Arkansas Tech University
MATH 2924: Calculus II
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7.7 Separable Differential Equations

An **ordinary differential equation** (abbreviated ODE) is an equation that involves an unknown function (the **dependent variable**) of a single variable, its **independent variable**, and one or more of its derivatives. The highest order derivative that appears in the equation is known as the **order** of the equation.

Example 7.7.1
Determine the order of each equation.
(a) $y' + 2ty = e^{-x^2}$
(b) $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y(t) = 0$
(c) $y'' + 3ty' + 2y = \sin(5t)$.

Solution.
(a) This is a first order differential equation because the highest derivative is the first derivative.
(b) and (c) are second order differential equations since the highest derivative in each equation is the second order derivative ■

A **solution** of a differential equation is a function that satisfies the equation: When you substitute this function and/or its derivatives into the differential equation, you get a true mathematical statement.

Example 7.7.2
Show that the function $y = 100 + e^{-t}$ is a solution to the differential equation

$$y' = 100 - y.$$

Solution.
Indeed, finding the first order derivative of y we have $y' = -e^{-t}$. Also, $100 - y = 100 - (100 + e^{-t}) = -e^{-t}$. Thus, $y' = 100 - y$ so that $y = 100 + e^{-t}$ is a solution to the given DE.■

Solving a differential equation means finding all possible solutions of the equation.

Example 7.7.3
Solve the differential equation:

$$y'' = -2t.$$

Solution.
Integrating twice, all the solutions have the form

$$y(t) = -\frac{t^3}{3} + C_1 t + C_2 \blacksquare$$

Note that the function of the previous example defines all the solutions to the differential equation. Such a function will be referred to as the **general solution.** The constants C_1 and C_2 are called the **parameters.** Specific values of C_1 and C_2 determine what is called a **particular solution**. To find a particular solution additional conditions on the values of the function or its derivatives must be given. Such conditions are called **initial conditions**. A differential equation together with a set of initial conditions is called an **initial value problem** (abbreviated IVP).

Example 7.7.4
Consider the differential equation $y''(t) - 1 = 0$.
(a) Find the general solution of this equation.
(b) Find the solution that satisfies the initial conditions $y(1) = 1$ and $y'(1) = 4$.

Solution.
(a) Integrating twice we find the general solution

$$y(t) = \frac{t^2}{2} + C_1 t + C_2.$$

(b) Since $y'(t) = t + C_1$ and $y'(1) = 4$ we find $4 = 1 + C_1$ so that $C_1 = 3$. Hence, $y(t) = \frac{t^2}{2} + 3t + C_2$. Now, the initial value $y(1) = 1$ implies $1 = \frac{1}{2} + 3 + C_2$. Solving for C_2 we find $C_2 = -\frac{5}{2}$. Hence, the solution to the IVP

$$\left\{ \begin{array}{c} y''(t) - 1 = 0 \\ y'(1) = 4, y(1) = 1 \end{array} \right.$$

is

$$y(t) = \frac{t^2}{2} + 3t - \frac{5}{2} \blacksquare$$

Separable Differential Equations
A first order differential equation is **separable** if it can be written with one variable only on the left and the other variable only on the right:

$$f(y)y' = g(t).$$

To solve this equation, we proceed as follows. Let $F(t)$ be an antiderivative of $f(t)$ and $G(t)$ be an antiderivative of $g(t)$. Then by the Chain Rule

$$\frac{d}{dt}F(y) = \frac{dF}{dy}\frac{dy}{dt} = f(y)y'.$$

Thus,

$$f(y)y' - g(t) = \frac{d}{dt}F(y) - \frac{d}{dt}G(t) = \frac{d}{dt}[F(y) - G(t)] = 0.$$

It follows that

$$F(y) - G(t) = C$$

which is equivalent to

$$\int f(y)y'dt = \int g(t)dt + C.$$

As you can see, the result is generally an implicit equation involving a function of y and a function of t. It may or may not be possible to solve this to get y explicitly as a function of t. For an initial value problem, substitute the values of t and y by t_0 and y_0 to get the value of C.

Example 7.7.5
Solve the IVP $yy' = 4\sin(2t), \quad y(0) = 1$.

Solution.
This is a separable differential equation. Integrating both sides we find

$$\int \left(\frac{y^2}{2}\right)' dt = 4\int \sin(2t)dt.$$

Thus,

$$y^2 = -4\cos(2t) + C.$$

Since $y(0) = 1$, we find $C = 5$. Now, solving explicitly for $y(t)$ we find

$$y(t) = \pm\sqrt{-4\cos t + 5}.$$

Since $y(0) = 1$, we find $y(t) = \sqrt{-4\cos t + 5}$. The interval of existence of the solution is the interval $-\infty < t < \infty$ ■

The Logistic Population Model of Verhulst
In this model, the population growth is limited to a maximum size of M,

called the **carrying capacity**. For this model, it is assumed that the rate of change of the population $y(t)$ is proportional to the product of the population and the amount by which y falls short of the maximal size $M - y$. That is,

$$\frac{dy}{dt} = ky(M - y) \tag{7.7.1}$$

for some positive constant k. Equation (7.7.1) is a separable differential equation.

Example 7.7.6
Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. If it is determined that the rate at which the virus spreads is proportional not only to the number $y(t)$ of students infected but also to the number of students not infected. Determine the number of infected students after 6 days given that the number of infected students after 4 days is 50.

Solution.
We first must find a formula for $y(t)$ which is the solution to the IVP

$$\frac{dy}{dt} = k(1000 - y)y, \quad y(0) = 1.$$

Separating the variable, we find

$$\begin{aligned}
\frac{dy}{y(1000-y)} =& kdt \\
\int \frac{dy}{y(1000-y)} =& \int kdt \\
\frac{1}{1000}\int \left(\frac{1}{y} + \frac{1}{1000-y}\right) dy =& \int kdt \\
\ln\left|\frac{y}{1000-y}\right| =& 1000kt + C' \\
\left|\frac{y}{1000-y}\right| =& e^{1000kt+C'} \\
\frac{y}{1000-y} =& Ce^{1000kt}.
\end{aligned}$$

Using the initial condition $y(0) = 1$, we find $C = \frac{1}{999}$. Solving for y, we find

$$y(t) = \frac{1000e^{1000kt}}{999 + e^{1000kt}}.$$

But $y(4) = 50$ so that

$$50 = \frac{1000}{1 + 999e^{-4000k}}.$$

Solving this equation for k, we find $k \approx 0.0009906$. Thus,

$$y(t) = \frac{1000}{1 + 999e^{-0.9906t}}.$$

Finally,

$$y(6) = \frac{1000}{1 + 999e^{-0.9906(6)}} \approx 276 \text{ students} \blacksquare$$

Mixing Models

All mixing problems we consider here will involve a "tank into which a certain mixture will be added at a certain input rate and the mixture will leave the system at a certain output rate. We shall always reserve $y = y(t)$ to denote the amount of substance in the tank at any given time t.

The differential equation involved here arises from the following natural relationship:

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}.$$

The main assumption that we will be using here is that the concentration of the substance in the liquid is uniform throughout the tank.

Consider a tank initially containing a volume V_0 of mixture (substance and liquid) of concentration c_0. Then the initial amount of the substance is given by $y_0 = c_0V_0$.

Suppose a mixture of concentration $c_i(t)$ flows into the tank at the volume rate $r_i(t)$. Then the substance is entering the tank at the rate $c_i(t)r_i$. Suppose that the well-mixed solution is pumped out of the tank at the volume rate $r_o(t)$. The concentration of this outflow is $\frac{y(t)}{V(t)}$ where $V(t)$ is the current volume of solution in the tank. Then clearly

$$\frac{dy}{dt} = c_i(t)r_i(t) - \frac{y(t)}{V(t)}r_o(t), \quad y(0) = y_0$$

and

$$\frac{dV}{dt} = r_i(t) - r_o(t).$$

Solving the last equation we find

$$V(t) = V_0 + \int_0^t (r_i(s) - r_o(s))ds.$$

Example 7.7.7
Consider a tank with volume 5000 liters containing 20 kg of salt. Suppose a solution with 0.03 kg/liter of salt flows into the tank at a rate of 25 liters/min. The solution in the tank is well-mixed. Solution flows out of the tank at a rate of 25 liters/min. How much salt will be in the tank at time t?

Solution.
Since $r_i = r_o$, we have $V(t) = V_0 = 5000$. If $y(t)$ is the amount of salt in the tank at any time t then

$$y' = 0.03 \times 25 - \frac{y}{5000} \times 25, \quad y(0) = 20$$

or

$$y' = \frac{150 - y(t)}{200}, \quad y(0) = 20.$$

This is a separable differential equation. Solving this equation, we find

$$\begin{aligned} \frac{dy}{150 - y} =& \frac{dt}{200} \\ \int \frac{dy}{150 - y} =& \int \frac{dt}{200} \\ -\ln|150 - y| =& \frac{t}{200} + C. \end{aligned}$$

Since $y(0) = 20$, we find $-\ln 130 = C$. Thus,

$$-\ln|150 - y| = \frac{t}{200} - \ln 130 \Longrightarrow \pm(150 - y) = 130e^{-\frac{t}{200}}.$$

But $y(0) = 20$ so that $150 - y(t) = 130e^{\frac{t}{200}}$ or $y(t) = 150 - 130e^{-\frac{t}{200}}$ ■