

## 11 Trigonometric Functions of Acute Angles

In this section you will learn (1) how to find the trigonometric functions using right triangles, (2) compute the values of these functions for some special angles, and (3) solve model problems involving the trigonometric functions.

First, let's review some of the features of right triangles. A triangle in which one angle is  $90^\circ$  is called a **right triangle**. The side opposite to the right angle is called the **hypotenuse** and the remaining sides are called the **legs** of the triangle.

Suppose that we are given an acute angle  $\theta$  as shown in Figure 11.1. Note that  $a \neq 0$  and  $b \neq 0$ .

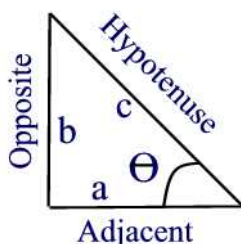


Figure 11.1

Associated with  $\theta$  are three lengths, the hypotenuse, the opposite side, and the adjacent side. We define the values of the trigonometric functions of  $\theta$  as ratios of the sides of a right triangle:

$$\begin{aligned} \sin \theta &= \frac{\textit{opposite}}{\textit{hypotenuse}} = \frac{b}{r} & \cos \theta &= \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{a}{r} & \tan \theta &= \frac{\textit{opposite}}{\textit{adjacent}} = \frac{b}{a} \\ \csc \theta &= \frac{\textit{hypotenuse}}{\textit{opposite}} = \frac{r}{b} & \sec \theta &= \frac{\textit{hypotenuse}}{\textit{adjacent}} = \frac{r}{a} & \cot \theta &= \frac{\textit{adjacent}}{\textit{opposite}} = \frac{a}{b} \end{aligned}$$

where  $r = \sqrt{a^2 + b^2}$  (Pythagorean formula).

The symbols  $\sin$ ,  $\cos$ ,  $\sec$ ,  $\csc$ ,  $\tan$ , and  $\cot$  are abbreviations of **sine**, **co-sine**, **secant**, **cosecant**, **tangent**, and **cotangent**. The above ratios are the same regardless of the size of the triangle. That is, the trigonometric functions defined above depend only on the angle  $\theta$ . To see this, consider

Figure 11.2.

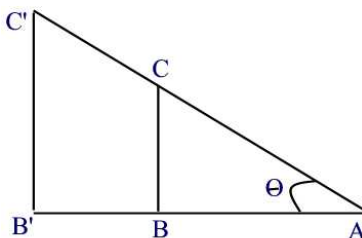


Figure 11.2

The triangles  $\triangle ABC$  and  $\triangle AB'C'$  are similar. Thus,  $\frac{|AB|}{|AB'|} = \frac{|AC|}{|AC'|} = \frac{|BC|}{|B'C'|}$ . For example, using the cosine function we have

$$\cos \theta = \frac{|AB|}{|AC|} = \frac{|AB'|}{|AC'|}.$$

The following identities, known as **reciprocal identities**, follow from the definition given above.

$$\begin{aligned} \sin \theta &= \frac{1}{\csc \theta}, & \cos \theta &= \frac{1}{\sec \theta}, & \tan \theta &= \frac{1}{\cot \theta}, \\ \csc \theta &= \frac{1}{\sin \theta}, & \sec \theta &= \frac{1}{\cos \theta}, & \cot \theta &= \frac{1}{\tan \theta}. \end{aligned}$$

**Example 11.1**

Find the exact value of the six trigonometric functions of the angle  $\theta$  shown in Figure 11.3.

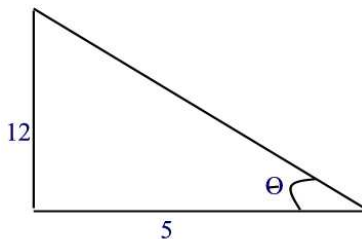


Figure 11.3

**Solution.**

By the Pythagorean Theorem, the length of the hypotenuse is  $\sqrt{144 + 25} = \sqrt{169} = 13$ . Thus,

$$\begin{aligned} \sin \theta &= \frac{12}{13} & \cos \theta &= \frac{5}{13} & \tan \theta &= \frac{12}{5} \\ \csc \theta &= \frac{13}{12} & \sec \theta &= \frac{13}{5} & \cot \theta &= \frac{5}{12} \blacksquare \end{aligned}$$

Given the value of one trigonometric function, it is possible to find the values of the remaining trigonometric functions of that angle.

**Example 11.2**

Suppose that  $\theta$  is an acute angle for which  $\cos \theta = \frac{5}{7}$ . Determine the values of the other five trigonometric functions.

**Solution.**

Since  $\cos \theta = \frac{5}{7}$ , the adjacent side has length 5 and the hypotenuse has length 7. See Figure 11.4. Using the Pythagorean Theorem, the opposite side has length  $\sqrt{49 - 25} = 2\sqrt{6}$ . Thus,

$$\begin{aligned} \sin \theta &= \frac{2\sqrt{6}}{7} & ; & & \cos \theta &= \frac{5}{7} \\ \sec \theta &= \frac{7}{5} & ; & & \tan \theta &= \frac{2\sqrt{6}}{5} \\ \csc \theta &= \frac{7}{2\sqrt{6}} = \frac{7\sqrt{6}}{12} & ; & & \cot \theta &= \frac{5}{2\sqrt{6}} = \frac{5\sqrt{6}}{12} \blacksquare \end{aligned}$$

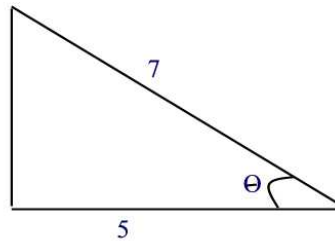


Figure 11.4

**Example 11.3**

Solve for  $y$  given that  $\tan 30^\circ = \frac{\sqrt{3}}{3}$ . (See Figure 8.5).

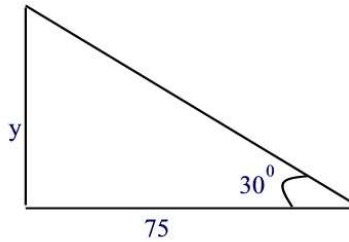


Figure 11.5

**Solution.**

According to Figure 11.5,  $y = 75 \tan 30^\circ = 75\left(\frac{\sqrt{3}}{3}\right) = 25\sqrt{3}$  ■

**Trigonometric Functions of Special Angles**

Next, we compute the trigonometric functions of some special angles. It's useful to remember these special trigonometric ratios because they occur often.

**Example 11.4**

Determine the values of the six trigonometric functions of the angle  $45^\circ$ . See Figure 11.6.

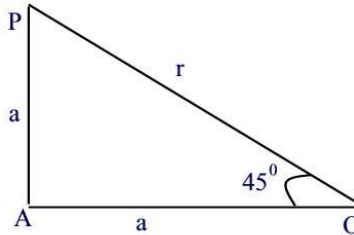


Figure 11.6

**Solution.**

Using Figure 11.6, the triangle  $OAP$  is a right isosceles triangle. By the Pythagorean Theorem we find that  $r^2 = 2a^2$  or  $r = a\sqrt{2}$ . Thus,

$$\begin{aligned} \sin 45^\circ &= \frac{\sqrt{2}}{2} & ; & \quad \csc 45^\circ = \sqrt{2} \\ \cos 45^\circ &= \frac{\sqrt{2}}{2} & ; & \quad \sec 45^\circ = \sqrt{2} \\ \tan 45^\circ &= 1 & ; & \quad \cot 45^\circ = 1 \blacksquare \end{aligned}$$

**Example 11.5**

Determine the trigonometric functions of the angles

- (a)  $\theta = 30^\circ$
- (b)  $\theta = 60^\circ$ .

**Solution.**

(a) Let  $ABC$  be an equilateral triangle with side of length  $a$ . Let  $P$  be the midpoint of the side  $AC$  and  $h$  the height of the triangle. See Figure 11.7. Using the Pythagorean Theorem we find  $h = a\frac{\sqrt{3}}{2}$ .

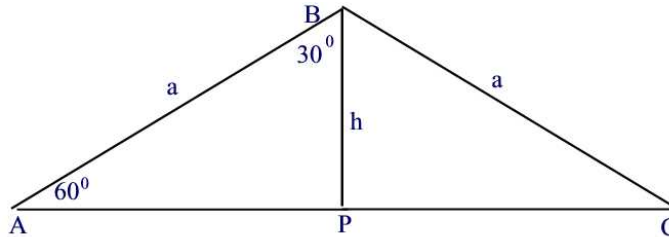


Figure 11.7

Thus,

$$\begin{aligned}\sin 30^\circ &= \frac{1}{2} ; & \cos 30^\circ &= \frac{\sqrt{3}}{2} ; & \tan 30^\circ &= \frac{\sqrt{3}}{3} \\ \csc 30^\circ &= 2 ; & \sec 30^\circ &= \frac{2\sqrt{3}}{3} ; & \cot 30^\circ &= \sqrt{3}.\end{aligned}$$

(b) Similarly,

$$\begin{aligned}\sin 60^\circ &= \frac{\sqrt{3}}{2} ; & \cos 60^\circ &= \frac{1}{2} ; & \tan 60^\circ &= \sqrt{3} \\ \csc 60^\circ &= \frac{2\sqrt{3}}{3} ; & \sec 60^\circ &= 2 ; & \cot 60^\circ &= \frac{\sqrt{3}}{3} \blacksquare\end{aligned}$$

### Example 11.6

Find the exact value of  $2 \sin 60^\circ - \sec 45^\circ \tan 60^\circ$ .

#### Solution.

Using the results of the previous two problems we find that

$$2 \sin 60^\circ - \sec 45^\circ \tan 60^\circ = 2\left(\frac{\sqrt{3}}{2}\right) - \sqrt{2}\sqrt{3} = \sqrt{3} - \sqrt{6}.\blacksquare$$

We follow the convention that when we write a trigonometric function, such as  $\sin t$ , then it is assumed that  $t$  is in radians. If we want to evaluate the trigonometric function of an angle measured in degrees we will use the degree notation such as  $\cos 30^\circ$ .

### Angles of Elevation and Depression

If an observer is looking at an object, then the line from the observer's eye to the object is known as the **line of sight**. If the object is above the horizontal then the angle between the line of sight and the horizontal is called the **angle of elevation**. If the object is below the horizontal then the angle between the line of sight and the horizontal is called the **angle of depression**. See Figure 11.8.

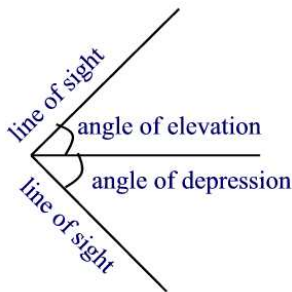


Figure 11.8

**Example 11.7**

From a point 115 feet from the base of a redwood tree, the angle of elevation to the top of the tree is  $64.3^\circ$ . Find the height of the tree to the nearest foot. See Figure 8.9.

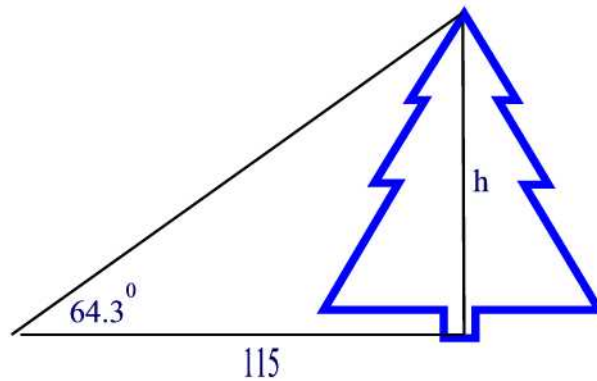


Figure 11.9

**Solution.**

According to Figure 11.9, we use the tangent function to find the height  $h$  of the tree:  $\tan 64.3^\circ = \frac{h}{115}$  so that  $h = 115 \tan 64.3^\circ \approx 238.952$  ft. ■

**Evaluating trigonometric functions with a calculator**

When evaluating trigonometric functions using a calculator, you need to set the calculator to the desired mode of measurement (degrees or radians). The functions sine, cosine, and tangent have a key in a standard scientific calculator. For the remaining three trigonometric functions the key  $x^{-1}$  is used in the process. For example, to evaluate  $\sec \frac{\pi}{8}$ , set the calculator to radian mode, then apply the following sequence of keystrokes:  $\pi$ ,  $\div$ , 8,  $\cos$ ,  $x^{-1}$  and enter to obtain  $\sec \frac{\pi}{8} \approx 1.0824$ .