15 Prime and Composite Numbers

Divides, Divisors, Factors, Multiples
In section 13, we considered the division algorithm: If a and b are whole numbers with $b \neq 0$ then there exist unique numbers q and r such that

$$a = bq + r, \quad 0 \leq r < b.$$

Of special interest is when $r = 0$. In this case, $a = bq$. We say that b **divides** a or b is a **divisor** of a. Also, we call b a **factor** of a and we say that a is a **multiple** of b. When b divides a we will write $a \mid b$.
If b does not divide a we will write $b \not| a$. For example, $2 \not| 3$.

Example 15.1
List all the divisors of 12.

Solution.
The divisors (or factors) of 12 are 1, 2, 3, 4, 6, 12 since $12 = 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4$.■

Next, we discuss some of the properties of "|".

Theorem 15.1
Let a, k, m, n be whole numbers with $a \neq 0$.
(a) If $a|m$ and $a|n$ then $a|(m + n)$.
(b) If $a|m$ and $a|n$ and $m \geq n$ then $a|(m - n)$.
(c) If $a|m$ then $a|km$.

Proof.
(a) Since $a|m$ and $a|n$ then we can find unique whole numbers b and c such that $m = ba$ and $n = ca$. Adding these equalities we find $m + n = a(b + c)$. But the set of whole numbers is closed under addition so that $b + c$ is also a whole number. By the definition of "|" we see that $a|(m + n)$.
(b) Similar to part (a) where $m + n$ is replaced by $m - n$.
(c) Since $a|m$ then $m = ba$ for some unique whole number b. Multiply both sides of this equality by k to obtain $km = (kb)a$. Since the set of whole numbers is closed with respect to multiplication then kb is a whole number. By the definition of "|" we have $a|km$.■

Practice Problems

Problem 15.1
(a) The number $162 = 2 \cdot 3^4$. How many different divisors does 162 have?
(b) Try the same process with $225 = 3^2 \cdot 5^2$.
(c) Based on your results in parts (a) - (b), if p and q are prime numbers and $a = p^m \cdot q^n$ then how many different divisors does n have?

Problem 15.2
(a) List all the divisors of 48.
(b) List all the divisors of 54.
(c) Find the largest common divisor of 48 and 54.

Problem 15.3
Let $a = 2^3 \cdot 3^1 \cdot 7^2$.
(a) Is $2^2 \cdot 7^1$ a factor of a? Why or why not?
(b) Is $2^1 \cdot 3^2 \cdot 7^1$ a factor of a? Why or why not?
(c) How many different factors does a possess?
(d) Make an orderly list of all the factors of a.

Problem 15.4
If n, b, and c are nonzero whole numbers and $n|bc$, is it necessarily the case that $n|b$ or $n|c$? Justify your answer.

Problem 15.5
Which of the following are true or false? Justify your answer in each case.
(a) $n|0$ for every nonzero whole number n.
(b) $0|n$ for every nonzero whole number n.
(c) $0|0$.
(d) $1|n$ for every whole number n.
(e) $n|n$ for every nonzero whole number n.

Problem 15.6
Find the least nonzero whole number divisible by each nonzero whole number less than or equal to 12.

Problem 15.7
If $42|n$ then what other whole numbers divide n?

Problem 15.8
If $2N = 2^6 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^7$, explain why $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ is a factor of N.

Prime and Composite Numbers
Any whole number a greater than 2 has at least two different factors, namely a and 1 since $a = 1 \cdot a$. If a and 1 are the only distinct factors of a then we call a a **prime** number. That is, a prime number is a number with only two distinct divisors 1 and the number itself. Examples of prime numbers are 2, 3, 5, 7, etc.
A number that is not prime is called **composite.** Thus, a composite number is a number that has more than two divisors. Examples of composite numbers are: 4, 6, 8, 9, etc.
The number 1 is called the **unit**. It is neither prime nor composite.

Example 15.2
List all the prime numbers less than 20.

Solution.
The prime numbers less than 20 are: 2, 3, 5, 7, 11, 13, 17, 19.■

Prime Factorization
Composite numbers can be expressed as the product of 2 or more factors greater than 1. For example, $260 = 26 \cdot 10 = 5 \cdot 52 = 26 \cdot 2 \cdot 5 = 2 \cdot 2 \cdot 5 \cdot 13$. When a composite number is written as the product of prime factors such as $260 = 2 \cdot 2 \cdot 5 \cdot 13$ then this product is referred to as the **prime factorization.**
Two procedures for finding the prime factorization of a number:
The Factor-Tree Method: Figure 15.1 shows two factor-trees for 260.

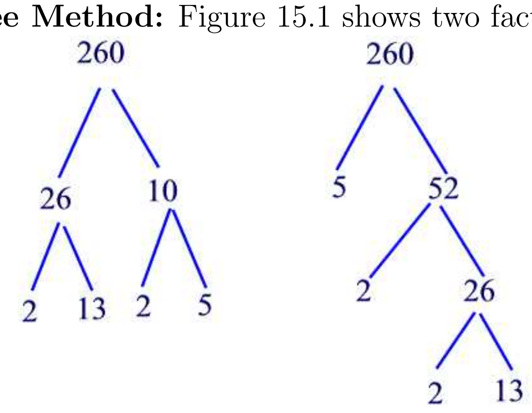


Figure 15.1

Note that a number can have different trees. However, all of them produce the same prime factorization except for order in which the primes appear in the products.
The **Fundamental Theorem of Arithmetic** also known as the **Unique Factorization Theorem** states that in general, if order of the factors are disregarded then the prime factorization is unique. More formally we have

Fundamemtal Theorem of Arithmetic
Every whole number greater than 1 can be expressed as the product of different primes in one and only one way apart from order.

The primes in the prime factorization are typically listed in increasing order from left to right and if a prime appears more than once, exponential notation is used. Thus, the prime factorization of 260 is $260 = 2^2 \cdot 5 \cdot 13$.

Prime-Divisor Method Besides the factor-tree method there is another method known as the "prime-divisor method". In this method, try all prime numbers in increasing order as divisors, beginning with 2. Use each prime number as a divisor as many times as needed. This method is illustrated in Figure 15.2

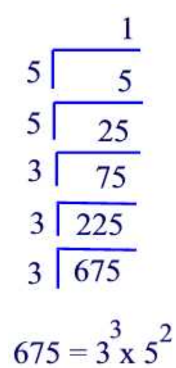
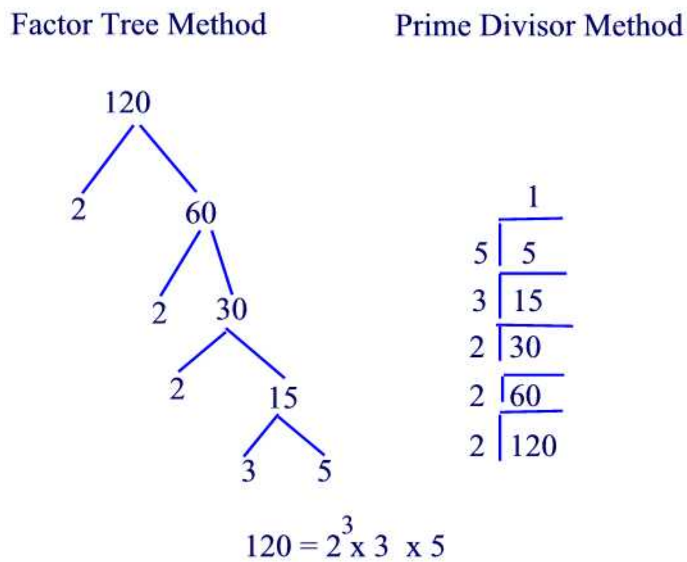


Figure 15.2

Example 15.3
Find the prime factorization of 120 using the two methods described above.

Solution.



Practice Problems

Problem 15.9
Eratosthenes, a Greek mathematician, developed the **Sieve of Eratosthenes** about 2200 years ago as a method for finding all prime numbers less than a given number. Follow the directions to find all the prime numbers less than or equal to 50.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50				

(a) Copy the list of numbers.
(b) Cross out 1 because 1 is not prime.
(c) Circle 2. Count 2s from there, and cross out $4, 6, 8, \cdots, 50$ because all

these numbers are divisible by 2 and therefore are not prime.
(d) Circle 3. Count 3s from there, and cross out all numbers not already crossed out because these numbers are divisible by 3 and therefore are not prime.
(e) Circle the smallest number not yet crossed out. Count by that number, and cross out all numbers that are not already crossed out.
(f) Repeat part (e) until there are no more numbers to circle. The circled numbers are the prime numbers.
(g) List all the prime numbers between 1 and 50.

Problem 15.10
List all prime numbers between 1 and 100 using the Sieve of Eratosthenes.

Problem 15.11
Extend the Sieve of Eratosthenes to find all the primes less than 200.

Problem 15.12
Write the prime factorizations of the following.
(a) 90 (b) 3155 (c) 84.

Problem 15.13
Find the prime factorization using both the factor-tree method and the prime divisor method.
(a) 495 (b) 320.

Problem 15.14
Twin primes are any two consecutive odd numbers, such as 3 and 5, that are prime. Find all the twin primes between 101 and 140.

Problem 15.15
(a) How many different divisors does $2^5 \cdot 3^2 \cdot 7$ have?
(b) Show how to use the prime factorization to determine how many different factors 148 has.

Problem 15.16
Construct factor trees for each of the following numbers.
(a) 72 (b) 126 (c) 264 (d) 550

Problem 15.17
Use the prime divisors method to find all the prime factors of the following numbers.
(a) 700 (b) 198 (c) 450 (d) 528

Problem 15.18
Determine the prime factorizations of each of the following numbers.
(a) 48 (b) 108 (c) 2250 (d) 24750

Problem 15.19
Show that if 1 were considered a prime number then every number would have more than one prime factorization.

Problem 15.20
Explain why $2^3 \cdot 3^2 \cdot 25^4$ is not a prime factorization and find the prime factorization of the number.

Determining if a Given Number is a Prime
How does one determine if a given whole number is a prime? To answer this question, observe first that if n is composite say with two factors b and c then one of its factor must be less than $\sqrt{n}$. For if not, that is, if $b > \sqrt{n}$ and $c > \sqrt{n}$ then

$$n = bc > \sqrt{n} \cdot \sqrt{n} = n,$$

that is $n > n$ which is impossible. Thus, if n is composite then either $b \leq \sqrt{n}$ or $c \leq \sqrt{n}$ or alternatively $b^2 \leq n$ or $c^2 \leq n$.
The above argument leads to the following test for prime numbers.

Theorem 15.2 *(Primality Test)*
If every prime factor of n is greater than $\sqrt{n}$ then n is composite. Equivalently, if there is a prime factor p of n such that $p^2 \leq n$ then n is prime.

Example 15.4
(a) Is 397 composite or prime?
(b) Is 91 composite or prime?

Solution.
(a) The possible primes p such that $p^2 \leq 397$ are 2,3,5,7,11,13,17, and 19. None of these numbers divide 397. So 397 is composite.

(b) The possible primes such that $p^2 \leq 91$ are 2, 3, 5, and 7. Since $7|91$ then by the above theorem 91 is prime.■

Practice Problems

Problem 15.21
Classify the following numbers as prime, composite or neither.
(a) 71 (b) 495 (c) 1

Problem 15.22
Without computing the results, explain why each of the following numbers will result in a composite number.
(a) $3 \times 5 \times 7 \times 11 \times 13$
(b) $(3 \times 4 \times 5 \times 6 \times 7 \times 8) + 2$
(c) $(3 \times 4 \times 5 \times 6 \times 7 \times 8) + 5$

Problem 15.23
To determine that 431 is prime, what is the minimum set of numbers you must try as divisors?

Problem 15.24
Use the Primality Test to classify the following numbers as prime or composite.
(a) 71 (b) 697 (c) 577 (d) 91.

Problem 15.25
What is the greatest prime you must consider to test whether 5669 is prime?