7.3 Volumes by the Method of Cylindrical Shells

We start this section by considering an example that motivates the method of cylindrical shells.

Example 7.3.1

Find the volume of the solid of revolution obtained by rotating the region bounded by \( y = 0 \) and \( y = x^2 - x^3 \) about the \( y \)-axis.

Solution.

The cross section to the solid of revolution in Figure 7.3.1 is a washer. The inner radius and the outer radius are the \( x \)-values of the equation \( x^2 - x^3 - y = 0 \) which is not easy to solve explicitly.

![Figure 7.3.1](image)

The above example motivates the need of a different method for finding the volume. Such a method is called the method of cylindrical shell.

Volume of a Cylindrical Shell

A cylindrical shell is a region contained between two cylinders of the same height with the same central axis. Consider a cylindrical shell with height \( h \), inner radius \( r_1 \) and outer radius \( r_2 \) as shown in Figure 7.3.2. The volume of the cylinder is calculated as follows:

\[
V = \pi r_2^2 h - \pi r_1^2 h
\]

\[
= 2\pi \left( \frac{r_1 + r_2}{2} \right) h(r_2 - r_1)
\]

\[
= 2\pi rh \Delta r
\]

where \( \Delta r = r_2 - r_1 \) is the thickness of the cylinder and \( r = \frac{r_1 + r_2}{2} \) is the average radius of the shell.
Now, let $S$ be a solid obtained by rotating the region shown in Figure 7.3.3(A) about the $y-$ axis. The resulting solid of revolution is shown in Figure 7.3.3(B).

Divide the interval $[a, b]$ into $n$ subintervals each of length $\Delta x = \frac{b-a}{n}$. Consider the $i^{th}$ interval and let $\overline{x}_i$ be the midpoint of this interval. When rotating the rectangle of width $\Delta x$ and height $f(\overline{x}_i)$ about the $y-$axis we generate a cylindrical shell (see Figure 7.3.4(A)) whose volume is

$$V_i = 2\pi \overline{x}_i f(\overline{x}_i) \Delta x.$$ 

Thus, an approximation to the volume of $S$ is given by the sum of these cylindrical shells (see Figure 7.3.4(B)):

$$V \approx \sum_{i=1}^{n} 2\pi \overline{x}_i f(\overline{x}_i) \Delta x.$$
This approximation gets better as \( n \to \infty \). Hence, we define the volume of \( S \) by

\[
V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i f(x_i) \Delta x = \int_{a}^{b} 2\pi x f(x) dx.
\]

![Figure 7.3.4](image)

**Example 7.3.2**
Find the volume of the solid in Example 7.3.1.

**Solution.**
The volume is given by

\[
V = \int_{0}^{1} 2\pi x(x^2 - x^3) dx = 2\pi \left[ \frac{x^4}{4} - \frac{x^5}{5} \right]_{0}^{1} = 0.1\pi \]

**Example 7.3.3**
Find the volume of the solid of revolution obtained by rotating the region between the curves \( y = x \) and \( y = x^2 \) about the \( y \)-axis.

**Solution.**
Figure 7.3.5 shows a rectangle that generates a cylindrical shell.
Thus,
\[ V = \int_{0}^{1} 2\pi x(x^2 - x^2)dx = 2\pi \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_{0}^{1} = \frac{\pi}{6} \]

**Example 7.3.4**

Use the method of cylindrical shells to find the volume of the solid of revolution obtained by rotating the region bounded by \( y = \sqrt{x} \) from \( x = 0 \) to \( x = 1 \) about the \( x \)-axis.

**Solution.**

Using Figure 7.3.6, we can write
\[ V = \int_{0}^{1} 2\pi y(1 - y^2)dy = 2\pi \left[ \frac{y^2}{2} - \frac{y^4}{4} \right]_{0}^{1} = \frac{\pi}{2} \]
Example 7.3.5
Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

Solution.
Using Figure 7.3.7, we can write

$$V = \int_0^1 2\pi (2 - x)(x - x^2)dx = 2\pi \left[ \frac{x^4}{4} - x^3 + x^2 \right]_0^1 = \frac{\pi}{2}$$