Let $n > 2N$. Show that $\frac{n}{2} < n - N$.
(e) Show that $\frac{na_n}{2} < \epsilon$.
(f) Show that $\lim_{n\to\infty} na_n = 0$.

Solution.
(a) If $a_N < 0$ for some N then $a_n \leq a_N < 0$ for all $n \geq N$. But then $\lim_{n\to\infty} a_n \neq 0$ which contradicts the fact that $\sum_{n=1}^{\infty} a_n$ is convergent.
(b) This follows from the fact that the sequence of partial sums is Cauchy.
(c) From (b), we have $a_{N+1}+a_{N+2}+\cdots+a_n < \epsilon$. But $a_n \leq \cdots \leq a_{N+2} \leq a_{N+1}$. Thus, $(n-N)a_n < \epsilon$.
(d) We have $n > 2N \to -\frac{n}{2} < -N \to \frac{n}{2} < n - N$.
(e) If $n > 2N > N$ we have $\frac{na_n}{2} < (n-N)a_n < \epsilon$.
(f) It follows from (e) that $\lim_{n\to\infty} \frac{na_n}{2} = 0$ and therefore $\lim_{n\to\infty} na_n = 2\lim_{n\to\infty} \frac{na_n}{2} = 0 \blacksquare$

Exercise 26.14
Let N be a positive integer. Suppose that $a_n = b_n$ for all $n \geq N$. Then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge. Thus, changing a finite number of terms in a series does not change whether or not it converges, although it may change the value of its sum if it does converge.

solution.
The proof follows from the equality $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1}(a_n - b_n) + \sum_{n=1}^{\infty} b_n$.■

Solutions to Section 27

Exercise 27.1 *(Comparison test))*
Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two series such that $0 \leq a_n \leq b_n$ for all $n \geq 1$. Let $\{S_n\}_{n=1}^{\infty}$ be the sequence of partial sums of $\{a_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ that of $\{b_n\}_{n=1}^{\infty}$.
(a) Show that the sequences $\{S_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ are increasing.
(b) Show that $S_n \leq T_n$ for all $n \geq 1$.
(c) Show that if $\{b_n\}_{n=1}^{\infty}$ is convergent then $\{S_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ are bounded.
(d) Show that if $\{b_n\}_{n=1}^{\infty}$ is convergent then $\{a_n\}_{n=1}^{\infty}$ is also convergent.
(e) Show that if $\{a_n\}_{n=1}^{\infty}$ is divergent then $\{b_n\}_{n=1}^{\infty}$ is also divergent.

Solution.
(a) $S_{n+1} - S_n = a_{n+1} \geq 0$ so that $S_{n+1} \geq S_n$. Thus, $\{S_n\}_{n=1}^{\infty}$ is increasing. Likewise, $\{T_n\}_{n=1}^{\infty}$ is increasing.
(b) Since $a_n \leq b_n$ for all $n \geq 1$, we must have $S_n \leq T_n$ for all $n \geq 1$.
(c) If $\{b_n\}_{n=1}^{\infty}$ is convergent then the sequence $\{T_n\}_{n=1}^{\infty}$ is also convergent. By Exercise 3.9, the sequence $\{T_n\}_{n=1}^{\infty}$ is bounded say by M. Since $S_n \leq T_n \leq M$ for all $n \geq 1$, the sequence $\{S_n\}_{n=1}^{\infty}$ is also bounded.
(d) Since $\{S_n\}_{n=1}^{\infty}$ is increasing and bounded from above, it must be convergent according to Exercise 5.5.
(e) This is just the contrapositive of (d) ■

Exercise 27.2
(a) Show that for $n \geq 1$ we have $\frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}$.
(b) Show that the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent.

Solution.
(a) Since $1 \geq 0$ we have $n+1 \geq n \to \frac{1}{n+1} \leq \frac{1}{n} \to \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}$.
(b) Since the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent so does the given series ■

Exercise 27.3
Show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-n+1}}$ is divergent.

Solution.
Indeed, for $n \geq 1$ we have $n^2 + (1-n) \leq n^2$ so that $\sqrt{n^2-n+1} \leq \sqrt{n^2} = n$. This implies that $0 \leq \frac{1}{n} \leq \frac{1}{\sqrt{n^2-n+1}}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (harmonic series), the comparison test asserts that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-n+1}}$ is divergent.■

Exercise 27.4 *(Limit Comparison Test)*
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with positive terms. Suppose that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$$

(a) Let $\epsilon = \frac{L}{2}$. Show that there exists a positive integer N such that

$$\left|\frac{a_n}{b_n} - L\right| < \frac{L}{2} \text{ for all } n \geq N.$$

(b) Use (a) to establish

$$\frac{L}{2}b_n < a_n < \frac{3}{2}Lb_n \text{ for all } n \geq N.$$

(c) Show that $\sum_{n=1}^{\infty} a_n$ is divergent if and only if $\sum_{n=1}^{\infty} b_n$ is divergent.

Solution.
(a) This follows from the definition of convergence of a sequence.
(b) The inequality in (a) is equivalent to

$$L - \frac{L}{2} < \frac{a_n}{b_n} < L + \frac{L}{2}$$

or

$$\frac{L}{2} < \frac{a_n}{b_n} < \frac{3}{2}L.$$

or

$$\frac{L}{2}b_n < a_n < \frac{3}{2}Lb_n, \quad n \geq N.$$

(c) If the series $\sum_{n=1}^{\infty} a_n$ converges then by the comparison test the series $\sum_{n=1}^{\infty} \frac{L}{2}b_n$ is convergent. By Exercise 26.8, the series $\sum_{n=1}^{\infty} b_n$ is also convergent.
Conversely, if $\sum_{n=1}^{\infty} b_n$ is convergent then $\sum_{n=1}^{\infty} \frac{3}{2}Lb_n$ is convergent and by the comparison test $\sum_{n=1}^{\infty} a_n$ is convergent. Similarly, $\sum_{n=1}^{\infty} a_n$ is divergent if and only if $\sum_{n=1}^{\infty} b_n$ is divergent.■

Exercise 27.5
Determine whether the series $\sum_{n=1}^{\infty} \frac{3n+1}{4n^3+n^2-2}$ converges or diverges.

Solution.
For large n we have $a_n = \frac{3n+1}{4n^3+n^2-2} \approx \frac{3n}{4n^3} = \frac{3}{4n^2}$. So let $b_n = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^3+n^2}{4n^3+n^2-2} = \frac{3}{4}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent then so does the series $\sum_{n=1}^{\infty} \frac{3n+1}{4n^3+n^2-2}$.
■

Exercise 27.6
Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of nonnegative terms. Show that if the series $\sum_{n=1}^{\infty} a_n$ is divergent so does the series $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$. Hint: Comparison test.

Solution.
Since the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded there is $M > 0$ such that $a_n \leq M$ for all $n \in \mathbb{N}$. Thus,

$$\frac{a_n}{1+a_n} \geq \frac{1}{1+M} a_n.$$

But the series $\sum_{n=1}^{\infty} a_n$ is divergent so does the series $\frac{1}{1+M}\sum_{n=1}^{\infty} a_n$. Hence, by the comparison test, the series $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ is divergent ■

Exercise 27.7
Use the limit comparison test to show that the series $\sum_{i=1}^{\infty} \frac{1}{2n+\ln n}$ is divergent.

Solution.
We have

$$\lim_{n\to\infty} \frac{\frac{1}{2n+\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+\ln n} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2} < 1$$

Since the Harmonic series is divergent, the given series is also divergent ■

Exercise 27.8
Suppose that $a_n \geq 0$ and that the series $\sum_{n=1}^{\infty} a_n$ diverges. Suppose that $\{a_n\}_{n=1}^{\infty}$ is unbounded. Show that $\lim_{n\to\infty} \frac{a_n}{1+a_n} \neq 0$. Hint: assume the contrary and get a contradction. Conclude that the series $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ is divergent.

Solution.
If $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0$ then we can find a positive integer N such that for $n \geq N$ we have $\frac{a_n}{1+a_n} < \frac{1}{2}$ which implies that $a_n < 1$ for all $n \geq N$. Let $M = a_1 + a_2 + \cdots + a_{N-1} + 1$. Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. This shows that the sequence is bounded contradicting the assumption that it is unbounded. Thus, $\lim_{n\to\infty} \frac{a_n}{1+a_n} \neq 0$ and the given series is divergent ■

Exercise 27.9
Suppose that $a_n \geq 0$ for all $n \in \mathbb{N}$ and that the series $\sum_{n=1}^{\infty} a_n$ converges.
(a) Show that there is a positive integer N such that $a_n < 1$ for all $n \geq N$.
(b) Show that the series $\sum_{n=1}^{\infty} a_n^2$ converges.

Solution.
(a) Since the series is convergent, we have $\lim_{n\to\infty} a_n = 0$. Thus, we can find a positive integer N such that $a_n < 1$ for all $n \geq N$.
(b) For $n \geq N$ we have $a_n^2 < a_n$. By the comparison test the series $\sum_{n=1}^{\infty} a_n^2$ converges ■

Exercise 27.10
Use comparison test to show that the series $\sum_{n=1}^{\infty}(\sqrt{n^2+1} - n)$ is divergent.

Solution.
We have $\sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n} > \frac{1}{1+\sqrt{2}} \frac{1}{n}$. The Harmonic series is divergent so that by the comparison test the given series is divergent ■

Solutions to Section 28

Exercise 28.1 *(Alternating Series Test)*
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that
(i) $a_n \geq a_{n+1}$, that is the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing.
(ii) $\lim_{n\to\infty} a_n = 0$.
Let $\{S_n\}_{n=1}^{\infty}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$. That is, $S_n = \sum_{k=1}^{n}(-1)^{k-1}a_k$.
(a) Show that for each $n \geq 1$ we have $S_{2n} \leq S_{2n+2}$. That is, the sequence $\{S_{2n}\}_{n=1}^{\infty}$ is increasing. Hint: Show that $S_{2n+2} - S_{2n} \geq 0$.
(b) Show that the sequence $\{S_{2n+1}\}_{n=1}^{\infty}$ is decreasing.
(c) Show that for all $n \geq 1$, we have $S_{2n} \leq a_1$. Hence, the sequence $\{S_{2n}\}_{n=1}^{\infty}$ is bounded from above. Conclude that the sequence $\{S_{2n}\}_{n=1}^{\infty}$ is convergent, say to L_1.
(d) Show that for all $n \geq 1$, we have $S_{2n+1} \geq (a_1 - a_2)$. Hence, the sequence $\{S_{2n+1}\}_{n=1}^{\infty}$ is bounded from below. Conclude that the sequence $\{S_{2n+1}\}_{n=1}^{\infty}$ is convergent, say to L_2.
(e) Show that $L_1 = L_2$. Hint: $S_{2n+1} = S_{2n} + a_{2n+1}$.
(f) Let $L = L_1 = L_2$. Show that $\lim_{n\to\infty} S_n = L$. We conclude that the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent. Hint: Look at the sequence $\{c_n\}_{n=1}^{\infty}$ in Exercise 10.4.

Solution.
(a) We have $S_{2n+2} - S_{2n} = a_{2n+1} - a_{2n+2} \geq 0$. Thus, the sequence $\{S_{2n}\}_{n=1}^{\infty}$ is increasing.
(b) We have $S_{2n+1} - S_{2n-1} = a_{2n+1} - a_{2n} \leq 0$. Thus, the sequence $\{S_{2n+1}\}_{n=1}^{\infty}$ is decreasing.
(c) We have

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1.$$

By Exercise 5.5, the sequence $\{S_{2n}\}_{n=1}^{\infty}$ is convergent, say to L_1.
(d) We have

$$S_{2n+1} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) + a_{2n+1} \geq (a_1 - a_2).$$

By Exercise **??**, the sequence $\{S_{2n+1}\}_{n=1}^{\infty}$ is convergent, say to L_2.
(e) Since $S_{2n+1} = S_{2n} + a_{2n+1}$, we have $L_2 = \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = L_1 + 0 = L_1$.
(f) Since $\{S_n\}_{n=1}^{\infty} = \{S_1, S_2, S_3, \cdots\}$ with $lim_{n\to\infty}S_{2n} = lim_{n\to\infty}S_{2n+1} = L$, by Exercise 10.4(c), $lim_{n\to\infty}S_n = L$ ■

Exercise 28.2
Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent.

Solution.
To see this, let $a_n = \frac{1}{n}$. Then $n < n+1$ implies that $\frac{1}{n+1} < \frac{1}{n}$ that is $a_{n+1} < a_n$. Also, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. Hence, by the previous result the given series is convergent.■

Exercise 28.3
Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)}$ is convergent.

Solution.
To see this, let $a_n = \frac{1}{n(n+1)}$. Now $n < n+2$ implies that $\frac{1}{n+2} < \frac{1}{n}$ and this implies that $\frac{1}{(n+1)(n+2)} < \frac{1}{n(n+1)}$ that is $a_{n+1} < a_n$. Also, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n(n+1)} = 0$. Hence, by the alternating series test the given series is convergent.■

Exercise 28.4
Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}$ converges or diverges.

Solution.
Since

$$\lim_{n\to\infty} (-1)^{n-1} \frac{n}{n+1} \neq 0$$

the series is divergent ■

Exercise 28.5
Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln(4n)}{n}$ converges or diverges.

Solution.
Let $f(x) = \frac{\ln(4x)}{x}$ for $x \geq 1$. Then $f'(x) = \frac{1-\ln(4x)}{x^2} < 0$ for $x \geq 1$. This shows that the sequence $\{\frac{\ln(4n)}{n}\}_{n=1}^{\infty}$ is decreasing. Moreover, we have by using L'Hôpital's rule

$$\lim_{n\to\infty} \frac{\ln(4n)}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Hence, the given series is convergent ■

Exercise 28.6
(a) Show that $\frac{n^n}{n!} \geq 1$ for all $n \geq 1$.
(b) Show that ther series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^n}{n!}$ is divergent.

Solution.
(a) We have

$$\frac{n^n}{n!} = \frac{n}{n}\frac{n}{n-1}\cdots\frac{n}{1} > 1.$$

(b) Since $\lim_{n\to\infty}(-1)^{n-1}\frac{n^n}{n!} \neq 0$, the given series is divergent ■

Exercise 28.7
Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^{n+1}+2^{n+1}}{3^n-n}$ diverges.

Solution.
We have

$$\begin{aligned}\lim_{n\to\infty}\frac{3^{n+1}+2^{n+1}}{3^n-n} &= \lim_{n\to\infty}\frac{3^n\left(3+2\left(\frac{2}{3}\right)^n\right)}{3^n\left(1-\frac{3^n}{n}\right)}\\ &= \lim_{n\to\infty}\frac{3+2\left(\frac{2}{3}\right)^n}{1-\frac{3^n}{n}}\\ &=\frac{3+0}{1-0} = 3\end{aligned}$$

Since $\lim_{n\to\infty}(-1)^{n-1}\frac{3^{n+1}+2^{n+1}}{3^n-n} \neq 0$ the given series is divergent ■

Solutions to Section 29

Exercise 29.1
Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(n+1)}$ is absolutely convergent.

Solution.
This follows from Exercise 26.1 ■

Exercise 29.2
Let $\sum_{n=1}^{\infty}a_n$ be an absolutely convergent series. Define the sequence $\sum_{n=1}^{\infty}b_n$ by $b_n = |a_n|$ and note that $a_n \leq b_n$. Show that the sequence $\sum_{n=1}^{\infty}a_n$ is convergent. That is, absolute convergence implies convergence.

Solution.
This is a direct consequence of the comparison test ■

Exercise 29.3
Give an example of a series that is convergent but not absolutely convergent.

Solution.
An example is the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(n+1)}$ ■

Exercise 29.4
Give an example of a series that is conditionally convergent.

Solution.
An example is the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ ■

Exercise 29.5
Suppose that $\sum_{n=1}^{\infty}a_n$ is absolutely convergent.
(a) Show that $0 \leq \frac{|a_n|+a_n}{2} \leq |a_n|$ and $0 \leq \frac{|a_n|-a_n}{2} \leq |a_n|$
(b) Show that the series $\sum_{n=1}^{\infty}\left(\frac{|a_n|+a_n}{2}\right)$ and $\sum_{n=1}^{\infty}\left(\frac{|a_n|-a_n}{2}\right)$ are convergent.

Solution.
(a) Since $-|a_n| \leq a_n$ we have $\frac{|a_n|+a_n}{2} \geq 0$. Since $a_n \leq |a_n|$, we have $a_n+|a_n| \leq 2|a_n|$ or $\frac{|a_n|+a_n}{2} \leq |a_n|$. Likewise, one can establish the second inequality.
(b) This follows from the comparison test and the fact that $\sum_{n=1}^{\infty}|a_n|$ is convergent ■

Exercise 29.6
(a) Show that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then the series $\sum_{n=1}^{\infty} a_n^2$ is also absolutely convergent.
(b) Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ for which $\sum_{n=1}^{\infty} a_n^2$ is divergent.

Solution.
(a) Since $\lim_{n\to\infty} |a_n| = 0$ we can find a positive integer N such that for $n \geq N$ we have $|a_n| < 1$. Thus, for $n \geq N$ we have $|a_n|^2 < |a_n|$. Thus, $\sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{N-1} |a_n|^2 + \sum_{n=N}^{\infty} |a_n|^2 < \sum_{n=1}^{N-1} |a_n|^2 + \sum_{n=N}^{\infty} |a_n|^2 < \sum_{n=1}^{N-1} |a_n|^2 + \sum_{n=1}^{\infty} |a_n|$. The series on the right is convergent so that by the comparison test the series $\sum_{n=1}^{\infty} |a_n|^2$ is convergent.
(b) By the alternating series test the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is convergent. But the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent ■

Exercise 29.7
Suppose that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $\{b_n\}_{n=1}^{\infty}$ is bounded. Show that $\sum_{n=1}^{\infty} a_n b_n$ is absolutely convergent (and thus convergent).

Solution.
Let $M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Thus, $\sum_{n=1}^{\infty} |a_n b_n| \leq M \sum_{n=1}^{\infty} |a_n|$. Since $\sum_{n=1}^{\infty} |a_n|$ is convergent, the series $M \sum_{n=1}^{\infty} |a_n|$ is convergent and so by the comparison test the series $\sum_{n=1}^{\infty} |a_n b_n|$ is convergent. Thus, $\sum_{n=1}^{\infty} a_n b_n$ is absolutely convergent and thus convergent ■

Exercise 29.8
Test the following series for absolute convergence, conditional convergence, or divergence.
(a) $\sum_{n=1}^{\infty} \frac{\sin n}{n2^n}$.
(b) $\sum_{n=1}^{\infty} (-1)^n \frac{5n}{n^2+2n}$.
(c) $\sum_{n=1}^{\infty} (-1)^n \frac{2^n - 2^{-n}}{2^n + 2^{-n}}$.

Solution.
(a) We have $\left|\frac{\sin n}{n2^n}\right| \leq \frac{1}{n2^n} \leq \frac{1}{2^n}$. The geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges so that by the comparison test the given series converges absolutely.
(b) By the limit comparison test one can show that the series of absolute value is divergent. By the alternating series test, the given series is convergent. Hence, the given series is conditionally convergent.
(c) The limit of the nth term is either 1 or -1 so that by the nth term test the series is divergent ■

Exercise 29.9
Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln 4n}{n}$ is absolutely convergent.

Solution.
By Exercise **??** we know that the series is convergent. However, since $\frac{\ln 4n}{n} > \frac{1}{n}$ and the Harmonic series is divergent, the series of absolute values is divergent. Hence, the given series is conditionally convergent ■

Exercise 29.10
Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ is monotone decreasing with $\lim_{n\to\infty} a_n = 0$. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence such that $|b_n| \leq a_n - a_{n+1}$ for all $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} b_n$ is absolutely convergent.

Solution.
Note first that the conditions on a_n imply that $a_n \geq 0$ for all $n \in \mathbb{N}$. Let $S_n = \sum_{n=1}^{n} |b_n|$. Since $|b_n| \leq a_n - a_{n+1}$ for each n we have

$$S_n \leq (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_n - a_{n+1}) = a_1 - a_{n+1} \leq a_1.$$

It follows that the sequence $\{S_n\}_{n=1}^{\infty}$ is bounded from above by a_1. Moreover, this sequence is increasing. Hence, by Exercise 5.5, the series $\sum_{n=1}^{\infty} |b_n|$ is convergent so that $\sum_{n=1}^{\infty} b_n$ converges absolutely ■

Solutions to Section 30

Exercise 30.1 *(Integral Test)*
Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and suppose that there is a function $f : [1, \infty) \to \mathbb{R}$ such that f is decreasing and positive with $f(n) = a_n$ for all $n \geq 1$.
(a) Show that $\{S_n\}_{n=1}^{\infty}$ is increasing.
(b) Define $F : [1, \infty) \to \mathbb{R}$ by $F(x) = \int_1^x f(t)dt$. Show that F is increasing.
(c) For $n \geq 2$ and $x \in [n-1, n]$, show that $a_n \leq f(x) \leq a_{n-1}$ and $a_n \leq \int_{n-1}^{n} f(x)dx \leq a_{n-1}$.
(d) Show that $S_n - a_1 \leq F(n) \leq S_{n-1}$.
(e) Suppose that $\int_1^{\infty} f(x)dx = L$. Since F is increasing we can write $F(n) \leq L$ for all $n \geq 1$. Show that $\{S_n\}_{n=1}^{\infty}$ is bounded. Hint: Use (d).
(f) Show that $\{S_n\}_{n=1}^{\infty}$ is convergent. Hence, $\sum_{n=1}^{\infty} a_n$ is convergent.
(g) Conversely, suppose that the series $\sum_{n=1}^{\infty} a_n$ converges to a number S. Show that for any positive integer $n \geq 2$ we have

$$F(n) \leq S.$$

(h) Show that for all $R \geq 1$ we have $F(R) \leq S$. Thus, $\int_1^{\infty} f(x)dx = \lim_{R\to\infty} \int_1^R f(x)dx$ is convergent. Hint: For any $R \geq 1$ we have $R \leq [R] + 1$ with $[R] + 1 \geq 2$.

Solution.
(a) We have $S_{n+1} - S_n = a_{n+1} > 0$ so that $S_{n+1} > S_n$.
(b) This follows from $F'(x) = f(x) > 0$.
(c) We have $a_n = f(n) \leq f(x) \leq f(n-1) = a_{n-1}$. Integrate from $n-1$ to n we obtain the desired result.
(d) We have $S_n - a_1 = a_2 + a_3 + cdots + a_n \leq \int_1^2 f(x)dx + \int_2^3 f(x)dx + \cdots + \int_{n-1}^{n} f(x)dx = \int_1^n f(x)dx = F(n) \leq a_1 + a_2 + \cdots + a_{n-1} = S_{n-1}$.
(e) Using (d) we have $S_n - a_1 \leq F(n) \leq L$ so that $S_n \leq L + a_1$ for all $n \geq 1$. That is, $\{S_n\}_{n=1}^{\infty}$ is bounded.
(f) This follows from Exercise 5.5.
(g) From (d), we have $F(n) \leq S_{n-1} \leq S$.
(h) We have $F(R) \leq F([R]+1) = \int_1^{[R]+1} f(x)dx \leq S$ for all $R \geq 1$. Hence, letting $R \to \infty$ we find that $\int_1^{\infty} f(x)dx \leq S$. That is, $\int_1^{\infty} f(x)dx$ is convergent.
Note that F is increasing and bounded from above so that $\lim_{R\to\infty} \int_1^R f(x)dx$ exists ■

Exercise 30.2 *(p-series)*
(a) Show that the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is convergent for $p>1$.
(b) Show that the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent for $p\le 1$.

Solution.
If $p<0$ then $\lim_{n\to\infty}\frac{1}{n^p}=\infty$ so that by the nth term test the series is divergent. If $p=0$ then the series is an infinite sum of 1 and so is divergent. So suppose that $p>1$. Let $f(x)=\frac{1}{x^p}$. Then $f'(x)=-\frac{p}{x^{p+1}}<0$ so that f is decreasing on $[1,\infty)$ and positive there. Moreover, $f(n)=\frac{1}{n^p}$. So by the integral test the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if and only if $\int_1^{\infty}\frac{dx}{x^p}$ is convergent. Now, we have

$$\int_1^R\frac{dx}{x^p}=\frac{R^{1-p}}{1-p}-\frac{1}{1-p}.$$

The improper integral exists if and only if $p>1$. It is divergent if $0<p<1$. Finally, for $p=1$ the given series is just the Harmonic series which is divergent ■

Exercise 30.3
Show that the series $\sum_{n=1}^{\infty}\frac{n}{(n^2+1)(\ln(n^2+1))^a}$ is convergent for all $a>1$. Hint: The integral test.

Solution.
It is easy to check that the integrand is nonnegative and decreasing. By the integral test we have

$$\begin{aligned}\int_1^{\infty}\frac{x}{(x^2+1)(\ln(x^2+1))^a}dx=&\lim_{R\to\infty}\int_1^R\frac{x}{(x^2+1)(\ln(x^2+1))^a}dx\\ =&\lim_{R\to\infty}\frac{1}{2(1-a)}\left[(\ln(x^2+1))^{1-a}\right]_1^R\\ =&\lim_{R\to\infty}\frac{1}{2(1-a)}\left[\frac{1}{(\ln(R^2+1)^{a-1}}-\frac{1}{(\ln 2)^{a-1}}\right]\\ =&-\frac{1}{2(1-a)(\ln 2)^{a-1}}\ ■\end{aligned}$$

Exercise 30.4
Use the integral test to test the convergence of the series $\sum_{n=4}^{\infty}\frac{1}{n\ln n\ln(\ln n)}$.

Solution.
We have
$$\int_4^\infty \frac{dx}{x\ln x \ln\ln x} = \int_{\ln\ln 4}^\infty \frac{du}{u} = \infty.$$
Thus, the given series is divergent ■

Exercise 30.5
Use the Integral Test to show that $\sum_{n=1}^\infty n^2 e^{-n^3}$ is convergent.

Solution.
Note that $f(x) > 0$ and $f'(x) < 0$ so that f is decreasing. By the Integral Test we have
$$\int_1^\infty x^2 e^{-x^3} dx = \lim_{R\to\infty}[-\frac{1}{3}e^{-R^3} + \frac{1}{3}e^{-1}] = \frac{1}{3}e^{-1}.$$
Since the improper integral is convergent so does the given series ■

Exercise 30.6
Use the integral test to show that the series $\sum_{n=1}^\infty e^{-n^2}$ is convergent.

Solution.
Let $f(x) = e^{-x^2} > 0$. Then $f'(x) = -2xe^{-x^2} < 0$ for $x \ge 1$. Now, for $x \ge 1 \to x^2 \ge x \to e^{-x^2} \le e^{-x}$. Since $\int_1^\infty e^{-x}dx = \frac{1}{e}$, the improper integral $\int_1^\infty e^{-x^2}dx$ is convergent. Hence, by the integral test, the given series is convergent ■

Exercise 30.7
Use the integral test to show that the series $\sum_{n=1}^\infty \frac{(\ln n)^2}{n}$ is divergent.

Solution.
Let $f(x) = \frac{(\ln x)^2}{x} \ge 0$ for $x \ge 1$. Moreover, $f'(x) = 2\ln x - \ln x)^2 x^2 < 0$ for $x \ge 8$. Thus,
$$\int_1^\infty \frac{(\ln x)^2}{x} \ge \int_8^\infty \frac{(\ln x)^2}{x}.$$
But
$$\int_8^\infty \frac{(\ln x)^2}{x} = \lim_{R\to\infty}[\frac{(\ln R)^3}{3}] = \infty.$$
Since the improper integral $\int_1^\infty \frac{(\ln x)^2}{x}$ is divergent, the given integral is divergent as well ■

Solutions to Section 31

Exercise 31.1 *(Ratio Test)*
Let $\sum_{n=1}^{\infty} a_n$ be a series of non-zero terms and suppose that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \geq 0$.
(a) Suppose $0 \leq L < 1$. Let $\epsilon = \frac{1-L}{2}$. Show that there is a positive integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \frac{1+L}{2} \text{ for all } n \geq N.$$

Hint: Use definition of convergence and Exercise 1.18.
(b) Let $r = \frac{1+L}{2}$. Show that $0 < r < 1$ and $|a_{N+k}| < r^k|a_N|$ for all $k = 1, 2, \cdots$.
(c) Find the value of the sum

$$\sum_{n=1}^{\infty} r^n|a_N|.$$

(d) Let $b_n = \sum_{k=1}^{n} |a_k|$. Show that the sequence $\{b_n\}_{n=1}^{\infty}$ is increasing.
(e) Let $M = b_N + \frac{r|a_N|}{1-r}$. Show that $|b_n| \leq M$ for all $n \geq 1$.
(f) Show that the sequence $\{b_n\}_{n=1}^{\infty}$ is convergent. What can you say about the sequence $\{a_n\}_{n=1}^{\infty}$?

Solution.
(a) Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, we can find a positive integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| - L \leq \left|\frac{a_{n+1}}{a_n} - L\right| < \frac{1-L}{2} \text{ for all } n \geq N.$$

This is equivalent to

$$\left|\frac{a_{n+1}}{a_n}\right| < \frac{L+1}{2} \text{ for all } n \geq N.$$

(b) Since $L < 1$ we have $1 + L < 2$ and therefore $r = \frac{1+L}{2} < 1$. Clearly, $r > 0$. By (a) we have $a_{N+1} < r|a_N|$. Suppose that $a_{N+k} < r^k|a_N$. Then $|a_{N+k+1}| < r|a_{N+K}| < r^{k+1}|a_N$. Hence, $|a_{N+k}| < r^k|a_N$ for all $k = 1, 2, \cdots$.
(c) The given sum is a geometric series with ration r and whose sum is given by

$$\sum_{n=1}^{\infty} r^n|a_N| = |a_N|\sum_{n=1}^{\infty} r^n = \frac{r|a_N}{1-r}.$$

(d) For any $n \geq 1$, we have $b_{n+1} = b_n + |a_{n+1}| \geq b_n$. This shows that the sequence $\{b_n\}_{n=1}^{\infty}$ is increasing.
(e) If $n \leq N$ then $|b_n| = \sum_{k=1}^{n} |a_k| < M$. If $n \geq N$ then $|b_n| = \sum_{k=1}^{N} |a_k| + \sum_{k=N}^{n} |a_k| \leq \sum_{k=1}^{N} |a_k| + \sum_{n=1}^{\infty} |a_n| = \sum_{k=1}^{N} |a_k| + \frac{r|a_N}{1-r} = M$. It follows that the sequence $\{b_n\}_{n=1}^{\infty}$ is convergent by Exercise **??**. Now, by Exercise 29.2 the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent ■

Exercise 31.2 *(Ratio Test)*
Let $\sum_{n=1}^{\infty} a_n$ be a series of non-zero terms and suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \geq 0$.
(a) Suppose $L > 1$. Let $\epsilon = L - 1$. Show that there is a positive integer N such that

$$L - \left|\frac{a_{n+1}}{a_n}\right| < \epsilon \text{ for all } n \geq N.$$

(b) Show that $|a_{n+1}| > |a_N| > 0$ for all $n \geq N$.
(c) Show that the series $\sum_{n=1}^{\infty} a_n$ is divergent. Hint: The nth term test.

Solution.
(a) Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, we can find a positive integer N such that

$$L - \left|\frac{a_{n+1}}{a_n}\right| < L - 1 \text{ for all } n \geq N.$$

(b) From (a) we conclude that $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for all $n \geq N$. Hence, $|a_{n+1}| > |a_N| > 0$ for all $n \geq N$.
(c) From (b) we see that $\lim_{n\to\infty} \neq 0$ so that by the nth term test the sequence $\sum_{n=1}^{\infty} a_n$ is divergent ■

Exercise 31.3
Consider the harmoninc series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we know it is divergent. Find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.
We have $\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1$ ■

Exercise 31.4
Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
(a) Show that this series is convergent.
(b) Find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution.
(a) This is a $p-$series with $p = 2 > 1$ so it is convergent.
(b) We have $\lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = 1$ ■

Exercise 31.5
Use the ratio test to determine the convergence of the series $\sum_{n=1}^{\infty}(-1)^n \frac{100^n}{n!}$.

Solution.
We have $\lim_{n\to\infty} \frac{\frac{100^{n+1}}{(n+1)!}}{\frac{100^n}{n!}} = \lim_{n\to\infty} \frac{100}{n+1} = 0 < 1$. By the ratio test the series is convergent ■

Exercise 31.6
Use the ratio test to determine the convergence of the series $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$. Hint: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Solution.
We have $\lim_{n\to\infty} \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n = \frac{2}{e} < 1$. By the ratio test the series is convergent ■

Exercise 31.7
Find $\lim_{n\to\infty} \frac{n!}{n^2}$.

Solution.
Let $a_n = \frac{n^2}{n!}$. By the ratio test we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n^2} = 0 < 1$$

so that the series $\sum_{n=1}^{\infty} a_n$ is convergent. Hence,

$$\lim_{n\to\infty} a_n = 0.$$

We conclude that

$$\lim_{n\to\infty} \frac{n!}{n^2} = \lim_{n\to\infty} \frac{1}{a_n} = \infty$$ ■

Exercise 31.8 *(nth root test)*
Consider a series $\sum_{n=1}^{\infty} a_n$. Define $L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$.
(a) Suppose first that $0 \geq L < 1$. Let $\epsilon = \frac{1-L}{2}$. Show that there is a positive integer N such that

$$|a_n|^{\frac{1}{n}} < \frac{1+L}{2} \text{ for all } n \geq N.$$

(b) Let $r = \frac{1+L}{2}$. Show that $0 < r < 1$ and $|a_n| < r^n$ for all $n \geq N$.
(c) Use (b) to conclude that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and hence convergent.

Solution.
(a) Since $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$, we can find a positive integer N such that

$$|a_n|^{\frac{1}{n}} - L \leq \left||a_n|^{\frac{1}{n}} - L\right| < \frac{1-L}{2} \text{ for all } n \geq N.$$

This is equivalent to

$$|a_n|^{\frac{1}{n}} < \frac{1+L}{2} \text{ for all } n \geq N.$$

(b) Since $0 \leq L < 1 \to 1 + L < 2 \to r = \frac{1+L}{2} < 1$. Moreover, by (a), we have $|a_n| < r^n$ for all $n \geq N$. (c) Since the goemetric series $\sum_{n=1}^{\infty} r^n$ is convergent for $0 < r < 1$, by the comparison test the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and hence convergent ■

Exercise 31.9 *(nth root test)*
Suppose that $L > 1$ in the previous exercise. Prove that the series $\sum_{n=1}^{\infty} a_n$ is divergent. Hint: nth term test.

Solution.
Let $\epsilon = L-1$. Then there is a positive integer N such that $||a_n|^{\frac{1}{n}} - L| < L-1$. This implies that $L - |a_n|^{\frac{1}{n}} < L - 1$ or $|a_n|^{\frac{1}{n}} > 1$. Hence, for $n \geq N$ we have $|a_n| > 1$. But this means that $\lim_{n\to\infty} \neq 0$. Hence, by the nth term test, the given series is divergent ■

Exercise 31.10
The root test is inconclusive if $L = 1$.
(a) We know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent. Show that $L = 1$.
(b) We know that the series $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ is conditionally convergent. Show that $L = 1$.
(c) We know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Show that $L = 1$.

Solution.
(a) The given series is a $p-$series with $p = 2$ so it is convergent by Exercise **??**. Let $y = \left(\frac{1}{n^2}\right)^{\frac{1}{n}}$. Then $\ln y = -\frac{2\ln n}{n} \to 0$ as $n \to \infty$. Thus, $y \to 1$ as $n \to \infty$. That is, $L = 1$.
(b) The given series is conditionally convergent by Exercise **??**. Let $y = \left(\frac{1}{n}\right)^{\frac{1}{n}}$. Then $\ln y = -\frac{\ln n}{n} \to 0$ as $n \to \infty$. Thus, $y \to 1$ as $n \to \infty$. That is, $L = 1$.
(c) The given series is the harmonic series so it is divergent. As in (b), $L = 1$ ■

Exercise 31.11
Use the root test to show that the series $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$ is divergent.

Solution.
We have
$$\lim_{n\to\infty} \left|\frac{n^n}{3^{1+2n}}\right|^{\frac{1}{n}} = \frac{1}{3}\lim_{n\to\infty} \frac{n}{9} = \infty.$$
By the root test the given series is divergent ■

Exercise 31.12
Use the root test to show that the series $\sum_{n=1}^{\infty} \left(\frac{5n-3n^3}{7n^3+2}\right)^n$ is absolutely convergent.

Solution.
We have
$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{5n-3n^3}{7n^3+2}\right| = \frac{3}{7} < 1.$$
By the root test the given series is convergent ■

Solutions to Section 32

Exercise 32.1
Define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{nx}{1+n^2x^2}$. Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the function $f(x) = 0$ for all $x \geq 0$.

Solution.
For all $x \geq 0$, $\lim_{n\to\infty} f_n(x) = 0$ ■

Exercise 32.2
For each positive integer n let $f_n : (0, \infty) \to \infty$ be given by $f_n(x) = nx$. Show that $\{f_n\}_{n=1}^{\infty}$ does not converge pointwise on D.

Solution.
This follows from the fact that $\lim_{n\to\infty} nx = \infty$ for all $x \in D$ ■

Exercise 32.3
For each positive integer n let $f_n : [0, 1] \to \infty$ be given by $f_n(x) = \frac{x}{n}$. Show that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the zero function. Hint: For a given ϵ, choose N such that $N > \frac{1}{\epsilon}$.

Solution.
Let $\epsilon > 0$ be given. Let N be a positive integer such that $N > \frac{1}{\epsilon}$. Then for $n \geq N$ we have

$$|f_n(x) - f(x)| = \frac{|x|}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

for all $x \in [0, 1]$ ■

Exercise 32.4
Define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{nx}{1+n^2x^2}$. By Exercise 32.1, this sequence converges pointwise to $f(x) = 0$. Let $\epsilon = \frac{1}{3}$. Show that there is no positive integer N with the property $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \geq 0$. Hence, the given sequence does not converge uniformly to $f(x)$.

Solution.
For any positive integer N and for $n \geq N$ we have

$$\left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right) \right| = \frac{1}{2} > \epsilon \ ■$$

Exercise 32.5
Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

(a) Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f.
(b) Show that the sequence $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly to f. Hint: Suppose otherwise. Let $\epsilon = 0.5$ and get a contradiction by using a point $(0.5)^{\frac{1}{N}} < x < 1$.

Solution.
(a) For all $0 \le x < 1$ we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$. Also, $\lim_{n\to\infty} f_n(1) = 1$. Hence, the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f.
(b) Suppose the contrary. Let $\epsilon = \frac{1}{2}$. Then there exist a positive integer N such that for all $n \ge N$ we have

$$|f_n(x) - f(x)| < \frac{1}{2}$$

for all $x \in [0,1]$. In particular, we have

$$|f_N(x) - f(x)| < \frac{1}{2}$$

for all $x \in [0,1]$. Choose $(0.5)^{\frac{1}{N}} < x < 1$. Then $|f_N(x) - f(x)| = x^N > 0.5 = \epsilon$ which is a contradiction. Hence, the given sequence does not converge uniformly ■

Exercise 32.6
Give an example of a sequence of continuous functions $\{f_n\}_{n=1}^{\infty}$ that converges pointwise to a discontinuous function f.

Solution.
See Exercise 32.5(a) ■

Exercise 32.7
Suppose that for each $n \ge 1$ the function $f_n : D \to \mathbb{R}$ is continuous in D. Suppose that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f. Let $a \in D$.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that if $n \ge N$ then $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in D$.

(b) Show that there is a $\delta > 0$ such that for all $|x - a| < \delta$ we have $|f_N(x) - f_N(a)| < \frac{\epsilon}{3}$
(c) Using (a) and (b) show that for $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$. Hence, f is continuous in D since a was arbitrary. Symbolically we write

$$\lim_{x\to a}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to a} f_n(x)$$

Solution.
(a) This follows from the definition of uniform convergence.
(b) This follows from the fact that f_N is continuous at $a \in D$.
(c) For $|x - a| < \delta$ we have $|f(x) - f(a)| = |f(a) - f_N(a) + f_N(a) - f_N(x) + f_N(x) - f(x)| \leq |f_N(a) - f(a)| + |f_N(a) - f_N(x)| + |f_N(x) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ ■

Exercise 32.8
Consider the interval $[0, 1]$ and let the rationals in this interval be labeled $r_1, r_2, \cdots$ arranged in increasing order. For each positive integer n we define the function $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \cdots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that f_n is Riemann integrable on $[0, 1]$. Hint: Remark 3.
(b) Show that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

(c) Show that f is not Riemann integrable.

Solution.
(a) Since f_n is bounded and discontinuous at a finite number of points, by Remark 3, f_n is Riemann integrable.
(b) If x is irrational then for every $n \geq 1$ we have $f_n(x) = 0$. Thus, $\lim_{n\to\infty} f_n(x) = 0 = f(x)$. Suppose now that x is rational. Then there is a positive integer k such that $x = r_k$. In this case, $f_n(x) = 1$ for all $n \geq k$. Hence, $\lim_{n\to\infty} f_n(x) = 1 = f(x)$. It follows that the given sequence converges pointwise to f.
(c) This follows from Exercise 20.5 ■

$$\lim_{n\to\infty}\int_D f_n(x)dx = \int_D \lim_{n\to\infty} f_n(x)dx = \int_D f(x)dx? \qquad (3)$$

That is, can we interchange limit and integration? The answer is no as seen in the next exercise.

Exercise 32.9
Consider the functions $f_n : [0,1] \to \infty$ defined by $f_n(x) = n^2xe^{-nx}$.
(a) Show that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to $f(x) = 0$. Hint: L'Hôpital's rule.
(b) Find $\lim_{n\to\infty}\int_0^1 f_n(x)dx$. Hint: Integration by parts.
(c) Show that $\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)dx$.

Solution.
(a) We have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n^2xe^{-nx} = \lim_{n\to\infty}\frac{n^2x}{e^{nx}} = 0$$

where we apply L'Hôpital's rule twice. Hence, $\{f_n\}_{n=1}^{\infty}$ converges pointwise to $f(x) = 0$.
(b) Using integration by parts we find

$$\int_0^1 n^2xe^{-nx}dx = n^2\left[-\frac{xe^{-nx}}{n} - \frac{1}{n^2}e^{-nx}\right]_0^1 = 1 - e^{-n}(n+1).$$

Hence,

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty}[1 - e^{-n}(n+1)] = 1.$$

(c) From (b) we have

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} f_n(x)dx \blacksquare$$

Exercise 32.10
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Riemann integrable functions on $[a,b]$ that converges uniformly to a f defined on $[a,b]$.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for all $n \geq N$ we have

$$|f_n(x) - f(x)| < \tfrac{\epsilon}{4(b-a)} \text{ for all } x \in [a,b].$$

(b) Let $n \geq N$. Show that there is a partition P of $[a, b]$ such that

$$U(f_n, P) - L(f_n, P) < \frac{\epsilon}{2}.$$

(c) Suppose $n \geq N$ and P as in (b). Show that

$$U(f, P) \leq U(f_n, P) + \frac{\epsilon}{4}$$

and therefore

$$L(f, P) \geq L(f_n, P) - \frac{\epsilon}{4}.$$

Hint: $|f(x)| \leq |f_n(x)| + \frac{\epsilon}{4(b-a)}$ and $|f_n(x)| \leq |f(x)| + \frac{\epsilon}{4(b-a)}$ (d) Conclude that $U(f, P) - L(f, P) < \epsilon$ and therefore f is Riemann integrable on $[a, b]$.

Solution.
(a) This is just the definition of uniform convergence.
(b) This follows from Exercise 20.6.
(c) Since $|f(x)| \leq |f_n(x)| + \frac{\epsilon}{4(b-a)}$ we have

$$U(f, P) \leq U(f_n, P) + U(\frac{\epsilon}{4(b-a)}, P) = U(f_n, P) + \frac{\epsilon}{4}.$$

Likewise, since $|f_n(x)| \leq |f(x)| + \frac{\epsilon}{2(b-a)}$

$$L(f, P) \geq L(f_n, P) + L(\frac{-\epsilon}{4(b-a)}, P) = L(f_n, P) - \frac{\epsilon}{4}.$$

Hence,

$$U(f, P) - L(f, P) \leq [U(f_n, P) - L(f_n, P)] + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

It follows that f is Riemann integrable on $[a, b]$ ■

Exercise 32.11
Let $\{f_n\}_{n=1}^{\infty}$ and f be as in the previous exercise.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that if $n \geq N$ then

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{b-a} \text{ for all } x \in [a, b].$$

(b) Show that for every $n \geq N$ we have

$$\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| < \epsilon.$$

Thus, (3) holds. Hint: Exercise 23.1 and Exercise 24.3

Solution.
(a) This follows from the definition of uniform convergence.
(b) Using (a) and Exercise 24.3 we find

$$\begin{aligned}\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| =& \left|\int_a^b [f_n(x) - f(x)]dx\right| \\ \leq& \int_a^b |f_n(x) - f(x)|dx \\ <& \frac{\epsilon}{b-a}\int_a^b dx = \epsilon \blacksquare\end{aligned}$$

Exercise 32.12
Give an example of a sequence of differentiable functions $\{f_n\}_{n=1}^{\infty}$ that converges pointwise to a non-differentiable function f.

Solution.
See Exercise 32.5(a) $\blacksquare$

Exercise 32.13
Consider the family of functions $f_n : [-1, 1]$ given by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$.
(a) Show that f_n is differentiable for each $n \geq 1$.
(b) Show that for all $x \in [-1, 1]$ we have

$$|f_n(x) - f(x)| \leq \frac{1}{\sqrt{n}}$$

where $f(x) = |x|$. Hint: Note that $\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2} \geq \frac{1}{\sqrt{n}}$.
(c) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for $n \geq N$ we have

$$|f_n(x) - f(x)| < \epsilon \text{ for all } x \in [-1, 1].$$

Thus, $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the non-differentiable function $f(x) = |x|$.

Solution.
(a) f_n is the composition of two differentiable functions so it is differentiable with derivative

$$f_n'(x) = x\left[x^2 + \frac{1}{n}\right]^{-\frac{1}{2}}.$$

(b) We have

$$|f_n(x) - f(x)| = \left|\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2}\right| = \left|\frac{(\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2})(\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2})}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}}\right|$$
$$= \frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}}$$
$$notag \leq \frac{\frac{1}{n}}{\frac{1}{\sqrt{n}}} \qquad (4)$$

(c) Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ we can find a positive integer N such that for all $n \geq N$ we have $\frac{1}{\sqrt{n}} < \epsilon$. Now the answer to the question follows from this and part (b) ■

Exercise 32.14
Give an example of a sequence of differentiable functions $\{f_n\}_{n=1}^{\infty}$ that converges uniformly to a a differentiable function f such that $\lim_{n\to\infty} f_n'(x) \neq f'(x) = [\lim_{n\to\infty} f_n(x)]'$. That is, one cannot, in general, interchange limits and derivatives. Hint: Exercise 32.3

Solution.
Exercise **??** with $\lim_{n\to\infty} f_n'(x) = g(x)$ where $g(1) = 1$ and $g(x) = 0$ for $0 \leq x < 1$ ■

Exercise 32.15
Let $\{f_n\}_{n=1}^{\infty}$ defined on a set D be uniformly Cauchy.
(a) Show that for each $x \in D$, the sequence of numbers $\{f_n(x)\}_{n=1}^{\infty}$ is convergent. Call the limit $f(x)$. Thus, we can define a function $f : D \to \mathbb{R}$ such

that $f(x) = \lim_{n\to\infty} f_n(x)$. Hint: Exercise 7.7
(b) Show that $\{f_n\}_{n=1}^\infty$ converges pointwise to f.
(c) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for all $m, n \geq N$ we have

$$|f_m(x) - f_n(x)| < \frac{\epsilon}{2} \text{ for all } x \in D.$$

(d) Fix $x \in D$. Show that there is a positive integer $m \geq N$ such that $|f_m(x) - f(x)| < \frac{\epsilon}{2}$.
(e) For the fixed x in (d), let $n \geq N$. Show that $|f_n(x) - f(x)| < \epsilon$.
(f) Conclude that $\{f_n\}_{n=1}^\infty$ converges uniformly to f.

Solution.
(a) Let $x \in D$. By uniform Cauchy, the sequence $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence of numbers in $\mathbb{R}$. By Exercise 7.7 it is convergent. Call its limit $f(x)$. Hence, we define a function $f : D \to \mathbb{R}$ by $f(x) = \lim_{n\to\infty} f_n(x)$.
(b) This follows from the definition of f.
(c) This follows from the definition of uniform Cauchy sequence.
(d) Fix $x \in D$. The result follows from the fact that $\{f_n\}_{n=1}^\infty$ converges pointwise to f.
(e) For $n \geq N$ we have

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \epsilon.$$

(f) Since the previous inequality is true for all $x \in D$ we conclude that $\{f_n\}_{n=1}^\infty$ converges uniformly to f ■

Exercise 32.16
Let $\{f_n\}_{n=1}^\infty$ be a sequence of differentiable functions on $[a, b]$ such that $\{f_n(c)\}_{n=1}^\infty$ converges for some $c \in [a, b]$. Assume also that $\{f_n'\}_{n=1}^\infty$ converges uniformly to g in $[a, b]$.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N_1 such that for all $m, n \geq N_1$ we have

$$|f_m'(x) - f_n'(x)| < \frac{\epsilon}{2(b-a)} \text{ for all } x \in [a, b].$$

(b) Show that there is a positive integer N_2 such that for all $m, n \geq N_2$ we have

$$|f_m(c) - f_n(c)| < \frac{\epsilon}{2}.$$

Hint: Exercise 7.3
(c) Show that for all $x \in [a,b]$ there is a d between c and x such that

$$f_m(x) - f_n(x) = f_m(c) - f_n(c) + (x-c)[f'_m(d) - f_n(d)].$$

Hint: Apply the Mean Value theorem to the function $f_m - f_n$ restricted to the interval $[c,x]$.
(d) Let $N = N_1 + N_2$. Use (a) - (c) to show that for $n \geq N$ we have

$$|f_m(x) - f_n(x)| < \epsilon \text{ for all } x \in [a,b].$$

That is, the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy.
(e) Show that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a a function f.

Solution.
(a) Let $\epsilon > 0$ be given. There is a positive integer N' such that for all $n \geq N_1$ we have

$$|f'_n(x) - g(x)| < \tfrac{\epsilon}{4(b-a)} \text{ for all } x \in [a,b].$$

Hence, for all $m,n \geq N_1$ we have

$$|f'_m(x) - f'_n(x)| \leq |f'_m(x) - g(x)| + |f'_n(x) - g(x)| < \frac{\epsilon}{4(b-a)} + \frac{\epsilon}{4(b-a)} = \frac{\epsilon}{2(b-a)}$$

(b) This follows from the fact that $\{f_n(c)\}_{n=1}^{\infty}$ is Cauchy by Exercise 7.3.
(c) This is a result from the MVT.
(d) For $n \geq N$ we have $|f_m(x) - f_n(x)| \leq |f_m(c) - f_n(c)| + (b-a)|f'_m(d) - f'_n(d)| < \frac{\epsilon}{2} + (b-a)\frac{\epsilon}{2(b-a)} = \epsilon$. Hence, the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy.
(e) This is a consequence of Exercise 32.15 ■

Exercise 32.17
In this exercise we want to show that f of the previous exercise is differentiable in $[a,b]$ and $f' = g$.
(a) Show that there is a positive integer N_1 such that for all $n \geq N_1$ we have

$$|f'_m(x) - f'_n(x)| < \tfrac{\epsilon}{3} \text{ for all } x \in [a,b].$$

(b) Let $x_0 \in [a, b]$. Use the MVT to the function $f_m - f_n$ to show the existence of a point d between x_0 and x such that

$$f_m(x) - f_n(x) = f_m(x) - f_n(x_0) + (x - x_0)[f'_m(d) - f'_n(d)].$$

(c) Use (a) and (b) to show that

$$\left|\frac{f_m(x) - f_n(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0}\right| < \frac{\epsilon}{3}.$$

(d) Show that

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0}\right| \leq \frac{\epsilon}{3}.$$

(e) Show that there is a positive integer N_2 such that for all $n \geq N_2$ we have

$$|f'_n(x_0) - g(x_0)| < \frac{\epsilon}{3}.$$

(f) Let $N = N_1 + N_2$. Show that there is a $\delta > 0$ such that

$$\text{If } 0 < |x - x_0| < \delta \text{ then } \left|\tfrac{f_N(x) - f_N(x_0)}{x - x_0} - f'_N(x_0)\right| < \tfrac{\epsilon}{3}.$$

(g) Use (d) - (f) to conclude that

$$\text{If } 0 < |x - x_0| < \delta \text{ then } \left|\tfrac{f(x) - f(x_0)}{x - x_0} - g(x_0)\right| < \epsilon.$$

That is, f is differentiable at x_0 with $f'(x_0) = g(x_0$.

Solution.
(a) Similar to (a) in the previous exercise.
(b) Easy.
(c) We have

$$\left|\frac{f_m(x) - f_n(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0}\right| = |f'_m(d) - f'_n(d)| < \frac{\epsilon}{3}.$$

(d) This follows by letting $m \to \infty$.
(e) This follows from the fact that $\{f'_n(c)\}_{n=1}^{\infty}$ converges to $g(x_0)$.

(f) This follows from the fact that f_N is differentiable at x_0.
(g) Suppose $0 < |x - x_0| < \delta$. Then

$$\begin{aligned}\left|\frac{f(x) - f(x_0)}{x - x_0} - g(x_0)\right| =& \left|\frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0}\right.\\ &+ \left.\frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N'(x_0) + f_N'(x_0) - g(x_0)\right|\\ <& \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

Hence, f is differentiable at x_0 with $f'(x_0) = g(x_0)$ ■

Exercise 32.18
Consider the sequence of functions $f_n(x) = x - \frac{x^n}{n}$ defined on $[0, 1)$.
(a) Does $\{f_n\}_{n=1}^{\infty}$ converge to some limit function? If so, find the limit function and show whether the convergence is pointwise or uniform.
(b) Does $\{f_n'\}_{n=1}^{\infty}$ converge to some limit function? If so, find the limit function and show whether the convergence is pointwise or uniform.

Solution.
(a) Let $\epsilon > 0$ be given. Let N be a positive integer such that $N > \frac{1}{\epsilon}$. Then for $n \geq N$

$$\left|x - \frac{x^n}{n} - x\right| = \frac{|x|^n}{n} < \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Thus, the given sequence converges uniformly (and pointwise) to the function $f(x) = x$.
(b) Since $\lim_{n\to\infty} f_n'(x) = 1$ for all $x \in [0, 1)$, the sequence $\{f_n'\}_{n=1}^{\infty}$ converges pointwise to $f(x) = 1$. However, the convergence is not uniform. To see this, let $\epsilon = \frac{1}{2}$ and suppose that the convergence is uniform. Then there is a positive integer N such that for $n \geq N$ we have

$$|1 - x^{n-1} - 1| = |x|^{n-1} < \frac{1}{2}.$$

In particular, if we let $n = N + 1$ we must have $x^N < \frac{1}{2}$ for all $x \in [0, 1)$. But $x = \left(\frac{1}{2}\right)^{\frac{1}{N}} \in [0, 1)$ and $x^N = \frac{1}{2}$ which contradicts $x^N < \frac{1}{2}$. Hence, the convergence is not uniform ■

Exercise 32.19
Suppose that each f_n is uniformly continuous on D and that $f_n \to f$ uniformly on D. Prove that f is uniformly continuous on D.

Solution.
Let $\epsilon > 0$ be given. By uniform convergence, there is a positive integer N such that $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in D$ whenever $n \geq N$. Now, since f_N is uniformly continuous on D we can find a $\delta > 0$ such that if $|x - y| < \delta$ then $|f_n(x) - f_N(y)| < \frac{\epsilon}{3}$. Thus, for any $x, y \in D$ such that $|x - y| < \delta$ we have

$$\begin{aligned} |f(x) - f(y)| =& |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\ \leq& |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ <& \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

Hence, f is uniformly continuous on D ■

Exercise 32.20
Let $f_n(x) = \frac{x^n}{1+x^n}$ for $x \in [0, 2]$.
(a) Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ on $[0, 2]$.
(b) Does $f_n \to f$ uniformly on $[0, 2]$?

Solution.
(a) The pointwise limit is

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } 1 < x \leq 2 \end{cases}$$

(b) The convergence cannot be uniform because if it were f would have to be continuous ■

Exercise 32.21
Prove that if $f_n \to f$ and $g_n \to g$ uniformly on a set D then $f_n + g_n \to f + g$ uniformly on D.

Solution.
Let $\epsilon > 0$ be given. Let N_1 be a positive integer such that $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $n \geq N_1$ and all $x \in D$. Likewise, let N_2 be a positive integer such that $|g_n(x) - g(x)| < \frac{\epsilon}{2}$ for all $n \geq N_2$ and all $x \in D$. Let $N = N_1 + N_2$. Then for $n \geq N$ we have

$$|(f_n(x) + g_n(x)) - (f(x) + g(x))| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $x \in D$. That is, $f_n + g_n \to f + g$ uniformly on D ■

Exercise 32.22
Prove that if $f_n \to f$ uniformly on a set D then $\{f_n\}_{n=1}^\infty$ uniformly Cauchy on D.

Solution.
Let $\epsilon > 0$ be given. Then there is a positive integer N such that if $n \geq N$ then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in D$. Thus, if $m, n \geq N$ we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $x \in D$. This shows that $\{f_n\}_{n=1}^\infty$ uniformly Cauchy on D ■

Exercise 32.23
Suppose that $\{f_n\}_{n=1}^\infty$ is uniformly convergent on a set D where each f_n is bounded on D, that is $|f_n(x)| \leq M_n$ for all $x \in D$. Show that there is a positive constant M such that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in D$.

Solution.
By the previous exercise $\{f_n\}_{n=1}^\infty$ is uniformly Cauchy on D. Let $\epsilon = 1$. Then there is a positive integer N such that

$$|f_n(x) - f_N(x)| < 1 \text{ for all } n \geq N \text{ and all } x \in D.$$

Hence,

$$|f_n(x)| < 1 + M_N \text{ for all } n \geq N \text{ and all } x \in D.$$

Let $M = 1 + M_1 + M_2 + \cdots + M_{N-1}$. Then $|f_n(x)| \leq M$ for all $x \in D$ and all $n \in \mathbb{N}$ ■

Exercise 32.24
Suppose that $f_n \to f$ and $g_n \to g$ uniformly on D. Moreover, suppose that $|f_n(x)| \leq M_n$ and $|g_n(x)| \leq M_n$ for all $n \in \mathbb{N}$ and all $x \in D$. Prove that $f_n g_n \to fg$ uniformly on D.

Solution.
By the previous exercise there exist $M_1 > 0$ and $M_2 > 0$ such that $|f_n(x)| \leq M_1$ and $|g_n(x)| \leq M_2$ for all $x \in D$ and all $n \in \mathbb{N}$. Let $M = M_1 + M_2$. Then $|f_n(x)| \leq M$ and $|g_n(x)| \leq M$ for all $x \in D$ and all $n \in \mathbb{N}$. Now, $|f(x)| - |f_n(x) - f(x)| \leq |f(x) - f_n(x) + f(x)| = |f_n(x)| \leq M \to |f(x)| \leq$

$M + |f_n(x) - f(x)|$ for all $n \in \mathbb{N}$ and all $x \in D$. By pointwise convergence we obtain $|f(x)| \leq M$ for all $x \in D$. Likewise, $|g(x)| \leq M$ for all $x \in D$. Let $\epsilon > 0$ be given. Then there is a positive integer N_1 such that if $n \geq N_1$ then $|f_n(x) - f(x)| < \frac{\epsilon}{2M}$ for all $x \in D$. Likewise, there is a positive integer N_2 such that if $n \geq N_2$ then $|g_n(x) - g(x)| < \frac{\epsilon}{2M}$ for all $x \in D$. Let $N = N_1 + N_2$. Then for $n \geq N$ we have

$$\begin{aligned}|f_n(x)g_n(x) - f(x)g(x)| \leq& |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)|\\ <& M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon\end{aligned}$$

for all $x \in D$. This shows that $f_ng_n \to fg$ uniformly on D ■

Exercise 32.25
Let $f_n(x) = x + \frac{1}{n}$ for all $x \in \mathbb{R}$ and $g_n(x) = \left(x + \frac{1}{n}\right)^2$.
(a) Show that $f_n \to f$ uniformly where $f(x) = x$.
(b) Show that g_n does not converge uniformly to the function $g(x) = x^2$.

Solution.
(a) Let $\epsilon > 0$ be given. Choose N such that $N > \frac{1}{\epsilon}$. Then for $n \geq N$ we have

$$|f_n(x) - f(x)| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

This shows that f_n converges uniformly to f.
(b) Suppose the contrary. Let $\epsilon = 1$. Then there positive integer N such that if $n \geq N$ we have

$$|g_n(x) - g(x)| = \left|\frac{2x}{n} + \frac{1}{n^2}\right| < 1$$

for all $n \geq N$ and all $x \in \mathbb{R}$. But if we choose $n = N$ and $x = N$ we obtain

$$|g_N(N) - g(N)| = \frac{2N^2 + 1}{N^2} > 1$$

a contradiction. Hence, g_n does not convergen to g uniformly ■

Exercise 32.26
Give an example of a sequence $\{f_n\}_{n=1}^\infty$ and a function f such that $f_n \to f$ uniformly but f_n^2 does not converge uniformly to f^2.

Solution.
Let $f_n(x) = x + \frac{1}{n}$ and use previous exercise ■

Exercise 32.27
Give an example of two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ such that $f_n \to f$ and $g_n \to g$ uniformly but $f_n g_n$ does not converge uniformly to fg. Thus, the condition of boundedness in Exercise 32.24 is crucial.

Solution.
Let $f_n(x) = g_n(x) = x + \frac{1}{n}$ and $f(x) = g(x) = x$ and use previous exercise ■

Solutions to Section 33

Exercise 33.1
Suppose that $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a power series that converges for $x = c$. Note that the series converges to a_0 if $c = a$. So we will assume that $c \neq a$.
(a) What is the value of the limit $\lim_{n\to\infty} a_n(c-a)^n$?
(b) Show that there is a positive integer N such that $|a_n(c-a)^n| < 1$ for all $n \geq N$.
(c) Let $M = \sum_{n=0}^{N-1} |a_n(c-a)^n| + 1$. Show that $|a_n(c-a)^n| \leq M$ for all $n \geq 0$.
(d) Let x be such that $|x-a| < |c-a|$. Show that for any $n \geq 0$ we have

$$|a_n(x-a)^n| \leq M\left|\frac{x-a}{c-a}\right|^n.$$

(e) Show that the series $\sum_{n=0}^{\infty} M\left|\frac{x-a}{c}\right|^n$ is convergent.
(f) Show that the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is absolutely convergent and hence convergent.
We conclude that if a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for $x = c$ it is convergent for any x satisfying $|x-a| < |c-a|$.

Solution.
(a) By the nth term test, $\lim_{n\to\infty} a_n(c-a)^n = 0$.
(b) From the definition of convergence of a sequence and part (a), there exists a positive integer N such that $|a_n(c-a)^n| < 1$ for all $n \geq N$.
(c) If $n \leq N-1$ then $|a_n(c-a)^n| \leq \sum_{n=0}^{N-1} |a_n(c-a)^n| < M$. If $n \geq N$ we have $|a_n(c-a)^n| < 1 < M$.
(d) For $n \geq 0$ we have

$$|a_n(x-a)^n| = |a_n(c-a)^n| \cdot \left|\frac{x-a}{c-a}\right|^n \leq M\left|\frac{x-a}{c-a}\right|^n.$$

(e) This series is a geometric series with ratio $\left|\frac{x-a}{c-a}\right| < 1$ so it is convergent.
(f) This follows from the comparison test and Exercise 29.2 ∎

Exercise 33.2
Suppose that $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a power series that diverges for $x = d$. Let x be a number satisfying $|x-a| > |d-a|$. Show that the assumption $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges at x leads to a contradiction. Hence, the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ must be divergent. Hint: Use Exercise 33.1.

Solution.
Suppose that $|x-a| > |d-a|$. If $\sum_{n=0}^{\infty} a_n(x-a)^n$ is convergent then by Exercise 33.1, the series $\sum_{n=0}^{\infty} a_n(d-a)^n$ is absolutely convergent and hence convergent. But this contradicts the fact that $\sum_{n=0}^{\infty} a_n(d-a)^n$ is divergent. Hence, $\sum_{n=0}^{\infty} a_n(x-a)^n$ must be divergent ■

Exercise 33.3
Consider a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$. Let C be the collection of all real numbers at which the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges. That is,

$$C = \{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n(x-a)^n \text{ converges}\}.$$

(a) Show that $C \neq \emptyset$.
(b) Explain in words the meaning that $C = \{a\}$.
(c) Explain in words the meaning that $C = (-\infty, \infty) = \mathbb{R}$.
(d) Suppose that $C \neq \{a\}$ and $C \neq \mathbb{R}$. That is, there is a real number $d \neq a$ such that $\sum_{n=0}^{\infty} a_n(d-a)^n$ diverges. Show that if $x \in C$ then $|x-a| \leq |d-a|$. Conclude that $\{|x-a| : x \in C\}$ is bounded from above with an upper bound M. What is the value of M?
(e) Show that there is a finite number R such that R is the least upper bound of $\{|x-a| : x \in C\}$. Thus, $|x-a| \leq R$ for all $x \in C$. Show that $R > 0$.
(f) Show that for any real number x such that $|x-a| > R$, the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is divergent.
(g) Show that for any real number x such that $|x-a| < R$, the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is convergent. Hint: Let $\epsilon = R - |x-a|$ and use the definition of supremum to show that there exist an $x_0 \in C$ such that $R - \epsilon < |x_0 - a| \leq R$.

Solution.
(a) Since the series converges at $x = a$ to a_0, we have $a \in C$ and so $C \neq \emptyset$.
(b) If $C = \{a\}$ the series converges only at $x = a$ and diverges for all $x \neq a$.
(c) If $C = (-\infty, \infty)$ then the series converges for all values of x.
(d) This follows from Exercise 33.2. Thus, for any $x \in C$ we have $|x-a| \leq |d-a|$. This shows that the set $\{|x-a| : x \in C\}$ bounded from above by $M = |d-a|$.
(e) The existence of R follows from the completeness axiom of $\mathbb{R}$. Since $|x-a| \leq R$ for all $x \in C$ and $a, d \in C$ with $d \neq a$ we conclude that $R > 0$.

(f) Let $x \in \mathbb{R}$ such that $|x-a| > R$. Then $x \notin C$ and therefore $\sum_{n=0}^{\infty} a_n(x-a)^n$ is divergent.
(g) Let $x \in \mathbb{R}$ such that $|x-a| < R$. Let $\epsilon = R - |x-a|$. By the definition of supremum, there exists an $x_0 \in C$ such that $R - \epsilon < |x_0 - a| \le R$. But this implies that $|x-a| < |x_0 - a|$. By Exercise 33.1, the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is (absolutely) convergent ■

Exercise 33.4
Find the radius of convergence of each of the following series:
(a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
(b) $\sum_{n=0}^{\infty} n!x^n$.
(c) $\sum_{n=0}^{\infty} x^n$.

Solution.
(a) By the ratio test the series converges for any value of x. Thus, $R = \infty$.
(b) By the ratio test the series converges only when $x = 0$. Thus, $R = 0$.
(c) This is a geomteric series that converges for $|x| < 1$ so that $R = 1$ ■

Exercise 33.5
Suppose that $\sum_{n=0}^{\infty} a_n(x-a)^n$ is a power series with $a_n \neq 0$ for all $n \ge 0$. Suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \ge 0.$$

(a) Find $\lim_{n\to\infty} \left| \frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n} \right|$.
(b) Suppose that $L = 0$. Show that $R = \infty$. That is, a power series converges for all $x \in \mathbb{R}$.
(c) Suppose that $L > 0$. Show that $R = \frac{1}{L}$.
(d) Suppose that $L = \infty$. Show that $R = 0$, that is, the series diverges for all $x \neq a$.

Solution.
(a) By the ratio test we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a)n} \right| |x-a| = L|x-a|.$$

(b) If $L = 0$ then $L|x-a| = 0 < 1$ for all $x \in \mathbb{R}$. Hence, $R = \infty$ and the power series converges for all $x \in \mathbb{R}$.

(c) A power series converges if and only if $L|x-a| < 1$ and diverges for $L|x-a| > 1$. Thus, $R = \frac{1}{L}$.
(d) If $L = \infty > 1$ then the series diverges for all $x \neq a$. That is, $R = 0$ ■

Exercise 33.6
Find the interval of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n\frac{(x-1)^n}{n}$.

Solution.
We have $\frac{1}{R} = \lim_{n\to\infty}\frac{n}{n+1} = 1$ so that $R = 1$. Hence, the series converges for $|x-1| < 1$ and diverges for $|x-1| > 1$. So the series converges for all x such that $0 < x < 2$. What about the endpoints $x = 0$ and $x = 2$? If we replace x by 0 we obtain the series $-\sum_{n=1}^{\infty}\frac{1}{n}$ which is divergent (harmonic series). If we replace x by 2 we obtain the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ which converges by the alternating series test. Thus, the interval of convergence is $0 < x \leq 2$ ■

Exercise 33.7
Find the interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{n}{n^2+2}x^n$.

Solution.
Let R be the radius of convergence. Then

$$\frac{1}{R} = \lim_{n\to\infty}\frac{n+1}{(n+1)^2+2}\cdot\frac{n^2+2}{n} = 1.$$

Thus, $R = 1$ so the series converges for $|x| < 1$ and diverges for $|x| > 1$. If $x = 1$ then the series becomes $\sum_{n=1}^{\infty}\frac{n}{n^2+2}$. Since

$$\lim_{n\to\infty}\frac{\frac{n}{n^2+2}}{\frac{1}{n}} = 1 > 0$$

and the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent, by the limit comparison test, the series $\sum_{n=1}^{\infty}\frac{n}{n^2+2}$ is also divergent.
If $x = -1$ then we get the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+2}$. By the alternating series test one can show that this series is convergent. Thus, the interval of convergence is $-1 \leq x < 1$ ■

Exercise 33.8
Find the interval of convergence of the power series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n^2+1}$.

Solution.
Use the absolute ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{|x|^n} = |x|\left(\frac{n^2+1}{n^2+2n+2}\right) \to |x| \ as \ n \to \infty.$$

Therefore the series converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. If $|x| = 1$, the ratio test gives no information, so we have to look at the endpoints separately:
• At $x = 1$ we have the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+1}$ an absolutely convergent series by the alternating series test.
• At $x = -1$ we have the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ an absolutely convergent series by comparison to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
Conclusion: This power series is absolutely convergent on $[-1, 1]$ and diverges everywhere else ■

Exercise 33.9
Find the interval of convergence of the power series $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{e}{2}\right)^n \frac{(x-1)^n}{n}$.

Solution.
Use the absolute ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{e^{n+1}|x-1|^{n+1}}{2^{n+1}(n+1)} \cdot \frac{2^n n}{e^n|x-1|^n} = \frac{e}{2} \cdot \frac{n}{n+1}|x-1| \to \frac{e}{2}|x-1| \ as \ n \to \infty.$$

Therefore the series converges absolutely if $|x-1| < \frac{2}{e}$ and diverges if $|x-1| > \frac{2}{e}$. If $|x-1| = \frac{2}{e}$, the ratio test gives no information, so we have to look at the endpoints separately:
• At $x - 1 = -\frac{2}{e}$ we have the series $-\sum_{n=1}^{\infty} \frac{1}{n}$ a divergent series.
• At $x - 1 = \frac{2}{e}$ we have the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ a conditionally convergent series.
Conclusion: This power series is absolutely convergent on $|x-1| < \frac{2}{e}$, conditionally convergent at $x = \frac{2}{e}$ and divergent everywhere else ■

Exercise 33.10
Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges if $x = -3$ and diverges if $x = 7$. Indicate which of the following statements must be true, cannot be true, or may be true.
(a) The power series converges if $x = -10$.

(b) The power series diverges if $x = 3$.
(c) The power series converges if $x = 6$.
(d) The power series diverges if $x = 2$.
(e) The power series diverges if $x = -7$.
(f) The power series converges if $x = -4$.

Solution.
The thing we know for sure is that the series converges for $|x| < 3$ and diverges for $|x| > 7$.
(a) Cannot be true.
(b) May be true.
(c) May be true.
(d) Cannot be true.
(e) May be true.
(f) May be true ■

Exercise 33.11
Give an example of a power series that converges on the interval $[-11, -3)$.

Solution.
An example is the series $\sum_{n=1}^{\infty} \frac{(x+7)^n}{n4^n}$ ■

Exercise 33.12
Determine all the values of the real number x for which the series

$$\sum_{n=1}^{\infty} \frac{x^n}{3^n n(\log(3n))^3}$$

converges

Solution.
The radius of convergence is

$$\left|\frac{a_n}{a_{n+1}}\right| = 3 \cdot \frac{n+1}{n} \left(\frac{\log(3n+3)}{\log 3n}\right)^3 \to 3 \; as \; n \to \infty.$$

So the series converges for all x satisfyin $|x| < 3$. If $x = -3$ the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(\log 3n)^3}$ which converges by the alernating series test. If $x = 3$ the series becomes $\sum_{n=1}^{\infty} \frac{1}{n(\log 3n)^3}$ which converges by the integral test. Thus, the given series converges on the interval $[-3, 3]$ and diverges elsewhere ■

Solutions to Section 34

Exercise 34.1
Find the Taylor series of $f(x) = \frac{1}{1-x}$, where $-1 < x < 1$.

Solution.
Finding successive derivatives we obtain

$$\begin{aligned} f'(x) =&(1-x)^{-2} & f'(0) = 1 \\ f''(x) =&2(1-x)^{-3} & f''(0) = 2 = 2! \\ f'''(x) =&3 \cdot 2(1-x)^{-4} & f'''(0) = 3! \\ &\vdots \\ f^{(n)}(x) =&n \cdot (n-1) \cdots 3 \cdot 2(1-x)^{-(n+1)} & f^{(n)}(0) = n! \end{aligned}$$

Thus,

$$P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}.$$

This is the nth partial sum of a geometric series that converges for $|x| < 1$. Moreover,

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

This shows that

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

for all $-1 < x < 1$ ■

Exercise 34.2
Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-\frac{2}{x^2}} & \text{if } x \neq 0 \end{cases}$$

(a) Find the Taylor polynomial of order n of f at $x = 0$.
(b) Show that $f(x) \neq \lim_{n\to\infty} P_n(x)$ for all x near 0. That is, the Taylor series of f about $x = 0$ does not converge to $f(x)$ for number very close to 0.

Solution.
(a) Using a graphing calculator we see that the derivative of f of any order is 0 at $x = 0$. Hence, for all $n \geq 0$ we have $P_n(x) = 0$ for x close to 0. Thus, for x close to 0 we have $f(x) = e^{-\frac{2}{x^2}} \neq 0 = \lim_{n\to\infty} P_n(x)$ ■

Exercise 34.3
(a) Show that the above result holds for $n = 0$. Hint: Apply the Fundamental Theorem of Caculus on the interval $[a, x]$.
(b) Suppose that the result holds for up to n. That is, for any $x \in [a, a+h]$ we can estimate $f(x)$ by $P_n(x)$ for x near a :

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x)$$

Suppose that f has continuous derivatives up to order $n+2$. Use integration by parts to show that

$$R_{n+1}(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + R_{n+2}(x).$$

Hence,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + R)n + 2(x).$$

Solution.
(a) By the Fundamental Theorem of Calculus we have $f(x) = f(a) + \int_a^x f'(t)dt$. Thus, the result is true for $n = 0$. (b) Suppose that the result holds up n. Suppose that $f^{(n+2)}$ exists and continuous. Using integration by parts we find

$$\begin{aligned}
\frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt = & -\left[\frac{f^{(n+1)}(t)}{(n+1)n!}(x-t)^{n+1}\right]_a^x + \frac{1}{n!}\int_a^x \frac{f^{(n+2)}(t)}{n+1}(x-t)^{n+1}dt \\
= & \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{1}{(n+1)!}\int_a^x f^{(n+2)}(t)(x-t)^{n+1}dt \\
= & \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + R_{n+2}(x)
\end{aligned}$$

■

Exercise 34.4 *(Lagrange's Form of Remainder)*
(a) Show that there exist $x_1, x_2 \in [a, x]$ such that $f^{(n+1)}(x_1) \leq f^{(n+1)}(t) \leq f^{(n+1)}(x_2)$ for all $t \in [a, x]$.
(b) Use (a) to show that

$$\frac{f^{(n+1)}(x_1)}{(n+1)!}(x-a)^{n+1} \leq R_{n+1}(x) \leq \frac{f^{(n+1)}(x_2)}{(n+1)!}(x-a)^{n+1}$$

where

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt.$$

(c) Show that

$$f^{(n+1)}(x_1) \leq R_{n+1}(x)\frac{(n+1)!}{(x-a)^{n+1}} \leq f^{(n+1)}(x_2).$$

(d) Show that there is a $c \in [a, x]$ such that

$$f^{(n+1)}(c) = R_{n+1}(x)\frac{(n+1)!}{(x-a)^{n+1}}$$

and therefore

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Solution.
(a) Since $f^{(n+1)}(t)$ is continuous on $[a, x]$, by Exercise 17.4 there exist $x_1, x_2 \in [a, x]$ such that

$$f^{(n+1)}(x_1) \leq f^{(n+1)}(t) \leq f^{(n+1)}(x_2) \text{ for all } t \in [a, x].$$

(b) This follows by multiplying the result in (a) by $(x-t)^n$ and then integrating from a to x.
(c) This follows by multiplying the result in (b) by the ration $\frac{(n+1)!}{(x-a)^{n+1}}$.
(d) This follows by applying the Intermediate Value Theorem to the function $f^{(n+1)}(t)$ on the interval $[a, x]$ ■

Exercise 34.5 *(Estimating $R_{n+1}(x)$)*
Suppose that there is $M > 0$ such that $|f^{(n+1)}(x)| \leq M$ for all $x \in [a, a+h]$.
(a) Show that for all $x \in [a, a+h]$ we have

$$|R_{n+1}(x)| \leq \frac{M}{(n+1)!|}|x-a|^{n+1}.$$

(b) Show that

$$\lim_{n\to\infty} R_{n+1}(x) = 0.$$

Hint: Exercise 1.14 and Squeeze rulw.

Solution.
(a) We have

$$|R_{n+1}(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right| \le \frac{M}{(n+1)!|}|x-a|^{n+1}.$$

(b) Since

$$\lim_{n\to\infty} \frac{(x-a)^{n+1}}{(n+1)!} = 0$$

and

$$-M\frac{(x-a)^{n+1}}{(n+1)!} \le R_{n+1}(x) \le M\frac{(x-a)^{n+1}}{(n+1)!}$$

the result follows from the squeeze rule ■

Exercise 34.6
Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that f, f', f'' exist and are continuous. Furthermore, $f \ge 0$ and $f'' \le 0$. Show that f is a constant function.

Solution.
Let $a \in \mathbb{R}$. Using Taylor Theorem we can write

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$$

for some c between a and x. But $f'' \le 0$ and $f \ge 0$ so that the last equality implies that

$$f(a) + f'(a)(x-a) = f(x) - \frac{f''(c)}{2}(x-a)^2 \ge f(x) \ge 0$$

for all $x \in \mathbb{R}$. But this true only when $f'(a) = 0$ and $f(a) \ge 0$. Since a was arbitrary, we conclude that $f'(x) = 0$ for all $x \in \mathbb{R}$. Hence, f is a constant function ■

Exercise 34.7
Find the Taylor polynomial of order n about 0 for $f(x) = e^x$, and write down the corresponding remainder term.

Solution.
Since the derivative of f of any order is just e^x we find

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_{n+1}(x)$$

where

$$R_{n+1}(x) = \frac{e^c}{(n+1)!}x^{n+1} \blacksquare$$

Exercise 34.8
Find the Taylor Polynomial of order 3 for the function $f(x) = \cos x$ centered at $x = \frac{\pi}{6}$.

Solution.
Simple calculation leads to

$$P_4(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{6}\right)^2 + \frac{1}{12}\left(x - \frac{\pi}{6}\right)^3 \blacksquare$$

Exercise 34.9
Find the Lagrange form of the remainder $R_{n+1}(x)$ for the function $f(x) = \frac{1}{1+x}$.

Solution.
By successive differentiation we find $f^{(n+1)}(x) = (-1)^{n+1}(n+1)!(1+x)^{-(n+2)}$. Hence,

$$R_{n+1}(x) = \frac{(-1)^{n+1}}{(1+c)^{n+2}}x^{n+1} \blacksquare$$

Exercise 34.10
Let $g(x)$ be a function such that $g(5) = 3, g'(5) = -1, g''(5) = 1$ and $g'''(5) = -3$.
(a) What is the Taylor polynomial of degree 3 for $g(x)$ near 5?
(b) Use (a) to approximate $g(4.9)$.

Solution.
(a) We have: $c_0 = g(5) = 3, c_1 = g'(5) = -1, c_2 = \frac{g''(5)}{2!} = \frac{1}{2}$, and $c_3 = \frac{g'''(5)}{3!} = -\frac{1}{2}$. Thus, $P_3(x) = 3 - (x-5) + \frac{1}{2}(x-5)^2 - \frac{1}{2}(x-5)^3$.
(b) $g(4.9) = 3 - (4.9-5) + \frac{1}{2}(4.9-5)^2 - \frac{1}{2}(4.9-5)^3 = 3.1675.\blacksquare$

Exercise 34.11
Suppose that the function $f(x)$ is approximated near $x = 0$ by a sixth degree Taylor polynomial

$$P_6(x) = 3x - 4x^3 + 5x^6.$$

Find the value of the following:
(a) $f(0)$ (b) $f'(0)$ (c) $f'''(0)$ (d) $f^{(5)}(0)$ (e) $f^{(6)}(0)$

Solution.
If

$$P_6(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6$$

then $c_0 = 0, c_1 = 3, c_2 = 0, c_3 = -4, c_4 = c_5 = 0$, and $c_6 = 5$.

(a) $f(0) = c_0 = 0$, (b) $f'(0) = c_1 = 3$, (c) $f'''(0) = 3!c_3 = -24$, (d) $f^{(5)}(0) = 5!c_5 = 0$, (e) $f^{(6)}(0) = 6!c_6 = 3600$.■

Exercise 34.12
Find the third degree Taylor polynomial approximating

$$f(x) = \arctan x,$$

near $a = 0$.

Solution.
We have

$$\begin{aligned} f(x) =& \arctan x, & c_0 &= f(0) = 0 \\ f'(x) =& \frac{1}{1+x^2}, & c_1 &= f'(0) = 1 \\ f''(x) =& -(1+x^2)^{-2}(2x), & c_2 &= \frac{f''(0)}{2!} = 0 \\ f'''(x) =& -2(1+x^2)^{-3}(4x^2) - 2(1+x^2)^{-2}, & c_3 &= \frac{f'''(0)}{3!} = -\frac{1}{3} \end{aligned}$$

Thus,

$$P_3(x) = x - \frac{1}{3}x^3.■$$

Exercise 34.13
Find the fifth degree Taylor polynomial approximating

$$f(x) = \ln(1+x),$$

near $a = 0$.

Solution.
We have

$$\begin{array}{rclrcrcr} f(x) & = & \ln(1+x), & c_0 & = & f(0) & = & 0 \\ f'(x) & = & \frac{1}{1+x}, & c_1 & = & f'(0) & = & 1 \\ f''(x) & = & -\frac{1}{(1+x)^2}, & c_2 & = & \frac{f''(0)}{2!} & = & -\frac{1}{2} \\ f'''(x) & = & \frac{2}{(1+x)^3}, & c_3 & = & \frac{f'''(0)}{3!} & = & \frac{1}{3} \\ f^{(4)}(x) & = & -\frac{6}{(1+x)^4}, & c_4 & = & \frac{f^{(4)}(0)}{4!} & = & -\frac{1}{4} \\ f^{(5)}(x) & = & \frac{24}{(1+x)^5}, & c_5 & = & \frac{f^{(5)}(0)}{5!} & = & \frac{1}{5} \end{array}$$

Thus,

$$P_5(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5. \blacksquare$$

Solutions to Section 35

Exercise 35.1
Let $f(x) = \cos x$.
(a) Using successive differentiation find a formula for $f^{(n)}(0)$.
(b) Show that

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}.$$

(c) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

(d) Show that

$$|R_{n+1}(x)| \le \frac{|x|^{n+1}}{(n+1)!}.$$

(e) Show that $\lim_{n\to\infty} R_{n+1}(x) = 0$. Hence, conclude that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

Solution.
(a) By successive differentiation we find

$$\begin{array}{rcl} f'(x) & = & -\sin x \\ f''(x) & = & -\cos x \\ f'''(x) & = & \sin x \\ f^{(4)}(x) & = & \cos x \\ \vdots & & \end{array}$$

We see that the derivatives go through a cycle of length 4 and then repeat that cycle forever. It follows that

$$f^{(k)}(0) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ (-1)^{\frac{k}{2}} & \text{if } k \text{ is even.} \end{cases}$$

(b) Hence,

$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} = \sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}.$$

(c) Now, consider the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

and let $a_n = \frac{(-1)^n}{(2n)!}$. Then

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0 < 1.$$

This shows that the series is convergent for all values of x.
(d) Since $|f^{(n+1)}(c)| \le 1$ we find that

$$|R_{n+1}(x)|\frac{|x|^{n+1}}{(n+1)!}.$$

(e) But

$$\lim_{n\to\infty}\frac{x^{n+1}}{(n+1)!} = 0$$

so that $\lim_{n\to\infty} R_{n+1}(x) = 0$. This implies that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots \blacksquare$$

Exercise 35.2
Let $f(x) = e^x$.
(a) Find $f^{(n)}(0)$ for all $n \ge 0$.
(b) Find an expression for $P_n(x)$.
(c) Consider the series

$$\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots.$$

Find the radius of convergence.
(d) Find an expression for $R_{n+1}(x)$ and show that

$$\lim_{n\to\infty} R_{n+1}(x) = 0.$$

Hence, conclude that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Solution.
(a) For all nonnegative integer k we have $f^{(k)}(x) = e^x$ and $f^{(k)}(0) = 1$.
(b) Thus, the nth Taylor polynomial is given by the expression

$$P_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

(c) Let $a_n = \frac{1}{n!}$. Then by the ratio test we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

Thus, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all values of x.
(d) It remains to show that the series converges to e^x. For that, we need to use Taylor Theorem. Write $f(x) = P_n(x) + E_n(x)$ where

$$|E_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}\right| = \frac{e^c}{(n+1)!}|x|^{n+1}$$

(e) Since $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ we find

$$\lim_{n\to\infty} R_{n+1}(x) = 0.$$

Hence,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \blacksquare$$

Exercise 35.3
Let $f(x) = \ln(1+x)$.
(a) Find $f^{(n)}(0)$ for all $n \geq 0$.

(b) Find an expression for $P_n(x)$.
(c) Consider the series

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots.$$

Find the radius of convergence.
(d) Show that

$$|R_{n+1}(x)| \leq \frac{1}{|1+c|^{n+1}} \cdot \frac{|x|^{n+1}}{(n+1)!}.$$

(e) Show that

$$\lim_{n\to\infty} R_{n+1}(x) = 0.$$

Hence, conclude that

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}, \quad -1 < x \leq 1.$$

Solution.
(a) Taking derivatives:

$$\begin{aligned}
f'(x) =&(1+x)^{-1} & f'(0) &= 1 = 0!\\
f''(x) =&-(1+x)^{-2} & f''(0) &= -1!\\
f'''(x) =&2(1+x)^{-3} & f'''(0) &= 2!\\
f^{(4)}(x) =&-6(1+x)^{-4} & f^{(4)}(0) &= -3!\\
\vdots & & &\\
f^{(n)}(x) =&(-1)^{n-1}(n-1)!(1+x)^{-n} & f^{(n)}(0) &= (-1)^{n-1}(n-1)!
\end{aligned}$$

(b) Hence,

$$P_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} = \sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k}.$$

(c) Letting $a_n = (-1)^{n-1}\frac{1}{n}$ and applying the ratio test we find that

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1.$$

Hence, the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ converges for all $-1 < x < 1$. By the alternating series test we know that $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent so the interval of convergence of the previous series is $-1 < x \le 1$.
(d) It remains to show that the series converges to $\ln(1+x)$. Using Taylor theorem, we can write $f(x) = P_n(x) + R_{n+1}(x)$, where

$$|R_{n+1}(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \le \frac{1}{|1+c|^{n+1}} \cdot \frac{|x|^{n+1}}{(n+1)}.$$

(e) Since $\lim_{n\to\infty}\frac{x^n}{n!} = 0$ we find

$$\lim_{n\to\infty} R_{n+1}(x) = 0.$$

Hence,

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}, \quad -1 < x \le 1 \blacksquare$$

Exercise 35.4
Find the Taylor series of $\frac{x}{e^x}$ about $x = 0$.

Solution.
Replacing x by $-x$ in the expansion of e^x we find

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \cdots + (-1)^n\frac{x^n}{n!} + \cdots$$

Now, multiplying both sides of this equality by x to obtain

$$\frac{x}{e^x} = xe^{-x} = x - x^2 + \frac{x^3}{2!} + \cdots + (-1)^n\frac{x^{n+1}}{n!} + \cdots$$

This series converges for all x $\blacksquare$

Exercise 35.5
Find the Taylor series of $f(x) = \frac{1}{1+x^2}$ about $x = 0$.

Solution.
Replacing x by $-x^2$ in Formula $(1-x)^{-1}$ we can write

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

This series converges for $-1 < x < 1$. $\blacksquare$

Exercise 35.6
Let $f(x) = \sin x$.
(a) Using successive differentiation find a formula for $f^{(n)}(0)$.
(b) Show that

$$P_{2n}(x) = P_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}.$$

(c) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots .$$

(d) Show that

$$|R_{n+1}(x)| \le \frac{|x|^{n+1}}{(n+1)!}.$$

(e) Show that $\lim_{n\to\infty} R_{n+1}(x) = 0$. Hence, conclude that

$$\sin x = \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}.$$

Solution.
(a) By successive differentiation we find

$$\begin{array}{rcl} f'(x) & = & \cos x \\ f''(x) & = & -\sin x \\ f'''(x) & = & -\cos x \\ f^{(4)}(x) & = & \sin x \\ \vdots & & \end{array}$$

We see that the derivatives go through a cycle of length 4 and then repeat that cycle forever. It follows that

$$f^{(k)}(0) = \begin{cases} 0 & \text{if } k \text{ is even} \\ (-1)^{\frac{k-1}{2}} & \text{if } k \text{ is odd.} \end{cases}$$

(b) Hence, for $n \ge 1$,

$$P_{2n}(x) = P_{2n1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}.$$

(c) Let $a_n = \frac{(-1)^n}{(2n+1)!}$. Then

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)} = 0 < 1.$$

This shows that the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$ is convergent for all values of x.
(d) Since $|f^{(n+1)}(c)| \le 1$ we find that

$$|R_{n+1}(x)|\frac{|x|^{n+1}}{(n+1)!}.$$

(e) But

$$\lim_{n\to\infty}\frac{x^{n+1}}{(n+1)!} = 0$$

so that

$$\lim_{n\to\infty} R_{n+1}(x) = 0.$$

This implies that

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \blacksquare$$

Exercise 35.7
Find the MacLaurin series of $\frac{x}{1-2x}$.

Solution.
We have

$$\frac{x}{1-2x} = x\sum_{n=0}^{\infty}(2x)^n = \sum_{n=1}^{\infty}4^{n-1}x^n$$

which is valid for all $-\frac{1}{2} < x < \frac{1}{2}$ $\blacksquare$

Exercise 35.8
Find the coefficient of $(x-2)^2$ in the Taylor series expansion of $f(x) = \frac{1}{x}$ about $x = 2$.

Solution.
The coefficient is $\frac{f''(2)}{2!} = \frac{1}{8}$ $\blacksquare$

Exercise 35.9
Find the Maclaurin series for the function $f(x) = x^6e^{-x^2}$. Give your answer in sigma notation.

Solution.
We have

$$x^6e^{-x^2} = x^6\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+6}}{n!} \blacksquare$$

Exercise 35.10
Compute each of the following sums in terms of known functions:
(a) $\sum_{n=0}^{\infty}\frac{(-1)^nx^{4n+1}}{n!}$
(b) $\sum_{n=1}^{\infty}\frac{(-1)^nx^{4n+1}}{(2n+1)!}$
(c) $\sum_{n=0}^{\infty}\frac{(-1)^nx^{6n}}{(2n+2)!}$

Solution.
(a) $\sum_{n=0}^{\infty}\frac{(-1)^nx^{4n+1}}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^nx^{4n}}{n!} = xe^{-x^4}$.
(b) $\sum_{n=1}^{\infty}\frac{(-1)^nx^{4n+1}}{(2n+1)!} = \frac{1}{x}\sum_{n=1}^{\infty}\frac{(-1)^n(x^2)^{2n+1}}{(2n+1)!} = \frac{\sin x^2-x^2}{x}$.
(c) $\sum_{n=0}^{\infty}\frac{(-1)^nx^{6n}}{(2n+2)!} = -\frac{1}{x^6}\sum_{n=1}^{\infty}\frac{(-1)^n(x^3)^{2n}}{(2n)!} = \frac{1-\cos x^3}{x^6} \blacksquare$

Exercise 35.11
The hyperbolic cosine of x is defined to be the function $\cosh x = \frac{e^x+e^{-x}}{2}$. Find the MacLaurin series of $\cosh x$.

Solution.
Using the Taylor expansion of e^x and e^{-x} one can easily see that

$$\cosh x = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} \blacksquare$$

Exercise 35.12
The hyperbolic sine of x is defined to be the function $\sinh x = \frac{e^x-e^{-x}}{2}$. Find the MacLaurin series of $\sinh x$.

Solution.
Using the Taylor expansion of e^x and e^{-x} one can easily see that

$$\sinh x = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} \blacksquare$$

Exercise 35.13 *(Binomial Series)*
Consider the function $f(x) = (1+x)^n$ where $n \in \mathbb{R}$.
(a) Using successive differentiation show that $f^{(k)}(0) = k(k-1)\cdots(k-n+1)$.
Thus, $\frac{f^{(k)}(0)}{k!} = C(n,k)$ where

$$C(n,k) = \tfrac{n!}{k!(n-k)!} \text{ and } C(n,0) = 1.$$

(b) Find the interval of convergence of the binomial series $(1+x)^n = \sum_{k=0}^{\infty} C(n,k)x^k$.

Solution.
(a) This is done by successive differentiation.
(b) Using the absolute ration test we find

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|n-k|}{k+1}|x| \to |x| \ as\ k \to \infty.$$

Thus, the interval of convergence if $|x| < 1$. Convergence at the endpoints depends on the values of n and needs to be checked every time ■

Exercise 35.14
Find the MacLaurin series of $f(x) = \frac{1}{\sqrt{x+1}}$.

Solution.
We have

$$(1+x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} C(-\frac{1}{2},k)x^k = 1 - \frac{1}{2}x + \frac{(1)(3)}{2^2 2!}x^2 - \frac{(1)(3)(5)}{2^3 3!}x^3 + \cdots \quad ■$$

Solutions to Section 36

Exercise 36.1
Suppose that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on D. For each $x \in D$ let $f(x) = \sum_{n=1}^{\infty} f_n(x)$. That is, $\{S_n\}_{n=1}^{\infty}$ converges uniformly to f.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that if $n \geq N$ we have

$$\left|\sum_{k=1}^{n} f_k(x) - f(x)\right| < \frac{\epsilon}{2}$$

for all $x \in D$.
(b) Show that for $n > m \geq N$ we have

$$\left|\sum_{k=m+1}^{n} f_k(x)\right| = \left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\right| < \epsilon$$

for all $x \in D$.

Solution.
(a) Since the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent, the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ is uniformly convergent, say to a function f. Thus for a given $\epsilon > 0$, there is a positive integer N such that for $n \geq N$ we have

$$|\textstyle\sum_{k=1}^{n} f_k(x) - f(x)| < \frac{\epsilon}{2} \text{ for all } x \in D.$$

(b) If $n > m \geq N$ then we have

$$\begin{aligned}\left|\sum_{k=m+1}^{n} f_k(x)\right| =& \left|\sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{m} f_k(x)\right| \\ \leq& \left|\sum_{k=1}^{n} f_k(x) - f(x)\right| + \left|\sum_{k=1}^{m} f_k(x) - f(x)\right| \\ <& \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \blacksquare\end{aligned}$$

Exercise 36.2 *(Weierstrass)*
For each integer $n \geq$, let $f_n : D \to \mathbb{R}$ be a continuous function that is bounded on D with $|f_n(x)| \leq M_n$ for all $x \in D$. Suppose that the series

of numbers $\sum_{n=1}^{\infty} M_n$ is convergent. For each positive integer n define the partial sum

$$S_n(x) = \sum_{k=1}^{n} f_k(x).$$

(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for all $m, n \geq N$ we have

$$\left|\sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k\right| < \epsilon.$$

Hint: The sequence $\{\sum_{k=1}^{n} M_k\}_{n=1}^{\infty}$ is Cauchy.
(b) Suppose that $n > m \geq N$. By (a) we have $|\sum_{k=m+1}^{n} M_k| < \epsilon$. Show that for all $x \in D$ we have

$$|S_n(x) - S_m(x)| < \epsilon$$

Hence, the sequence $\{S_n\}_{n=1}^{\infty}$ is uniformly Cauchy.
(c) Conclude that the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent. Hint: Exercise 32.15

Solution.
(a) Since the series $\sum_{n=1}^{\infty} M_n$ is convergent, the sequence $\{\sum_{k=1}^{n} M_k\}_{n=1}^{\infty}$ is convergent and hence it is Cauchy. Thus, for a fixed ϵ we can find a positive integer N such that for $m, n \geq N$ we have

$$\left|\sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k\right| < \epsilon.$$

(b) We have $|S_n(x) - S_m(x)| = \left|\sum_{k=m+1}^{n} f_k(x)\right| \leq \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=m+1}^{n} M_k = |\sum_{k=m+1}^{n} M_k| < \epsilon$. Hence, the sequence $\{S_n\}_{n=1}^{\infty}$ is uniformly Cauchy.
(c) This follows from Exercise 32.15 ■

Exercise 36.3
Use Weierstrass M test to show that the series $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$ converges uniformly on $[-2, 2]$.

Solution.
Note first that for all $x \in [-2, 2]$ we have $\left|\frac{x^n}{3^n}\right| = \left|\frac{x}{3}\right|^n \leq \left(\frac{2}{3}\right)^n$. Let $f_n(x) = \frac{x^n}{3^n}$ and $M_n = \left(\frac{2}{3}\right)^{\cdot}$ The series $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series. Hence, by Weierstrass M test the given series is uniformly convergent in $[-2, 2]$ ■

Exercise 36.4
Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Let $0 < c < R$ and $D = [-c, c]$.
(a) Define $f_n(x) = a_n x^n$ and $M_n = |a_n c^n|$. Clearly, f_n is continuous in D and $M_n > 0$ for all integer $n \geq 0$. Show that $\sum_{n=0}^{\infty} M_n$ converges. Hint: Exercise 33.1(f)
(b) Let $x \in D$. Show that if $x \in [0, c]$ then $|g_n(x)| \leq M_n$. Hint: x^n is increasing for $x \geq 0$.
(c) Answer the same question if $x \in [-c, 0]$. (d) Conclude that the series is uniformly convergent on D.

Solution.
(a) Since $0 < c < R$ the series $\sum_{n=0}^{\infty} a_n c^n$ is absolutely convergent by Exercise 33.1(f). That is, the series $\sum_{n=0}^{\infty} M_n$ is convergent.
(b) If $x \in [0, c]$ then $x \geq 0$ and $(x^n)' \geq 0$ so that x^n is increasing in $[0, c]$. Since $0 \leq x \leq c$ w conclude that $|f_n(x)| = |a_n||x|^n \leq |a_n|c^n = M_n$.
(c) If $x \in [-c, 0]$ then $-x \in [0, c]$ so that $|f_n(x)| = |a_n||x|^n = |a_n|| - x|^n \leq |a_n|c^n = M_n$.
(d) This follows from Weiestrass M test ■

Exercise 36.5
Show that the following series converges uniformly.

$$\sum_{n=0}^{\infty} \frac{x^2}{3^n(x^2+1)}.$$

Solution.
We have

$$0 \leq \frac{x^2}{3^n(x^2+1)} \leq \frac{1}{3^n}.$$

Since the series $\sum_{n=0}^{\infty} \frac{1}{3^n}$ converges, the given series converges uniformly by the Weierstrass M test ■

Exercise 36.6
Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence with $|a_n| \leq M$ for all $n \in \mathbb{N}$. Show that ther series $\sum_{n=1}^{\infty} \frac{a_n}{n^x}$ converges uniformly for all $x \geq c > 1$.

Solution.
We use the Weierstrass M-test. For $x \geq c$ we have $\left|\frac{a_n}{n^x}\right| \leq \frac{M}{n^c}$. Because $c > 1$, the series $\sum_{n=1}^{\infty} \frac{M}{n^c}$ converges (p-series with $p > 1$). Thus, by the Weierstrass M=test, the given series converges uniformly and absolutely for $x \geq c$ ■

Exercise 36.7
Show that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges uniformly for all $x \in \mathbb{R}$.

Solution.
We have

$$0 \leq \left|\frac{\sin nx}{n^2}\right| \leq \frac{1}{n^2}$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, the Weierstrass M-test asserts that the given series converges uniformly for all $x \in \mathbb{R}$.

Exercise 36.8
Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions defined on a set D such that $|f_{n+1}(x) - f_n(x)| \leq M_n$ for all $x \in D$ and $n \in \mathbb{N}$. Assume that $\sum_{n=1}^{\infty} M_n$ is convergent. Show that the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on D.

Solution.
By the Weierstrass M-Test, we know that the series $\sum_{n=1}^{\infty}(f_{n+1}(x) - f_n(x))$ is uniformly convergent on D say to a function f. The nth partial sum of this series is $f_{n+1}(x) - f_1(x)$. Hence, $f_n \to f + f_1$ uniformly on D ■

Exercise 36.9
Show that the series $\sum_{n=1}^{\infty} \frac{x}{(1+x)^n}$ converges uniformly on $[1, 2]$.

Solution.
We have $x \leq 2 \to 3x \leq 2 + 2x \to \frac{x}{1+x} \leq \frac{2}{3}$. Thus,

$$\frac{x}{(1+x)^n} \leq \frac{x^n}{(1+x)^n} \leq \left(\frac{2}{3}\right)^n.$$

Since the series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges, by the Weierstrass M-test the given series is uniformly convergent on $[1, 2]$ ■

Exercise 36.10
Prove that $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ converges uniformly on any bounded interval $[a, b]$.

Solution.
We have

$$\left|\sin\left(\frac{x}{n^2}\right)\right| \leq \frac{|x|}{n^2} \leq \frac{M}{n^2}$$

where $M = |a|+|b|$. Since the series $\sum_{n=1}^{\infty} \frac{M}{n^2}$ is convergent, by the Weierstrass M-test the given series is uniformly convergent ■

Exercise 36.11
Show that the series $\sum_{n=1}^{\infty} \frac{1}{3^n} \cos\left(\frac{x}{3^n}\right)$ converges uniformly on $\mathbb{R}$.

Solution.
We have

$$\frac{1}{3^n} \cos\left(\frac{x}{3^n}\right) \leq \frac{1}{3^n}.$$

The result now follows from Weierstrass M-test ■

Solutions to Section 37

Exercise 37.1
Let $c \in D$. Let $R_0 > 0$ be a number such that $|c-a| < R_0 < R$. By Exercise 36.4, the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges uniformly on the interval $[a-R_0, a+R_0]$.
(a) Let $\epsilon > 0$ be given. Show that there is a positive integer N such that for all $n > m \geq N$ we have

$$\left|\sum_{k=0}^{n} a_k(x-a)^k - \sum_{k=0}^{n} a_k(x-a)^k\right| = \left|\sum_{k=m+1}^{n} a_k(x-a)^k\right| < \tfrac{\epsilon}{3} \text{ for all } x \in [a-R_0, a+R_0].$$

Hint: Exercise 36.1
(b) Show that there is a $\delta_1 > 0$ such that if $|x-a| < \delta_1$ then

$$\left|\sum_{k=0}^{N} a_k(x-a)^k - \sum_{k=0}^{N} a_k(c-a)^k\right| < \frac{\epsilon}{3}.$$

(c) Let $\delta = \min\{\delta_1, R_0 - |c-a|\}$. Show that for $|x-a| < \delta$ we have

$$|f(x) - f(c)| < \epsilon.$$

Hence, the function $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is continuous on D.

Solution.
(a) This follows from Exercise 36.1.
(b) Since $\sum_{k=0}^{N} a_k(x-a)^k$ is a polynomial, it is continuous at c.
(c) Suppose that $|x-a| < \delta$. Then $|f(x)-f(c)| = \left|\left(\sum_{k=0}^{N} a_k(x-a)^k + \sum_{k=N+1}^{\infty} a_k(x-a)^k\right) - \left(\sum_{k=}^{N}\right.\right.$
$\left|\sum_{k=0}^{N} a_k(x-a)^k - \sum_{k=0}^{N} a_k(c-a)^k\right| + \left|\sum_{k=N+1}^{\infty} a_k(x-a)^k\right| + \left|\sum_{k=N+1}^{\infty} a_k(c-a)^k\right| <$
$\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. Hence, the function $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is continuous at c. Since c was arbitrary, $f(x)$ is continuous in D ■

Exercise 37.2
Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ where the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$. Let $F(x) = \int_a^x f(t)dt$. Suppose that $a - R < x \leq a$. A similar result holds for $a \leq x < a + R$. (a) Show that $\{S_n\}_{n=1}^{\infty}$ converges uniformly to f on $[x, a]$.
(b) Evaluate $\int_x^a S_n(t)dt$.
(c) Show that the power series $\sum_{n=0}^{\infty} \frac{a_n(x-a)^{n+1}}{n+1}$ has radius of convergence R.
(d) Show that $F(x) = \sum_{n=0}^{\infty} \frac{a_n(x-a)^{n+1}}{n+1}$. Hint: Exercise 32.11

Solution.
(a) See Remark **??** (2).
(b) By integration we have

$$\int_x^a S_n(t)dt = \int_x^a \left(\sum_{k=0}^n a_k(t-a)^k\right) dt = \left[\sum_{k=0}^n \frac{a_k(t-a)^{k+1}}{k+1}\right]_x^a = -\sum_{k=0}^n \frac{a_k(x-a)^{k+1}}{k+1}.$$

(c) Let R' be the radius of convergence of $\sum_{n=0}^\infty \frac{a_n(x-a)^{n+1}}{n+1}$. We have

$$\begin{aligned}\frac{1}{R'} &= \lim_{n\to\infty}\left|\frac{\frac{a_{n+1}(x-a)^{n+2}}{n+2}}{\frac{a_n(x-a)^{n+1}}{n+1}}\right| \\ &= \lim_{n\to\infty}\frac{n+1}{n+2}\cdot\lim_{n\to\infty}\left|\frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n}\right| = \frac{1}{R}\end{aligned}$$

Hence, $R' = R$.
(d) By Exercise 32.11 limit and integration can be interchanged. Hence, we have $F(x) = \int_a^x f(t)dt = -\int_x^a \lim_{n\to\infty} S_n(t)dt = -\lim_{n\to\infty}\int_x^a S_n(t)dt = \lim_{n\to\infty}\sum_{k=0}^n \frac{a_k(x-a)^{k+1}}{k+1} = \sum_{n=0}^\infty \frac{a_n(x-a)^{n+1}}{n+1}$ ■

Exercise 37.3
Let $f(x) = \sum_{n=0}^\infty a_n(x-a)^n$ where the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$.
(a) Show that the power series $g(x) = \sum_{n=1}^\infty na_n(x-a)^{n-1}$ has radius of convergence R.
(b) Let $G(x) = \int_a^x g(t)dt$. Show that $G(x) = f(x) - a_0$ for $|x-a| < R$. Hint: Exercise 37.2.
(c) Show that $g(x) = f'(x)$ for all $|x-a| < R$. Hint: Exercise 25.2

Solution.
(a) Let R' be the radius of convergence of $g(x)$. We have

$$\begin{aligned}\frac{1}{R'} &= \lim_{n\to\infty}\left|\frac{(n+1)a_{n+1}(x-a)^n}{na_n(x-a)^{n-1}}\right| \\ &= \lim_{n\to\infty}\frac{n+1}{n}\cdot\lim_{n\to\infty}\left|\frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n}\right| = \frac{1}{R}\end{aligned}$$

Hence, $R' = R$.
(b) Integrating term-by-term we find $G(x) = \sum_{n=1}^\infty a_n(x-a)^n = f(x) - a_0$.
(c) By Exercise 25.2, we have $g(x) = G'(x) = f'(x)$ ■

Exercise 37.4
Show that $\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$ has radius of convergence 2 and show that the series converges uniformly to a continuous function on $[-2, 2]$.

Solution.
Using the absolute ratio test we find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n}{n+1}\right)^2 \left|\frac{x}{2}\right| \to \left|\frac{x}{2}\right| \ as\ n \to \infty.$$

Thus, the series converges for $|x| < 2$ so that the radius of convergence is 2. Now,

$$\left|\frac{x^n}{n^2 2^n}\right| \leq \frac{2^n}{n^2 2^n} = \frac{1}{n^2}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, the given series converges uniformly on $[-2, 2]$. Since each function $f_n(x) = \frac{x^n}{n^2 2^n}$ is continuous, by Exercise **??**, the series converges to a continuous function ■

Exercise 37.5
Let $g(x) = \sum_{n=1}^{\infty} \frac{\sin(3x)}{3^n}$. Prove that the series converges for all $x \in \mathbb{R}$ and that $g(x)$ is continuous everywhere.

Solution.
Since

$$0 \leq \left|\frac{\sin(3x)}{3^n}\right| \leq \frac{1}{3^n}$$

and the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent, the given series converges uniformly for all $x \in \mathbb{R}$. By Exercise 37.1, $g(x)$ is continuous everywhere ■

Exercise 37.6
Show that $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$ converges to a continuous function for all $x \in \mathbb{R}$.

Solution.
Notice that $x^2 \geq 0$ always. So $n^2 + x^2 \geq n^2 + 1$ which gives $0 < \frac{1}{n^2+x^2} \leq \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by Weierstrass M-test, the given series converges uniformly for all $x \in \mathbb{R}$.
Recall a theorem saying that if a sequence of continuous function converges uniformly, the limit function is also continuous (Exercise **??**). Using this and observe that $f_n(x) = \frac{1}{n^2+x^2}$ are continuous (as denominator are non-zero), we have $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$ converges to a continuous function for all real number x ■

Exercise 37.7
Find the Taylor series about $x = 0$ of $\cos x$ from the series of $\sin x$.

Solution.
We know that $\frac{d}{dx}(\sin x) = \cos x$, so we start the Taylor series of $\sin x$ and differentiate this series term by term we get the series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \blacksquare$$

Exercise 37.8
Find the Taylor's series about $x = 0$ for $\arctan x$ from the series for $\frac{1}{1+x^2}$.

Solution
Integrating term by term of the series $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ we find

$$\arctan x = \int \frac{dx}{1+x^2} = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

where $-1 < x < 1$. Since $\arctan 0 = 0$ then $C = 0$ and therefore

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \blacksquare$$

Exercise 37.9
Use the first 500 terms of series of $\arctan x$ and a calculator to estimate the numerical value of π.

Solution.
Substituting $x = 1$ in to the series for $\arctan x$ gives

$$\pi = 4 \arctan 1 = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots)$$

By using 500 terms of this series one finds

$$\pi \approx 3.140. \blacksquare$$

Exercise 37.10
Estimate the value of $\int_0^1 \sin(x^2)dx$.

Solution.
The integrand has no antiderivative expressible in terms of familiar functions. However, we know how to find its Taylor series: we know that

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots$$

Now if we substitute $t = x^2$, we have

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

In spite of the fact that we cannot antidifferentiate the function, we can antidifferentiate the Taylor series:

$$\begin{aligned}
\int_0^1 \sin(x^2)dx &= \int_0^1 (x^2 - \tfrac{x^6}{3!} + \tfrac{x^5}{5!} - \cdots)dx \\
&= (\tfrac{x^3}{3} - \tfrac{x^7}{7\cdot 3!} + \tfrac{x^{11}}{11\cdot 5!} - \cdots)|_0^1 \\
&= \tfrac{1}{3} - \tfrac{1}{7\cdot 3!} + \tfrac{1}{11\cdot 5!} - \tfrac{1}{15\cdot 7!} + \cdots \\
&\approx 0.31026 \blacksquare
\end{aligned}$$