

16 Tests of Divisibility

Sometimes it is handy to know if one number is divisible by another just by looking at it or by performing a simple test. The purpose of this section is to discuss some of the rules of divisibility. Such rules have limited use except for mental arithmetic. We will test the divisibility of a number by the numbers 2 through 11, excluding 7.

Two important facts are needed in this section. The first one states that if $a|m$ and $a|n$ then $a|(m+n)$, $a|(m-n)$, and $a|km$ for any whole numbers a, m, n, k with $a \neq 0$. The second fact is about the expanded representation of a number in base 10. That is, if $n = d_{k-1} \cdots d_2 d_1 d_0$ is a whole number with k digits then

$$n = d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + d_1 \times 10^1 + d_0.$$

Divisibility Tests for 2, 5, and 10

The divisibility tests for 2, 5, and 10 are grouped together because they all require checking the unit digit of the whole number.

Theorem 16.1

(a) $n = d_{k-1} \cdots d_2 d_1 d_0$ is divisible by 2 if and only if $2|d_0$, i.e., $d_0 \in \{0, 2, 4, 6, 8\}$. That is, n is divisible by 2 if and only if the unit digit is either 0, 2, 4, 6, or 8.

(b) $n = d_{k-1} \cdots d_2 d_1 d_0$ is divisible by 5 if and only if $5|d_0$, i.e., $d_0 \in \{0, 5\}$. That is, n is divisible by 5 if and only if the unit digit is either 0 or 5.

(c) $n = d_{k-1} \cdots d_2 d_1 d_0$ is divisible by 10 if and only if $10|d_0$, i.e., $d_0 = 0$.

Proof.

(a) Suppose that n is divisible by 2. Since $2|10^i$ for $1 \leq i \leq k-1$, 2 divides the sum $(d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + d_1 \times 10^1)$ and the difference $n - (d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + d_1 \times 10^1) = d_0$. That is, $2|d_0$.

Conversely, suppose that $2|d_0$. Since $2|(d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + d_1 \times 10^1)$, 2 divides the sum $2|(d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + d_1 \times 10^1 + d_0)$. That is, $2|n$.

(b) The exact same proof of part (a) works by replacing 2 by 5.

(c) The exact same proof of part (a) works by replacing 2 by 10. ■

Example 16.1

Without dividing, determine whether each number below is divisible by 2, 5

and/or 10.

(a) 8,479,238 (b) 1,046,890 (c) 317,425.

Solution.

(a) Since the unit digit of 8,479,238 is 8, this number is divisible by 2 but not by 5 or 10.

(b) Since the unit digit of 1,046,890 is 0, this number is divisible by 2, 5, and 10.

(c) Since the unit digit of 317,425 is 5, this number is divisible by 5 but not by 2 or 10. ■

Divisibility Tests for 3 and 9

The divisibility tests for 3 and 9 are grouped together because they both require computing the sum of the digits.

Theorem 16.2

(a) $n = d_{k-1} \cdots d_2 d_1 d_0$ is divisible by 3 if and only if $3|(d_{k-1} + \cdots + d_2 + d_1 + d_0)$.

(b) $n = d_{k-1} \cdots d_2 d_1 d_0$ is divisible by 9 if and only if $9|(d_{k-1} + \cdots + d_2 + d_1 + d_0)$.

Proof.

(a) Suppose that $3|n$. Write $[9]_i = 99 \cdots 9$ where the 9 repeats i times. For example, $[9]_3 = 999$. With this notation we have $10^i = [9]_i + 1$. Hence

$$\begin{aligned} d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + d_1 \times 10^1 + d_0 &= \\ d_{k-1} \times ([9]_{k-1} + 1) + \cdots + d_2([9]_2 + 1) + d_1([9]_1 + 1) + d_0 &= \\ d_{k-1} \times [9]_{k-1} + \cdots + d_2 \times [9]_2 + d_1 \times [9]_1 + (d_{k-1} + \cdots + d_2 + d_1 + d_0) \end{aligned}$$

Since $3|[9]_i$ for any i , $3|(d_{k-1} \times [9]_{k-1} + \cdots + d_2 \times [9]_2 + d_1 \times [9]_1)$. Therefore, $3|[n - (d_{k-1} \times [9]_{k-1} + \cdots + d_2 \times [9]_2 + d_1 \times [9]_1)]$. That is, $3|(d_{k-1} + \cdots + d_2 + d_1 + d_0)$.

Conversely, suppose that $3|(d_{k-1} + \cdots + d_2 + d_1 + d_0)$. Then $3|[(d_{k-1} \times [9]_{k-1} + \cdots + d_2 \times [9]_2 + d_1 \times [9]_1) + (d_{k-1} + \cdots + d_2 + d_1 + d_0)]$. That is, $3|n$.

(b) The same exact proof of (a) works by replacing 3 by 9 since $9|[9]_i$. ■

Example 16.2

Use the divisibility rules to determine whether each number is divisible by 3 or 9.

(a) 468,172 (b) 32,094.

Solution.

(a) Since $4 + 6 + 8 + 1 + 7 + 2 = 28$ which is divisible by 3, 468,172 is divisible by 3. Since $9 \nmid 28$ the number 468,172 is not divisible by 9.

(b) Since $3 + 2 + 0 + 9 + 4 = 18$ and $3|18, 9|18$ the number 32,094 is divisible by both 3 and 9. ■

Divisibility by 4 and 8

The following theorem deals with the divisibility by 4 and 8.

Theorem 16.3

(a) $n = d_{k-1} \cdots d_2 d_1 d_0$ is divisible by 4 if and only if $4|(d_1 d_0)$ where $(d_1 d_0)$ is the number formed by the last two digits of n .

(b) $n = d_{k-1} \cdots d_2 d_1 d_0$ is divisible by 8 if and only if $8|(d_2 d_1 d_0)$ where $(d_2 d_1 d_0)$ is the number formed by the last three digits of n .

Proof.

(a) Suppose that $4|n$. Write n in the form

$$n = d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + (d_1 d_0).$$

Since $4|10^i$ for $2 \leq i \leq k-1$, we have $4|(d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2)$. Hence, $4|(n - d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2) = (d_1 d_0)$.

Conversely, suppose that $4|(d_1 d_0)$. Since $4|(d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2)$, $4|(d_{k-1} \times 10^{k-1} + \cdots + d_2 \times 10^2 + (d_1 d_0)) = n$.

(b) The proof is similar to (a) and is omitted. ■

Example 16.3

Use divisibility rules to test each number for the divisibility by 4 and 8.

(a) 1344 (b) 410,330

Solution.

(a) Since $4|44$ we have $4|1344$. Since $8|344$ we have $8|1344$.

(b) Since $4 \nmid 30$, $4 \nmid 410, 330$. Similarly, since $8 \nmid 330$, we have $8 \nmid 410, 330$. ■

Divisibility by 6

The divisibility by 6 follows from the following result.

Theorem 16.4

Let a and b be two whole numbers having only 1 as a common divisor and n a nonzero whole number. $a|n$ and $b|n$ if and only if $ab|n$.

Proof.

Suppose that $ab|n$. Then there is a unique nonzero whole number k such that $n = k(ab)$. Using associativity of multiplication we can write $n = (ka)b$. This means that $b|n$. Similarly, $n = (kb)a$. That is, $a|n$.

Conversely, suppose that $a|n$ and $b|n$. Write the prime factorizations of a, b , and n .

$$\begin{aligned} a &= p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k} \\ b &= p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} \\ n &= p_1^{w_1} p_2^{w_2} \cdots p_k^{w_k} \end{aligned}$$

where p_1, p_2, \dots, p_k are distinct prime factors. Thus

$$ab = p_1^{t_1+s_1} p_2^{t_2+s_2} \cdots p_k^{t_k+s_k}.$$

Since $a|n$, we find $t_1 \leq w_1, t_2 \leq w_2, \dots, t_k \leq w_k$. Similarly, since $b|n$, we have $s_1 \leq w_1, s_2 \leq w_2, \dots, s_k \leq w_k$. Now, since a and b have no common divisor different from 1, if $t_1 \neq 0$ then $s_1 = 0$ otherwise p_1 becomes a common divisor. Similarly, if $s_1 \neq 0$ then we must have that $t_1 = 0$. This shows that $s_i + t_i$ is either equal to s_i or to t_i . Hence, $s_1 + t_1 \leq w_1, s_2 + t_2 \leq w_2, \dots, s_k + t_k \leq w_k$. We conclude from this that $ab|n$. ■

If we let $a = 2$ and $b = 3$ in the previous theorem and use the fact that $6 = 2 \times 3$ we obtain the following result.

Theorem 16.5

A nonzero whole number n is divisible by 6 if and only if n is divisible by both 2 and 3.

Example 16.4

Use divisibility rules to test each number for the divisibility by 6.

(a) 746,988 (b) 4,201,012

Solution.

(a) Since the unit digit is 8, the given number is divisible by 2. Since $7 + 4 + 6 + 9 + 8 + 8 = 42$ and $3|42$ we conclude that $6|746,988$.

(b) The given number is divisible by 2 since it ends with 2. However, $4 + 2 + 0 + 1 + 0 + 1 + 2 = 10$ which is not divisible by 3 then $6 \nmid 4,201,012$. ■

Divisibility by 11

Theorem 16.6

A nonzero whole number is divisible by 11 if and only if the difference of the sums of the digits in the even and odd positions in the number is divisible by 11.

Proof.

For simplicity we will prove the theorem for $n = d_4d_3d_2d_1d_0$. In this case, note that $10 = 11 - 1$, $100 = 99 + 1$, $1000 = 1001 - 1$, and $10000 = 9999 + 1$. The numbers 11, 99, 1001, and 9999 are all divisible by 11. Thus, n can be written in the following form

$$n = 11 \cdot q + d_4 - d_3 + d_2 - d_1 + d_0.$$

Now, suppose that $11|n$. Since $11|11 \cdot q$ we have $11|(n - 11 \cdot q)$ i.e., $11|(d_4 - d_3 + d_2 - d_1 + d_0)$.

Conversely, suppose that $11|(d_4 - d_3 + d_2 - d_1 + d_0)$. Since $11|11 \cdot q$ we have $11|(11 \cdot q + d_4 - d_3 + d_2 - d_1 + d_0)$, i.e., $11|n$. ■

Example 16.5

Is the number 57, 729, 364, 583 divisible by 11?

Solution.

Since $(3 + 5 + 6 + 9 + 7 + 5) - (8 + 4 + 3 + 2 + 7) = 35 - 24 = 11$ and 11 is divisible by 11, the given number is divisible by 11. ■

Practice Problems

Problem 16.1

Using the divisibility rules discussed in this section, explain whether 6,868,395 is divisible by 15.

Problem 16.2

The number a and b are divisible by 5.

- Is $a + b$ divisible by 5? Why?
- Is $a - b$ divisible by 5? Why?
- Is $a \times b$ divisible by 5? Why?
- Is $a \div b$ divisible by 5? Why?

Problem 16.3

If 21 divides n , what other numbers divide n ?

Problem 16.4

Fill each of the following blanks with the greatest digit that makes the statement true:

- (a) $3|74_$
- (b) $9|83_45$
- (c) $11|6_55$.

Problem 16.5

When the two missing digits in the following number are replaced, the number is divisible by 99. What is the number?

$$85_ _1.$$

Problem 16.6

Without using a calculator, test each of the following numbers for divisibility by 2, 3, 4, 5, 6, 8, 9, 10, 11.

- (a) 746,988
- (b) 81,342
- (c) 15,810
- (d) 4,201,012
- (e) 1,001
- (f) 10,001.

Problem 16.7

There will be 219 students in next year's third grade. If the school has 9 teachers, can we assign each teacher the same number of students?

Problem 16.8

Three sisters earn a reward of \$37,500 for solving a mathematics problem. Can they divide the money equally?

Problem 16.9

What three digit numbers are less than 130 and divisible by 6?

Problem 16.10

True or false? If false, give a counter example.

- (a) If a number is divisible by 5 then it is divisible by 10
- (b) If a number is not divisible by 5 then it is not divisible by 10
- (c) If a number is divisible by 2 and 4 then it is divisible by 8
- (d) If a number is divisible by 8 then it is divisible by 2 and 4
- (e) If a number is divisible by 99 then it is divisible by 9 and 11.

Problem 16.11

Test each number for divisibility by 2, 3, and 5. Do the work mentally.

- (a) 1554 (b) 1999 (c) 805 (d) 2450

Problem 16.12

Are the numbers of the previous problem divisible by

- (a) 0 (b) 10 (c) 15 (d) 30

Problem 16.13

Is 1,927,643,001,548 divisible by 11? Explain.

Problem 16.14

At a glance, determine the digit d so that the number 87,543,24 d is divisible by 4. Is there more than one solution?

Problem 16.15

Determine the digit d so that the number 6,34 d ,217 is divisible by 11.

Problem 16.16

Find the digit d so that the number 897,650,243,28 d is divisible by 6.

Problem 16.17

- (a) Determine whether 97,128 is divisible by 2, 4 and 8.
- (b) Determine whether 83,026 is divisible by 2, 4, and 8.

Problem 16.18

Use the divisibility tests to determine whether each of the following numbers is divisible by 3 and divisible by 9.

- (a) 1002 (b) 14,238

Problem 16.19

The store manager has an invoice of 72 four-function calculators. The first and last digits on the receipt are illegible. The manager can read \$67.9_. What are the missing digits, and what is the cost of each calculator?

Problem 16.20

The number 57,729,364,583 has too many digits for most calculator to display. Determine whether this number is divisible by each of the following.

- (a) 2 (b) 3 (c) 5 (d) 6 (e) 8 (f) 9 (g) 10 (h) 11