3.6 Hyperbolic Functions

Certain even and odd combinations of the exponential functions $e^x$ and $e^{-x}$ arise so frequently in mathematics and its applications that they are given special names. They are referred to collectively as the hyperbolic functions, and individually as the hyperbolic sine (sinh), hyperbolic cosine (cosh), etc.

**DEFINITION OF THE HYPERBOLIC FUNCTIONS**

\[
\begin{align*}
\sinh x &= \frac{e^x - e^{-x}}{2} \\
\cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{\sinh x}{\cosh x} \\
\text{csch} x &= \frac{1}{\sinh x} \\
\text{sech} x &= \frac{1}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x}
\end{align*}
\]

Evaluate each of the following:

1) \( \sinh 0 \)  
\[
\frac{e^0 - e^{-0}}{2} = 0
\]

2) \( \cosh 4 \)  
\[
\frac{e^4 + e^{-4}}{2} = \left( \frac{e^4 + \frac{1}{e^4}}{2} \right) e^4 
\]

3) \( \tanh 1 \)  
\[
\frac{e^1 - e^{-1}}{2} = \frac{e^{-1}}{e^1 + e^{-1}} 
\]
The hyperbolic functions satisfy a number of identities that are similar to well-known trig functions. Some of them are as follows:

\[
\begin{align*}
\sinh(-x) &= -\sinh(x) \quad &\cosh(-x) &= \cosh(x) \\
\cosh^2 x - \sinh^2 x &= 1 \quad &1 - \tanh^2 x &= \text{sech}^2 x
\end{align*}
\]

Prove that \( \cosh^2 x - \sinh^2 x = 1 \)

The previous proof gives some insight into why these are called hyperbolic functions. Whereas a point \( P(\cos t, \sin t) \) lies on the unit circle for any real number \( t \), (satisfying the equation \( x^2 + y^2 = 1 \)), the point \( P(\cosh t, \sinh t) \) lies on the right branch of the hyperbola \( x^2 - y^2 = 1 \)
The derivatives of the hyperbolic functions are easily computed. For example,

\[
\frac{d}{dx} \sinh x = \frac{d}{dx} \left( \frac{e^x - e^{-x}}{2} \right) = \frac{1}{2} \left( e^x + e^{-x} \right) = \cosh x
\]

\[
\frac{d}{dx} \cosh x = \frac{d}{dx} \left( \frac{e^x + e^{-x}}{2} \right) = \frac{1}{2} \left( e^x - e^{-x} \right) = \sinh x
\]
Naturally, any of these can be combined with the Chain Rule. For example, if \( u \) is a function of \( x \), then

\[
\frac{d}{dx} (\sinh u) = \cosh u \frac{du}{dx}
\]

Ex: Find \( y' \) for each of the following:

\[
y = \sinh(\ln x) \quad y = \cosh(e^{5x})
\]

\[
y' = \frac{\cosh(\ln x) \cdot 1}{x} \quad y' = \frac{\sinh(e^{5x}) \cdot e^{5x} \cdot 5}{5e^{5x} \sinh(e^{5x})}
\]
Additionally, as with all other functions, hyperbolic functions can be combined with other functions as products or quotients, requiring us to use the product or quotient rule.

Ex: Find $y'$ for each of the following:

$y = (6x^5 - 7)\sinh(x^2)$

$$y' = (6x^5 - 7) \cdot \cosh(x^2) \cdot 2x + \sinh(x^2) \cdot 30x^4$$

$$= 2x \left[ (6x^5 - 7) \cosh(x^2) + 15x^3 \sinh(x^2) \right]$$

$y = \frac{\cosh(\ln x)}{(5x - 4)^3}$

$$y' = (5x - 4)^3 \cdot \sinh(\ln x) \cdot \frac{1}{x} - \cosh(\ln x) \cdot 3(5x - 4)^2 \cdot \frac{5}{x}$$

$$= (5x - 4)^3 \left( \frac{(5x - 4) \sinh(\ln x)}{x} - 15 \cosh(\ln x) \right) \cdot \frac{5}{x} \cdot \frac{1}{(5x - 4)^4}$$

$$= \left( \frac{(5x - 4) \sinh(\ln x) - 15x \cosh(\ln x)}{x (5x - 4)^4} \right)$$
\[ f(x) = \sinh(x^2) \cosh(e^{3x}) \]

\[ f'(x) = \sinh(x^2) \sinh(e^{3x}) 3e^{3x} + \cosh(e^{3x}) \cosh(x) \cdot 2x \]

\[ = 3e^{3x} \sinh(x^2) \sinh(e^{3x}) + 2x \cosh(e^{3x}) \cosh(x) \]
Note that $y = \sinh x$ and $y = \tanh x$ are odd, one-to-one functions, whereas $y = \cosh x$ is an even function, and thus is not one-to-one. However if we restrict the domain of $y = \cosh x$ to $x \geq 0$, we get a one-to-one function. Thus, we get the inverse hyperbolic trig functions:

\[
\begin{align*}
    y &= \sinh^{-1}x \iff \sinh y = x \\
    y &= \cosh^{-1}x \iff \cosh y = x \quad \text{and} \quad y \geq 0 \\
    y &= \tanh^{-1}x \iff \tanh y = x
\end{align*}
\]

Since the hyperbolic trig functions are defined in terms of exponential functions, it should not be surprising that the inverse hyperbolic trig functions are defined in terms of logarithms, as follows:

\[
\begin{align*}
    \sinh^{-1}x &= \ln(x + \sqrt{x^2 + 1}) \quad x \in \mathbb{R} \\
    \cosh^{-1}x &= \ln(x + \sqrt{x^2 - 1}) \quad x \geq 1 \\
    \tanh^{-1}x &= \frac{1}{2} \ln \left(\frac{1 + x}{1 - x}\right) \quad -1 < x < 1
\end{align*}
\]

Evaluate each of the following:

1) $\sinh^{-1}(1)$
\[
\ln \left(\frac{1 + \sqrt{2}}{2}\right) \approx 0.881
\]

2) $\cosh^{-1}(3)$
\[
\ln \left(3 + \sqrt{3^2 - 1}\right) = \ln (3 + 2\sqrt{2})
\]

3) $\tanh^{-1}(1/2)$
\[
\frac{1}{2} \ln \left(\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}\right) = \frac{1}{2} \ln \left(\frac{3}{1}\right) = \frac{1}{2} \ln 3
\]
Also, it should not be surprising that the inverse hyperbolic trig functions are differentiable functions.

### Derivatives of Inverse Hyperbolic Functions

\[
\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{1 + x^2}} \quad \quad \quad \frac{d}{dx} \cosh^{-1} x = -\frac{1}{|x|\sqrt{x^2 - 1}}
\]

\[
\frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2} \quad \quad \quad \frac{d}{dx} \coth^{-1} x = \frac{1}{1 - x^2}
\]

Again, any of these can be combined with the Chain Rule. For example, if \( u \) is a function of \( x \), then \[
\frac{d}{dx} (\sinh^{-1} u) = \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx}
\]

Ex: Find \( y' \) for each of the following:

\[
y = \sinh^{-1}(4x^3)
\]

\[
y' = \frac{1 \cdot 12x^2}{\sqrt{1 + (4x^3)^2}} = \frac{12x^2}{\sqrt{1 + 16x^6}}
\]

\[
y = \cosh^{-1}(e^{\sin x})
\]

\[
y' = \frac{1 \cdot e^{\sin x} \cdot \cos x}{\sqrt{(e^{\sin x})^2 - 1}} = \frac{e^{\sin x} \cdot \cos x}{\sqrt{e^{2\sin x} - 1}}
\]
Additionally, as with all other functions, inverse hyperbolic functions can be combined with other functions, requiring us to use the product, quotient, or chain rule.

Ex: Find $y'$ for each of the following:

$y = (\ln x)\sinh^{-1}(\ln x)$

$$y' = \ln x \cdot \frac{1}{\sqrt{1+(\ln x)^2}} \cdot \frac{1}{x} + \sinh^{-1}(\ln x) \cdot \frac{1}{x}$$

$$= \frac{1}{x} \left[ \frac{\ln x}{\sqrt{1+(\ln x)^2}} + \sinh^{-1}(\ln x) \right]$$

$y = (\cosh^{-1}(5x))^3$

$$y' = 3(\cosh^{-1}(5x))^2 \cdot \frac{1}{\sqrt{(5x)^2-1}} \cdot 5$$

$$= \frac{15(\cosh^{-1}(5x))^2}{\sqrt{25x^2-1}}$$