Arkansas Tech University
MATH 1203: Trigonometry
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In this chapter we introduce the trigonometric functions. These functions can be viewed in two different but equivalent ways. The first way is to view them as functions of real numbers, the other as functions of angles. In both ways, the functions assign the same value to a given real number. The difference is that in the first way, the real number is the length of an arc along the unit circle whereas in the second the number is the measure of an angle. The reason of studying both approaches is due to the fact that different applications require that we view these functions differently. For example, the first approach is needed when modeling harmonic motion. The second is needed when measuring the sides of a triangle.

10 Angles and Arcs

As stated in the introduction above, the two approaches of defining trigonometric functions involve the notions of angles and arcs.
In this section you will learn (1) to identify and classify angles, (2) to measure angles in both degrees and radians, (3) to convert between the units, (4) to find the measures of arcs spanned by angles, (5) to find the area of a circular sector, and (6) to measure linear and angular speeds, given a situation representing a circular motion.
Angles appear in a lot of applications. Let's mention one situation where angles can be very useful. Suppose that you are standing at a point 100 feet away of the Washington monument and you would like to approximate the height of the monument. Assuming that your height is negligeable compared to the height of the monument so that you can be identified by a point on the horizontal line. If you know the amount of opening between the line of sight, i.e. the line connecting you to the top of the monument, and the horizontal line then by applying a specific trigonometric function to that opening you will be able to estimate the height of the monument. The "opening" between the line of sight and the horizontal line gives an example of an angle.
An **angle** is determined by rotating a ray (or a half-line) from one position, called the **initial side**, to a terminal position, called the **terminal side**, as shown in Figure 10.1 below. The point V is called the **vertex** of the angle.

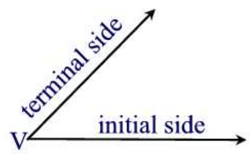

Figure 10.1

If the initial side is the positive $x-$axis then we say that the angle is in **standard position.** See Figure 10.2.

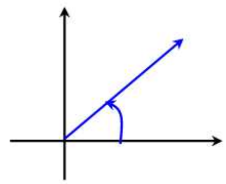

Figure 10.2

Angles that are obtained by a counterclockwise rotation of the initial side are considered **positive** and those that are obtained clockwise are **negative** angles. See Figure 10.3.

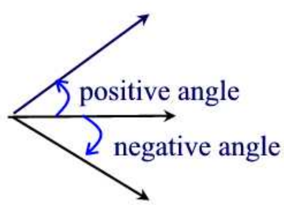

Figure 10.3

Most of the time, we will use Greek lowercase letters such as α (alpha), β (beta), γ (gamma) , etc. to denote angles. If α is an angle obtained by rotating an initial ray $\overrightarrow{OA}$ to a terminal ray $\overrightarrow{OB}$ then we sometimes denote that by writing $\alpha = \angle AOB$.

Angle Measure

The **measure of an angle** is determined by the amount of rotation from the initial side to the terminal side, this is how much the angle "opens". There are two commonly used measures of angles: **degrees** and **radians**

• **Degree Measure:**
If we rotate counterclockwise a ray about a fixed vertex and then return back to its initial position then we say that we have a one complete **revolution.** The angle in this case is said to have measure of 360 degrees, in symbol 360°. Thus, 1° is $\frac{1}{360}$th of a revolution. See Figure 10.4.

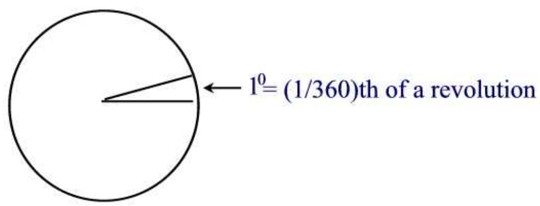

Figure 10.4

Example 10.1
Draw each of the following angles in standard positions: (a) 225° (b) −90° (c) 180°.

Solution.
The specified angles are drawn in Figure 10.5 below ■

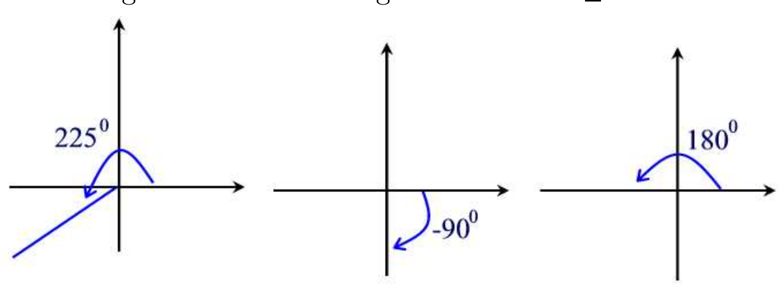

Figure 10.5

Remark 10.1
A protractor can be used to measure angles given in degrees or to draw an angle given in degree measure ■

Now, each degree can be divided into 60 equal parts, each called a **minute**. Thus,

$$1^\circ = 60' \ and \ \ 1' = (\frac{1}{60})^\circ.$$

Similarly, each minute can be divided into 60 equal parts, called **seconds**. Thus,

$$1' = 60'' \quad and \quad 1'' = (\frac{1}{60})' = (\frac{1}{3600})^\circ.$$

By introducing the minutes and seconds units one can now convert a decimal degree to a degree-minute-second format (DMS) as shown in the next example.

Example 10.2
Convert 32.519° to the form $D^\circ M' S''$.

Solution.

$$\begin{aligned}
32.519^\circ &= 32^\circ + 0.519^\circ \\
&= 32^\circ + (1^\circ)(0.519) \\
&= 32^\circ + (60')(0.519) \\
&= 32^\circ + 31.14' \\
&= 32^\circ 31' + 0.14' \\
&= 32^\circ 31' + (1')(0.14) \\
&= 32^\circ 31' + (60'')(0.14) \\
&= 32^\circ 31' 8.4'' \blacksquare
\end{aligned}$$

Example 10.3
Convert $50^\circ 6' 21''$ to the nearest ten-thousandth of a degree.

Solution.

$$\begin{aligned}
50^\circ 6' 21'' &= 50^\circ + 6(1') + 21(1'') \\
&= 50^\circ + (\tfrac{6}{60})^\circ + (\tfrac{21}{3600})^\circ \\
&\approx 50^\circ + .1^\circ + .0058^\circ \\
&= 50.1058^\circ \blacksquare
\end{aligned}$$

Remark 10.2
Angles represented in the DMS form are very useful in applications. For example, latitude describes the position of a point on the earth's surface in relation to the equator. A point on the equator has latitude of 0°. The north pole has a latitude of 90°. For example, New York City has latitude of $40^\circ 45' N$ $\blacksquare$

• **Radian Measure:**
A more natural method of measuring angles used in calculus and other

branches of mathematics is the **radian** measure. The amount an angle opens is measured along the arc of the unit circle with its center at the vertex of the angle.(An angle whose vertex is the center of a circle is called a **central angle**.) One **radian**, abbreviated **rad**, is defined to be the measure of a central angle that intercepts an arc s of length one unit. See Figure 10.6.

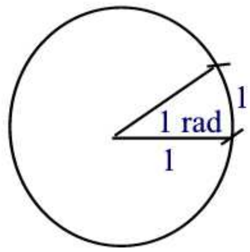

Figure 10.6

Since one complete revolution measured in radians is 2π radians and measured in degrees is $360°$, we have the following conversion formulas:

$$1° = \frac{\pi}{180} \text{ rad} \approx 0.01745 \text{ rad and } 1 \text{ rad} = \left(\frac{180}{\pi}\right)° \approx 57.296°.$$

Example 10.4
Complete the following chart.

degree	30°	45°	60°	90°	180°	270°
radian						

Solution.

degree	30°	45°	60°	90°	180°	270°
radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

By the conversion formulas, we have, for example $30° = 30(1°) = 30\left(\frac{\pi}{180}\right) = \frac{\pi}{6}$. In a similar way we convert the remaining angles. ■

Example 10.5
Convert each angle in degrees to radians: (a) $150°$ (b) $-45°$.

Solution.
(a) $150° = 150(1°) = 150(\frac{\pi}{180}) = \frac{5\pi}{6}$ rad.
(b) $-45° = -45(1°) = -45(\frac{\pi}{180}) = -\frac{\pi}{4}$ rad. ■

Example 10.6
Convert each angle in radians to degrees: (a) $-\frac{3\pi}{4}$ (b) $\frac{7\pi}{3}$.

Solution.
(a) $-\frac{3\pi}{4} = -\frac{3\pi}{4}(1\ rad) = -\frac{3\pi}{4}(\frac{180}{\pi})^\circ = -135^\circ$.
(b) $\frac{7\pi}{3} = \frac{7\pi}{3}(\frac{180}{\pi})^\circ = 420^\circ$ ■

Remark 10.3
When no unit of an angle is given then the angle is assumed to be measured in radians ■

Classification of Angles
Some types of angles have special names:(See Figure 10.7)
1. A 90° angle is called a **right** angle.
2. A 180° angle is called a **straight** angle.
3. An angle between 0° and 90° is called an **acute** angle.
4. An angle between 90° and 180° is called an **obtuse** angle.
5. Two acute angles are **complementary** if their sum is 90°.
6. Two positive angles are **supplementary** if their sum is 180°.
7. Angles in standard positions with terminal sides that lie on a coordinate axis are called **quadrantal angles.** Thus, the angles $0^\circ, \pm 90^\circ, \pm 180^\circ, etc$ are quadrantal angles.

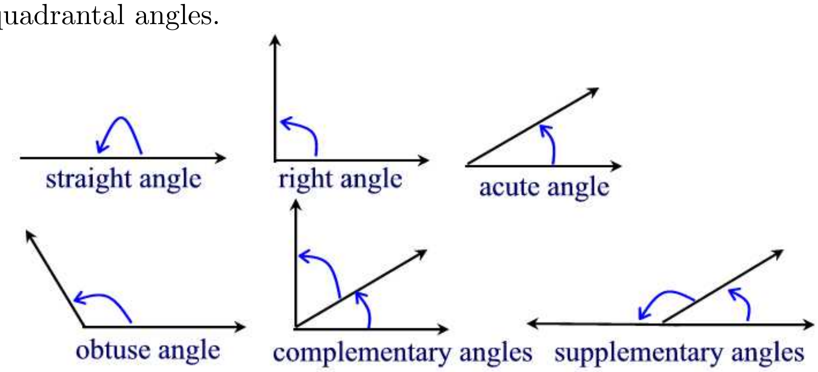

Figure 10.7

Example 10.7
Suppose that $\theta = 41^\circ 28'$. Determine the measure of an angle that is:
(a) Complementary to θ (b) Supplementary to θ.

Solution.
(a) $90^\circ - 41^\circ 28' = 89^\circ 60' - 41^\circ 28' = 48^\circ 32'$.
(b) $180^\circ - 41^\circ 28' = 179^\circ 60' - 41^\circ 28' = 138^\circ 32'$. ■

Remark 10.4
Non quadrantal angles are classified according to the quadrant that contains

the terminal side. For example, when we say that an angle is in Quadrant III then by that we mean that the terminal side of the angle lies in the third quadrant ■

Two angles in standard positions with the same terminal side are called **coterminal**(See Figure 10.8.) We can find an angle that is coterminal to a given angle by adding or subtracting one revolution. Thus, a given angle has many coterminal angles. For instance, $\alpha = 36°$ is coterminal to all of the following angles: $396°, 756°, -324°, -684°$

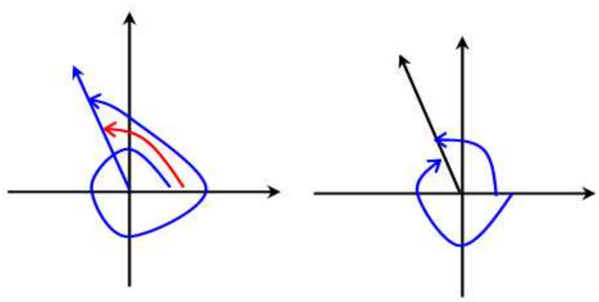

Figure 10.8

Example 10.8
Find a coterminal angle for the following angles, given in standard positions: (a) $530°$ (b) $-400°$.

Solution.
(a) A positive angle coterminal with $530°$ is obtained by adding a multiple of $360°$. For example, $530° + 360° = 890°$. A negative angle coterminal with $530°$ is obtained by subtracting a multiple of $360°$. For example, $530° - 720° = -190°$.
(b) A positive angle is $-400° + 720° = 320°$ and a negative angle is $-400° + 360° = -40°$. ■

Length of a Circular Arc
A circular arc swept out by a central angle is the portion of the circle which is opposite an interior angle. We discuss below a relationship between a central angle θ, measured in radians, and the length of the arc s that it intercepts.

Theorem 10.1
For a circle of radius r, a central angle of θ radians subtends an arc whose length s is given by the formula:

$$s = r\theta$$

Proof.
Suppose that $r > 1$. (A similar argument holds for $0 < r < 1$.) Draw the unit circle with the same center C (See Figure 10.9).

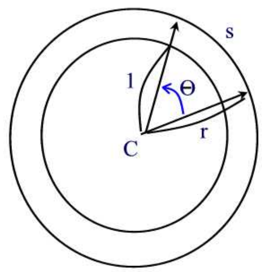

Figure 10.9

By definition of radian measure, the length of the arc determined by θ on the unit circle is also θ. From elementary geometry, we know that the ratio of the measures of the arc lengths are the same as the ratio of the corresponding radii. That is,

$$\frac{r}{1} = \frac{s}{\theta}.$$

Now the formula follows by cross-multiplying. ■

The above formula allows us to define the radian measure using a circle of any radius r. (See Figure 10.10).

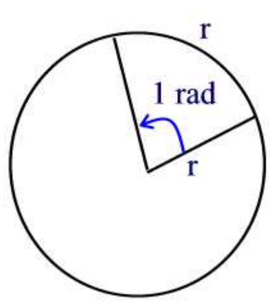

Figure 10.10

Example 10.9
Find the length of the arc of a circle of radius 2 meters subtended by a central angle of measure 0.25 radians.

Solution.
We are given that $r = 2\ m$ and $\theta = 0.25\ rad$. By the previous theorem we have:

$$s = r\theta = 2(0.25) = 0.5\ m \blacksquare$$

Example 10.10
Suppose that a central angle of measure $30°$ is subtended by an arc of length $\frac{\pi}{2}$ feet. Find the radius r of the circle.

Solution.
Substituting in the formula $s = r\theta$ we find $\frac{\pi}{2} = r\frac{\pi}{6}$. Solving for r to obtain $r = 3$ feet. $\blacksquare$

Circular Motion
Consider an object moving along a circle of radius r with a constant speed. Let s denote the distance traveled in time t along this circle and let θ be the central angle, measured in radians, corresponding to s. There are two ways to describe the motion of the object- linear and angular speed. The **linear speed** v of the object is the rate at which the distance traveled is changing. It is defined by the formula

$$v = \frac{s}{t}$$

The **angular speed** ω is the rate at which the central angle is changing. It is given by

$$\omega = \frac{\theta}{t}.$$

Since $s = r\theta$ then we have the following relationship between v and ω

$$v = \frac{s}{t} = \frac{r\theta}{t} = r\omega.$$

Example 10.11
The second hand of a clock is 10.2 centimeters long. Find the linear speed of the tip of the second hand.

Solution.
The distance traveled by the tip of the second hand in one revolution is

$$s = 2\pi(10.2) = 20.4\pi\ cm.$$

Therefore, the linear speed is

$$v = \frac{20.4\pi}{60} \approx 1.068cm/sec \blacksquare$$

Example 10.12
A hard disk in a computer rotates at 3600 revolutions per minute. Find the angular speed of the disk in radians per second.

Solution.
We have

$$\begin{aligned} 3600\ rev/minute &= \tfrac{3600\ rev}{1\ minute}\left(\tfrac{2\pi\ rad}{1\ rev}\right)\left(\tfrac{1\ minute}{60\ seconds}\right) \\ &= \tfrac{120\pi\ radians}{1\ second} \approx 377\ radians/second \blacksquare \end{aligned}$$

Area of a circular sector
A circular sector swept out by an interior angle is the portion of the interior of the circle which is between the two radii, and the circular arc. The area of a circle with radius r is known to be πr^2. This area corresponds to an arc of length $2\pi r$. Let θ be a central angle subtended by an arc of length $r\theta$. See Fig 10.11. The area of the circular sector corresponding to this arc is then

$$A = \frac{\pi r^3\theta}{2\pi r} = \frac{1}{2}r^2\theta.$$

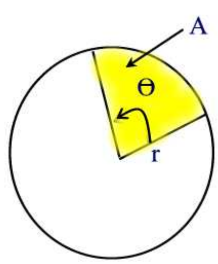

Figure 10.11

Example 10.13
Find the area of a circular sector of radius 10 meters and with central angle $\theta = \frac{\pi}{3}rad$.

Solution.
Substituting in the formula of A yields

$$A = \frac{1}{2}(10)^2(\frac{\pi}{3}) = \frac{50\pi}{3}m^2 \blacksquare$$