Solutions to Practice Problems

Exercise 3.7
Consider the set $A = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}$.
(a) Show that $A$ is bounded from above. Find the supremum. Is this supre-
mum a maximum of $A$?
(b) Show that $A$ is bounded from below. Find the infimum. Is this infimum
a minimum of $A$?

Solution.
(a) Clearly, $\frac{1}{2}$ is an upper bound of $A$. Let $M > 0$ be an upper bound of
$A$. We will show that $\frac{1}{2} \leq M$. Suppose the contrary. That is, suppose that
$M < \frac{1}{2}$. Since $M$ is an upper bound of $A$, we have $\frac{(-1)^n}{n} \leq M$ for all $n \in \mathbb{N}$.
In particular, letting $n = 2$ we obtain $\frac{1}{2} \leq M < \frac{1}{2}$ which is imposssibe. Thus,
$\frac{1}{2} \leq M$ so that $\sup\{A\} = \frac{1}{2}$. Since the supremum is an element of $A$ we
conclude that $\frac{1}{2}$ is also the maximum of $A$.
(b) Clearly, $-1$ is a lower bound of $A$. Let $m$ be a lower bound of $A$. We will
show that $m \leq -1$. Suppose the contrary. That is, suppose that $m > -1$.
Letting $n = 1$ we find that $-1 = \frac{(-1)^1}{1} \geq m > -1$, which is impossible.
Therefore, we must have $m \leq -1$. This establishes that $\inf\{A\} = -1$. Since
$-1$ is in $A$, it is the minimum of $A$.

Exercise 3.8
Consider the set $A = \{x \in \mathbb{R} : 1 < x < 2\}$.
(a) Show that $A$ is bounded from above. Find the supremum. Is this supre-
mum a maximum of $A$?
(b) Show that $A$ is bounded from below. Find the infimum. Is this infimum
a minimum of $A$?

Solution.
(a) Clearly, 2 is an upper bound of $A$. Let $M > 1$ be an upper bound of
$A$. We will show that $2 \leq M$. Suppose the contrary. That is, suppose that
$1 < M < 2$. Let $r$ be a rational number such that $M < r < 2$. Then $r \in A$
and $M < r$ which contradicts the fact that $M$ is an upper bound of $A$. Hence,
we must have $2 \leq M$ so that $\sup\{A\} = 2$. Since the supremum is not an
element of $A$ we conclude that 2 is not a maximum of $A$.
(b) Clearly, 1 is a lower bound of $A$. Let $m$ be a lower bound of $A$. We will
show that $m \leq 1$. Suppose the contrary. That is, suppose that $1 < m < 2$. 
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Let $r$ be a rational number such that $1 < r < m$. Then $r \in A$ and $r < m$ which contradicts the fact that $m$ is a lower bound of $A$. Thus, we must have $1 \leq m$ so that $\inf \{A\} = 1$. Since 1 is not in $A$, it is not a minimum of $A$.

Exercise 3.9
Consider the set $A = \{x > 0 : x^2 > 4\} = \{x > 0 : x > 2\}$.
(a) What is a lower bound of $A$?
(b) Let $L$ be a lower bound of $A$ such that $L > 2$. Let $y = \frac{L+2}{2}$. Show that $2 < y < L$.
(c) Show that $y \in A$ and $L \leq y$. Show that this leads to a contradiction. Hence, we must have $L \leq 2$ which means that 2 is the infimum of $A$.

Solution.
(a) Since $2 \leq x$ for all $x \in A$, 2 is a lower bound of $A$.
(b) Since $L > 2$ we have $L + 2 > 4$ and this implies $y = \frac{L+2}{2} > 2$. Also, $y = \frac{L+2}{2} < \frac{L+2}{2} = L$.
(c) Since $y > 2$ we have $y^2 > 4$ so that $y \in A$. But $L$ is a lower bound of $A$ so we must have $L \leq y$. But this contradicts $y < L$ from (b). It follows that 2 is the greatest lower bound of $A$.

Exercise 3.10
Show that for any real number $x$ there is a positive integer $n$ such that $n > x$.

Solution.
Let $a = 1$ and $b = x$ in the Archimedean property.

Exercise 3.11
Let $a$ and $b$ be any two real numbers such that $a < b$.
(a) Let $w$ be a fixed positive irrational number. Show that there is a rational number $r$ such that $a < wr < b$.
(b) Show that $wr$ is irrational. Hence, between any two distinct real numbers there is an irrational number.

Solution.
(a) Since $a < b$, we have $\frac{a}{w} < \frac{b}{w}$. By Exercise 3.6, there is a rational number $r$ such that $\frac{a}{w} < r < \frac{b}{w}$ or $a < rw < b$.
(e) If $rw = s$ with $s$ rational then $w = \frac{s}{r}$ which is a rational, a contradiction. Hence, $rw$ is irrational.
Exercise 3.12
Suppose that $\alpha = \sup A < \infty$. Let $\epsilon > 0$ be given. Prove that there is an $x \in A$ such that $\alpha - \epsilon < x$.

Solution.
Suppose the contrary. That is, $\alpha - \epsilon \geq x$ for all $x \in A$. In this case, $\alpha - \epsilon$ is an upper bound of $A$. Thus, we must have $\alpha \leq \alpha - \epsilon$ which is impossible.

Exercise 3.13
Suppose that $\beta = \inf A < \infty$. Let $\epsilon > 0$ be given. Prove that there is an $x \in A$ such that $\beta + \epsilon > x$.

Solution.
Suppose the contrary. That is, $\beta + \epsilon \leq x$ for all $x \in A$. In this case, $\beta + \epsilon$ is a lower bound of $A$. Thus, we must have $\beta + \epsilon \leq \beta$ which is impossible.

Exercise 3.14
For each of the following sets $S$ find $\sup \{S\}$ and $\inf \{S\}$ if they exist. You do not need to justify your answer.
(a) $S = \{x \in \mathbb{R} : x^2 < 5\}$.
(b) $S = \{x \in \mathbb{R} : x^2 > 7\}$.
(c) $S = \{-\frac{1}{n} : n \in \mathbb{N}\}$.

Solution.
(a) Note that $S = \{x \in \mathbb{R} : -\sqrt{5} < x < \sqrt{5}\}$. So $\sqrt{5}$ is an upper bound of $S$. Let $M$ be an upper bound of $S$. Suppose that $M < \sqrt{5}$. Let $r$ be a rational number such that $M < r < \sqrt{5}$. Then $r \in S$ and $M < r$. But this contradicts the fact that $M$ is an upper bound of $S$. Thus, $\sqrt{5} \leq M$ so that $\sup \{S\} = \sqrt{5}$.
Likewise one can show that $\inf \{S\} = -\sqrt{5}$.
(b) $\sup \{S\} = \infty$ and $\inf \{S\} = -\infty$.
(c) $\sup \{S\} = 0$ and $\inf \{S\} = -1$.

Exercise 3.15
(a) Show that for any positive numbers $a$ and $b$ we have $\frac{a+b}{2} \geq \sqrt{ab}$.
(b) Let $a_i > 0$ for $i = 1, 2, \cdots, n$. Suppose that $\sqrt{a_1 a_2 \cdots a_n} = 1$. Use (a) to show that $(1 + a_1)(1 + a_2)\cdots(1 + a_n) \geq 2^n$. 


Solution.
(a) We have \((\sqrt{a} - \sqrt{b})^2 \geq 0 \rightarrow a + b \geq 2\sqrt{ab} \rightarrow \frac{a+b}{2} \geq \sqrt{ab}\).
(b) For \(i = 1, \cdots, n\) we have \(\frac{1+a_i}{2} \geq \sqrt{a_i}\). Multiplying these \(n\) inequalities we find
\[
\left(\frac{1 + a_1}{2}\right) \left(\frac{1 + a_2}{2}\right) \cdots \left(\frac{1 + a_n}{2}\right) \geq \sqrt{a_1a_2\cdots a_n} = 1
\]
Hence,
\[
(1 + a_1)(1 + a_2)\cdots (1 + a_n) \geq 2^n \]  ■

Exercise 3.16
Consider the numbers \(s_1, s_2, \cdots\) where \(s_1 = \sqrt{2}\) and \(s_{n+1} = \sqrt{2 + s_n}\) for \(n \in \mathbb{N}\). Show that each of these numbers is irrational.

Solution.
We proof this claim by induction on \(n\). For \(n = 1\) we have \(s_1 = \sqrt{2}\) which is an irrational number. Suppose that \(s_k\) is irrational for \(k = 1, 2, \cdots, n\). We want to show that \(s_{n+1}\) is irrational. Suppose the contrary, then \(s_n = s_{n+1}^2 - 2\) is rational which contradicts the assumption that \(s_n\) is irrational. Hence, \(s_{n+1}\) must be irrational ■

Exercise 3.17
An algebraic number is a number that satisfies a polynomial equation with integer coefficients. Show that \(x = \sqrt{1 + \sqrt{5}}\) is an algebraic number.

Solution.
We have \(x = \sqrt{1 + \sqrt{5}} \implies x^3 = 1 + \sqrt{5} \implies (x^3 - 1)^2 = 5 \implies x^6 - 2x^3 - 4 = 0\) ■

Exercise 3.18
Let \(A \subset \mathbb{R}\). Let \(f, g : A \rightarrow \mathbb{R}\) be such that \(|f(x)| \leq M_1\) and \(|g(x)| \leq M_2\) for all \(x \in A\). Show the following
(a) \(\sup\{f(x) + g(x) : x \in A\} \leq \sup\{f(x) : x \in A\} + \sup\{g(x) : x \in A\}\).
(b) \(\inf\{f(x) + g(x) : x \in A\} \geq \inf\{f(x) : x \in A\} + \inf\{g(x) : x \in A\}\).
(c) \(\sup\{-f(x) : x \in A\} = -\inf\{f(x) : x \in A\}\)
(d) \(\sup\{f(x) - g(x) : x \in A\} \leq \sup\{f(x) : x \in A\} - \inf\{g(x) : x \in A\}\).
Solution.
(a) For all \( x \in A \), we have
\[
f(x) + g(x) \leq \sup\{f(x) : x \in A\} + \sup\{g(x) : x \in A\}.
\]
Thus, \( \sup\{f(x) : x \in A\} + \sup\{g(x) : x \in A\} \) is an upper bound of \( \{f(x) + g(x) : x \in A\} \). But \( \sup\{f(x) + g(x) : x \in A\} \) is the smallest upper bound of \( \{f(x) + g(x) : x \in A\} \) so that
\[
\sup\{f(x) + g(x) : x \in A\} \leq \sup\{f(x) : x \in A\} + \sup\{g(x) : x \in A\}.
\]

(b) Similar argument to (a).
(c) We have \( -f(x) \leq -\inf\{f(x) : x \in A\} \) for all \( x \in A \) so that \( -\inf\{f(x) : x \in A\} \) is an upper bound of \( \{-f(x) : x \in A\} \). But \( \sup\{-f(x) : x \in A\} \) is the smallest upper bound of \( \{-f(x) : x \in A\} \). Hence,
\[
\sup\{-f(x) : x \in A\} \leq -\inf\{f(x) : x \in A\}.
\]
Suppose that \( \sup\{-f(x) : x \in A\} < -\inf\{f(x) : x \in A\} \). Let \( \epsilon = -\inf\{f(x) : x \in A\} - \sup\{-f(x) : x \in A\} > 0 \). By Exercise 3.13, there is an \( a \in A \) such that \( \inf\{f(x) : x \in A\} + \epsilon > f(a) \). But this implies that \( -\inf\{f(x) : x \in A\} - \epsilon < -f(a) \leq \sup\{-f(x) : x \in A\} \). This leads to the contradiction \( \sup\{-f(x) : x \in A\} < \sup\{-f(x) : x \in A\} \). Hence, the equality must hold.
(d) We have
\[
\sup\{f(x) - g(x) : x \in A\} = \sup\{f(x) + (-g(x)) : x \in A\}
\leq \sup\{f(x) : x \in A\} + \sup\{-g(x) : x \in A\}
= \sup\{f(x) : x \in A\} - \inf\{g(x) : x \in A\} \quad \blacksquare
\]

Exercise 3.19
Let \( A \subset \mathbb{R} \). Let \( f, g : A \to \mathbb{R} \) be such that \( |f(x)| \leq M_1 \) and \( |g(x)| \leq M_2 \) for all \( x \in A \).
(a) Show that
\[
\{f(x) - g(y) : x, y \in A\} = \{|f(x) - g(y)| : x, y \in A\} \cup \{-|f(x) - g(y)| : x, y \in A\}.
\]
(b) Show that
\[
\sup\{|f(x) - g(y)| : x, y \in A\} = \sup\{f(x) - g(y) : x, y \in A\}.
\]
Solution.
(a) Let \( x, y \in A \). If \( f(x) \geq f(y) \) then \( f(x) - f(y) = |f(x) - f(y)| \in \{|f(x) - g(y)| : x, y \in A\} \). If \( f(x) \leq f(y) \) then \( f(x) - f(y) \leq 0 \) so that \( f(x) - f(y) = -|f(x) - f(y)| \in \{-|f(x) - g(y)| : x, y \in A\} \). Hence,
\[
\{f(x) - g(y) : x, y \in A\} \subseteq \{|f(x) - g(y)| : x, y \in A\} \cup \{-|f(x) - g(y)| : x, y \in A\}.
\]
The other inclusion is trivial.
(b) Note that \( \{|f(x) - g(y)| : x, y \in A\} \) consists of nonnegative numbers whereas \( \{-|f(x) - g(y)| : x, y \in A\} \) consists of negative numbers. Hence, the result is clear.

Exercise 3.20
(a) For each \( n \in \mathbb{N} \) we define \( I_n = [n, \infty) \). Show that \( \bigcap_{n=1}^{\infty} I_n = \emptyset \). Hint: Archimedean property.
(b) For each \( n \in \mathbb{N} \) we define \( J_n = [-\frac{1}{n}, \infty) \). Show that \( \bigcap_{n=1}^{\infty} J_n \neq \emptyset \).

Solution.
(a) Let \( a > 0 \). By the Archimedean property, there is a positive integer \( n \) such that \( n > y \). That is, \( y \not\in [n, \infty) \) and so \( y \not\in \bigcap_{n=1}^{\infty} I_n \). Since \( y \) was arbitrary, we must have \( \bigcap_{n=1}^{\infty} I_n = \emptyset \).
(b) \( 0 \in \bigcap_{n=1}^{\infty} J_n \).

Exercise 3.21 (Nested interval theorem)
For each \( n \in \mathbb{N} \) let \( I_n = [a_n, b_n] \). Suppose that \( I_{n+1} \subset I_n \) for each \( n \in \mathbb{N} \). Show that \( \bigcap_{n=1}^{\infty} I_n \neq \emptyset \).

Solution.
(a) We have \( a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_2 \leq b_1 \). Hence, for each \( n \in \mathbb{N} \), \( b_n \) is an upper bound of the set \( \{a_1, a_2, \cdots\} \). By completeness of \( \mathbb{R} \) there is a finite number \( \alpha \) such that \( \alpha = \sup\{a_1, a_2, \cdots\} \). Fix \( n \in \mathbb{N} \). By the definition of \( \alpha \) we have \( a_n \leq \alpha \). Now, since \( b_n \) is an upper bound of \( \{a_1, a_2, \cdots\} \), again by the definition of \( \alpha \) we have \( \alpha \leq b_n \). Hence, \( \alpha \in I_n \) for all \( n \in \mathbb{N} \) so that \( \bigcap_{n=1}^{\infty} I_n \neq \emptyset \).

Exercise 3.22
Show that \( \sqrt{2} + \frac{1}{\sqrt{2}} \) is irrational.
Solution.
Suppose the contrary. Then $\sqrt{2} + \frac{1}{\sqrt{2}} = \frac{p}{q}$ where $p$ and $q$ are non-zero integers. Multiply through by $\sqrt{2}$ to obtain $3 = \sqrt{2}^2$. Hence, $\sqrt{2} = \frac{2q}{p}$ which is a rational. But this contradicts the fact that $\sqrt{2}$ is irrational.

Exercise 3.23 (Cauchy-Schwarz inequality)
Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
(a) Show that $x^2 + y^2 \geq 2xy$ for all $x, y \in \mathbb{R}$.
(b) Show that
$$2 \frac{x_1y_1}{\sqrt{x_1^2 + x_2^2 \sqrt{y_1^2 + y_2^2}}} \leq \frac{x_1^2}{x_1^2 + x_2^2} + \frac{y_1^2}{y_1^2 + y_2^2}$$
and
$$2 \frac{x_2y_2}{\sqrt{x_2^2 + x_2^2 \sqrt{y_1^2 + y_2^2}}} \leq \frac{x_2^2}{x_1^2 + x_2^2} + \frac{y_2^2}{y_1^2 + y_2^2}$$
(c) Show that
$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2 \sqrt{y_1^2 + y_2^2}}$$

Solution.
(a) This follows from the fact that $(x - y)^2 \geq 0$.
(b) For the first inequality let $x = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$ and $y = \frac{y_1}{\sqrt{y_1^2 + y_2^2}}$. Likewise, for the second inequality, let $x = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$ and $y = \frac{y_2}{\sqrt{y_1^2 + y_2^2}}$.
(b) This follows by adding the two inequalities in (b).

Exercise 3.24
For each $n \in \mathbb{N}$ let $I_n = (-\frac{1}{n}, \frac{1}{n})$. Show that $\bigcap_{n=1}^{\infty} I_n = \{0\}$.

Solution.
Clearly, $0 \in \bigcap_{n=1}^{\infty} I_n$. Suppose that $x \in \bigcap_{n=1}^{\infty} I_n$ with $x \neq 0$. If $x > 0$ then by the Archimedean property we can find a positive integer $n$ such that $nx > 1$. But this implies that $x \notin (-\frac{1}{n}, \frac{1}{n})$ and therefore $x \notin \bigcap_{n=1}^{\infty} I_n$. If $x < 0$ then $-x > 0$ and we can find a positive integer $n$ such that $n(-x) > 1$ so that $x < -\frac{1}{n}$. Again this implies that $x \notin \bigcap_{n=1}^{\infty} I_n$. Hence, $\bigcap_{n=1}^{\infty} I_n$ consists of $\{0\}$. ■