2.7 Related Rates

In this section, we consider situations where more than one quantity in a situation is changing with respect to time. Our strategy is to find an equation that relates the quantities, identify what information we are given and what we are looking for, and then use the Chain Rule to differentiate both sides with respect to time. We use Leibniz notation that helps us identify the different rates of change and see the relationships more clearly.

Example: Suppose the radius of a sphere is increasing at a rate of 3 mm/second. How fast is the volume increasing when the diameter is 10 mm?

First, we need a formula for the volume of a sphere, \( V = \frac{4}{3} \pi r^3 \).

The information we are given is \( \frac{dr}{dt} = 3 \) and diameter = 10. We are looking for \( \frac{dV}{dt} \).

Note that \( \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \).

\[
\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}
\]

\[
\frac{dV}{dt} = 4\pi r^2 \cdot 3 = 12\pi r^2
\]

\[
\left. \frac{dV}{dt} \right|_{r=5} = 12\pi \cdot 5^2 = 300\pi \text{ mm}^3/\text{sec}.
\]
Example: Suppose a spherical cell is growing at a rate of 500 $\mu$m$^3$/day. At what rate is its radius increasing at the moment when the radius is 10 $\mu$m?

Again, the needed formula is for the volume of a sphere: $V = \frac{4}{3}\pi r^3$.

\[
\begin{align*}
V &= \frac{4}{3}\pi r^3 \\
\frac{dV}{dt} &= 4\pi r^2 \\
\left.\frac{dr}{dt}\right|_{r=10} &= \frac{500}{4\pi \cdot 10^2} \\
&= \frac{5}{4\pi} \text{ $\mu$m/ day} \\
&\approx 40 \text{ $\mu$m/ day}
\end{align*}
\]
Example: Suppose water is rising in a cylindrical tank with radius 10 m at a rate of 4 m³/minute. How fast is the height of the water increasing?

This time we need the formula for the volume of a cylinder: \( V = \pi r^2 h \)

\[
\frac{dV}{dt} = 100\pi
\]

\[
V = \pi \cdot 10^2 \cdot h
\]

\[
\frac{dV}{dh} = 100\pi
\]

\[
\frac{dV}{dt} = \pi \cdot 100\pi \cdot \frac{dh}{dt}
\]

\[
4 = \frac{100\pi}{100\pi} \cdot \frac{dh}{dt}
\]

\[
\frac{1}{25\pi} \text{ m/min. } \approx 0.013 \text{ m/min.}
\]
Example: Each side of a cube is increasing at a rate of 2 cm/sec. At what rate is the volume of the cube increasing at the moment that the side length is 10 cm?  

\[
\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}
\]

when \( x = 10 \) \( \frac{dx}{dt} = 2 \) \( \frac{dV}{dt} = ? \)

\[
V = x^3
\]

\[
\frac{dV}{dx} = 3x^2
\]

\[
\frac{dV}{dt} = 3x^2 \cdot 2 = 6x^2
\]

\[
\left. \frac{dV}{dt} \right|_{x=10} = 6 \cdot 10^2 = 600 \text{ cm}^3/\text{sec}
\]

Now determine the rate at which the surface area of the cube is increasing at that same point in time.

\[
\frac{dA}{dt} = ?
\]

\[
A = 6x^2
\]

\[
\frac{dA}{dx} = 12x
\]

\[
\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = 12x \cdot 2 = 24x
\]

\[
\left. \frac{dA}{dt} \right|_{x=10} = 24(10) = 240 \text{ cm}^2/\text{sec}
\]
Example: Suppose \( y = \sqrt{6x-5} \), where \( x \) and \( y \) are functions of \( t \).

\[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}
\]

a) If \( \frac{dx}{dt} = 2 \), find \( \frac{dy}{dt} \) when \( x = 1 \).

\[
\frac{dy}{dt} = \frac{3}{\sqrt{6x-5}} \cdot 2 = \frac{6}{\sqrt{6\cdot1-5}} = 6
\]

b) If \( \frac{dy}{dt} = 4 \), find \( \frac{dx}{dt} \) when \( x = 5 \).

\[
\frac{dx}{dt} = \frac{dy}{dt} \cdot \frac{dx}{dt} \cdot \frac{1}{\sqrt{6x-5}}
\]

\[
\frac{4\sqrt{6\cdot5-5}}{3} = \frac{dx}{dt}
\]

\[
\frac{dx}{dt} \bigg|_{x=5} = \frac{4\sqrt{6\cdot5-5}}{3} = \frac{20}{3}
\]
Example: The length of a rectangle is increasing at a rate of 7 cm/sec, and its width is increasing at a rate of 2 cm/sec. When the length is 15 cm and the width is 9 cm, how fast is the area of the rectangle increasing?

\[
\frac{dA}{dt} = 15 \cdot 2 + 9 \cdot 7
\]

\[
\frac{dA}{dt} = 93 \text{ cm}^2/\text{sec}
\]
Example: Suppose \( \frac{1}{x} + \frac{1}{y} = \frac{1}{z} \) where \( x, y, \) and \( z \) are all functions of \( t \).

Find \( dz/dt \) when \( dx/dt = 5, ay/au = 4, \) and \( (x, y, z) = (3, 6, 2) \).

First, solve the equation for \( z \).

\[
\frac{1}{x} + \frac{1}{y} = \frac{1}{z} \quad \text{LCD: } xyz
\]

\[
yz + xz = xy
\]

\[
z = \frac{xy}{y + x}
\]

\[
\frac{dx}{dt} = 5, \quad \frac{dy}{dt} = 4
\]

\[
(x, y, z) = (3, 6, 2)
\]

\[
\frac{dz}{dt} = \frac{(y + x)(x \cdot \frac{dx}{dt} + y \cdot \frac{dx}{dt}) - xy(\frac{dy}{dt} + \frac{dx}{dt})}{(y + x)^2}
\]

\[
= \frac{(6 + 3)(3 \cdot 4 + 6 \cdot 5) - 3 \cdot 6(4 + 5)}{(6 + 3)^2}
\]

\[
= \frac{9(42) - 18(9)}{9^2} = \frac{9(42 - 18)}{9^2} = \frac{9 \cdot 24}{3} = \frac{8}{\frac{8}{3}}
\]