5.4 The Definite Integral of a Derivative

We start this section by showing that the definite integral of a rate of change gives the total change of the function. We define the total change of a function \( F(t) \) from \( t = a \) to \( t = b \) to be the difference \( F(b) - F(a) \). Suppose that \( F(t) \) is continuous in \([a, b]\) and differentiable in \((a, b)\). Divide the interval \([a, b]\) into \( n \) equal subintervals each of length \( \Delta t = \frac{b-a}{n} \). Let \( a = t_0 < t_1 < \cdots < t_n = b \) be the partition points of the subdivision. Then on the interval \([t_0, t_1]\) the change in \( F \) can be estimated by the formula

\[
F'(t_0) \approx \frac{F(t_0 + \Delta t) - F(t_0)}{\Delta t}
\]

or

\[
F(t_0 + \Delta t) - F(t_0) \approx F'(t_0)\Delta t.
\]

That is

\[
F(t_1) - F(t_0) \approx F'(t_0)\Delta t.
\]

On the interval \([t_1, t_2]\) we get the estimation

\[
F(t_2) - F(t_1) = F'(t_1)\Delta t.
\]

Continuing in this fashion we find that on the interval \([t_{n-1}, t_n]\) we have

\[
F(t_{n-1}) - F(t_n) \approx F'(t_{n-1})\Delta t.
\]

Adding all these approximations we find that

\[
F(t_n) - F(t_0) \approx \sum_{i=0}^{n-1} F'(t_i)\Delta t.
\]

Letting \( n \to \infty \) we see that

\[
F(b) - F(a) = \int_a^b F'(t)dt.
\]

Example 5.4.1

The amount of waste a company produces, \( W \), in metric tons per week, is approximated by \( W = 3.75e^{-0.008t} \), where \( t \) is in weeks since January 1, 2000. Waste removal for the company costs \$15/ton. How much does the company pay for waste removal during the year 2000?
Solution.
The amount of tons produced during the year 2000 is just the definite integral $\int_0^{52} W(t) dt$. Using a calculator we find that

\[
\text{Total waste during the year} = \int_0^{52} 3.75e^{-0.008t} dt \approx 159 \text{ tons}
\]

The cost to remove this quantity is $159 \times 15 = 2385 \$$

Remark 5.4.1
When using $\int_a^b f(t) dt$ in applications then its units is the product of the units of $f(t)$ with the units of $t$. 