4 Applications of Functions to Economics

The goal of this section is to exhibit some functions used in business and economics.

The cost function $C$ gives the cost $C(q)$ of manufacturing a quantity $q$ of some good. A linear cost function has the form

$$C(q) = mq + b,$$

where the vertical intercept $b$ is called the fixed costs, i.e. the costs incurred even if nothing is produced, and the slope $m$ is called the variable costs per unit.

Example 4.1
What is the cost function of manufacturing a product with fixed costs of $400 and variable costs of $40 per item, assuming the function is linear?

Solution.
The cost function is

$$C(q) = 40q + 400.$$

Example 4.2
Values of a linear cost function are shown below. What are the fixed costs and the variable costs per units? Find a formula for the cost function.

<table>
<thead>
<tr>
<th>$q$</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(q)$</td>
<td>5000</td>
<td>5020</td>
<td>5040</td>
<td>5060</td>
<td>5080</td>
</tr>
</tbody>
</table>

Solution.
The fixed costs are $b = C(0) = 5000$, the variable costs are

$$m = \frac{5020 - 5000}{5 - 0} = 4/\text{unit}$$

The cost function is

$$C(q) = 4q + 5000.$$

A revenue function $R$ gives the total revenue $R(q)$ from the sale of a quantity $q$ at a unit price $p$ dollars. Thus, $R(q) = pq$. 


Example 4.3
A company that makes a certain brand of chairs has fixed costs of $5,000 and variable costs of $30 per chair. The company sells the chairs for $50 each. Find formulas for the cost and revenue functions.

Solution.
The cost function is \( C(q) = 30q + 5000 \). The revenue function is \( R(q) = pq = 50q \).

In any business, decisions are made based on the profit function. Profit is defined to be revenue minus cost. That is

\[ P(q) = R(q) - C(q). \]

The break-even point is the point where the profit is zero, i.e. \( R(q) = C(q) \).

Example 4.4
A company has cost and revenue functions, in dollars, given by \( C(q) = 6,000 + 10q \) and \( R(q) = 12q \).

(a) Graph the functions \( C(q) \) and \( R(q) \) on the same coordinate axes.
(b) Find the break-even point and illustrate it graphically.
(c) When does the company make a profit? Loses money?

Solution.
(a) The graph is given in Figure 9.

(b) The break-even point is the point of intersection of the two lines. To find the point, set \( 12q = 10q + 6000 \) and solve for \( q \) to find \( q = 3000 \). Thus, the break-even point is the point \( (3000, 36000) \).
(c) The company makes profit for \( q > 3000 \) and loses money for \( 0 \leq q < 3000 \).

Marginal Analysis
In economics and business the term marginal stands for a rate of change. Marginal analysis is an area of economics concerned with estimating the effect on quantities such as cost, revenue, and profit when the level of production is changed by a unit amount. For example, if \( C(q) \) is the cost of producing \( q \) units of a certain commodity, then the marginal cost, \( MC(q) \), is
the additional cost of producing one more unit and is given by the difference 
\( MC(q) = C(q + 1) - C(q) \). The marginal cost is the same as the variable costs 
per unit introduced earlier in the lecture.

We define similarly, the **marginal revenue** \( MR(q) = R(q + 1) - R(q) \) and the **marginal profit** \( MP(q) = P(q + 1) - P(q) \). Note that

\[
MP(q) = P(q + 1) - P(q) = R(q + 1) - C(q + 1) - [R(q) - C(q)] \\
= [R(q + 1) - R(q)] - [C(q + 1) - C(q)] = MR(q) - MC(q).
\]

**Example 4.5**

Let \( C(q) \) represent the cost, \( R(q) \) the revenue, and \( P(q) \) the total profit, in dollars, of producing \( q \) units.

(a) If \( MC(50) = 75 \) and \( MR(50) = 84 \), approximately how much profit is 
 earned by the 51st item?

(b) If \( MC(90) = 71 \) and \( MR(90) = 68 \), approximately how much profit is earned 
by the 91st item?

**Solution.**

(a) \( MP(50) = MR(50) - MC(50) = 84 - 75 = $9 \). A gain of $9.

(b) \( MP(90) = MR(90) - MC(90) = 68 - 71 = -$3 \). A loss by 3 dollars.

**The Depreciation Function**

An important application of linear functions in financial modeling is the depreciation function.

In a financial setting, a linear function with negative slope is called a **depreciation function**.

**Example 4.6**

A new sports car costs $40,000 and depreciates $3000 per year.

(a) Determine an equation for the depreciation function.

(b) How much will the car be worth in 5 years?

**Solution.**

Since the rate of depreciation is constant, the depreciation function is a linear function, say, \( V(t) = mt + b \). Since \( b = V(0) \), we find \( b = 40,000 \). Also, \( m = -3000 \) per year. Thus, \( V(t) = -3000t + 40,000 \).

(b) The question here is equivalent to finding \( V(5) \) which is \( V(5) = -3000(5) + 40,000 = $25,000 \).

**Supply and Demand Curves**

The quantity \( q \) manufactured and sold depends on the unit price \( p \). In general, 
when the price goes up then manufacturers are willing to supply more of the 
product whereas consumers are going to reduce their buyings. Since consumers 
and manufacturers react differently to changes in price, there are two curves 
relating \( p \) and \( q \).
The supply curve is the quantity that producers are willing to make at a given price. Thus, increasing price will increase quantity. The demand curve is the quantity that will be bought by consumers at a given price. Thus, decreasing price will increase quantity. Even though quantity is a function of price, it is the tradition to use the vertical axis for the variable \( p \) and the horizontal axis for the variable \( q \). The supply and demand curves intersect at a point \((q^*, p^*)\) called the point of equilibrium. We call \( p^* \) the equilibrium price and \( q^* \) the equilibrium quantity.

**Example 4.7**
Find the equilibrium point for the supply function \( S(p) = 3p - 50 \) and the demand function \( D(p) = 100 - 2p \).

**Solution.**
Setting the equation \( S(p^*) = D(p^*) \) to obtain \( 3p^* - 50 = 100 - 2p^* \). By adding \( 2p^* + 50 \) to both sides we obtain \( 5p^* = 150 \). Solving for \( p^* \) we find \( p^* = \$30 \).
Substituting this value in \( S(p) \) we find \( q^* = 3(30) - 50 = 40 \).

The impact of Taxes on Equilibrium
Now let us consider the previous problem again. Suppose that the government imposes a $5 tax per item on the supplier. How does this increase affect the equilibrium price \( p^* \)? The consumers pay \( p \) dollars per unit. The suppliers receive only \( p - 5 \) dollars per unit because $5 goes to the government as taxes. By imposing the $5 tax per item the new quantity to be supplied is now given by

\[
S(p - 5) = 3(p - 5) - 50 = 3p - 65.
\]

However, the demand function is the same. Calculating the equilibrium price we find

\[
\begin{align*}
3p - 65 & = 100 - 2p \\
5p & = 165 \\
p & = \$33.
\end{align*}
\]

Thus, the previous equilibrium price has increased by $3. This means that the consumer’s share of the $5 tax is $3 whereas the supplier share is $2. That is, even though the tax was imposed on the producer, some of the tax is passed on to the consumer in terms of higher prices.