9 Equivalence Relations

In the study of mathematics, we deal with many examples of relations between elements of various sets. For example, in working with the integers, we encounter relations such as "x is less than y". Notice the importance of the ordering of the elements of the set in this relation. That is, "x less than y" is not the same as "y less than x." A relation such as the one just mentioned can be described by the following definition

Definition 9.1
A relation $\sim$ on a nonempty set $S$ is a subset of the Cartesian product $S \times S$. Thus, if $(a, b) \in \sim$ then we write $a \sim b$.

Example 9.1
1. On the set $\mathbb{Z}$ of integers, $\sim = \{(x, 2x) : x \in \mathbb{Z}\}$ is a relation on $\mathbb{Z}$. Note that $a \sim b$ if and only if $b = 2a$.
2. A mapping between from a nonempty set $S$ into $S$ is a relation.

We now consider some properties which a given relation $\sim$ on a set $S$ may or may not have.

Definition 9.2
Let $\sim$ be a relation on a nonempty set $S$. We say that $\sim$ is:

- reflexive if and only if for all $a \in S$, we have $a \sim a$;
- symmetric if and only if for $a, b \in S$ if $a \sim b$ then $b \sim a$;
- transitive if and only if whenever $a, b, c \in S$ such that $a \sim b$ and $b \sim c$ then $a \sim c$.

Example 9.2
Let $S$ be a nonempty set and $\mathcal{P}(S)$ be the collection of all subsets of $S$. Let $\sim$ be the relation defined by

$$A \sim B \iff A \subseteq B, \text{ where } A, B \in \mathcal{P}(S).$$

Then $\sim$ is reflexive and transitive but not symmetric since it is not always true that if $A \subseteq B$ then $B \subseteq A$. (For example, $\{2\} \subseteq \{1, 2\}$ but $\{1, 2\} \nsubseteq \{2\}$.)
Just as there were different classes of functions (one-to-one, onto, and one-to-one correspondence), there are also special classes of relations. One of the most useful kind of relations (besides functions, which of course are also relations) are those called equivalence relations which we define next.

**Definition 9.3**
A relation \( \sim \) on a set \( S \) which is reflexive, symmetric, and transitive is called an **equivalence relation**.

**Example 9.3**
1. The equality ”=” relation between real numbers or sets.
2. The relation ”is similar to” on the set of all triangles.
3. The relation ”\( \geq \)” between real numbers is not an equivalence relation, because although it is reflexive and transitive, it is not symmetric. e.g. \( 7 \geq 5 \) does not imply that \( 5 \geq 7 \).

The following example is important in applications to combinatorics.

**Example 9.4**
Let \( S \) be a nonempty set and \( G \) be a subgroup of \( Sym(S) \). Define the relation \( \sim \) on \( S \) by

\[
a \sim b \iff \alpha(a) = b \text{ for some } \alpha \in G.
\]

Then \( \sim \) is an equivalence relation on \( S \). Indeed,

\( \sim \) **is reflexive**: If \( a \in S \) then \( \iota_S(a) = a \). Since \( G \) is a subgroup and \( \iota_S \in G \) then \( a \sim a \).

\( \sim \) **is symmetric**: Let \( a, b \in S \) such that \( a \sim b \). Then there is an \( \alpha \in G \) such that \( \alpha(a) = b \). Since \( G \) is a group then \( \alpha^{-1} \in G \). Moreover, \( a = \iota_S(a) = (\alpha^{-1} \circ \alpha)(a) = \alpha^{-1}(\alpha(a)) = \alpha^{-1}(b) \). Thus, \( b \sim a \).

\( \sim \) **is transitive**: Let \( a, b, c \in S \) such that \( a \sim b \) and \( b \sim c \). Then there exist permutations \( \alpha \) and \( \beta \) in \( G \) such that \( \alpha(a) = b \) and \( \beta(b) = c \). Since \( G \) is a group then \( G \) is closed under composition and therefore \( \beta \circ \alpha \in G \). Moreover, \( (\beta \circ \alpha)(a) = \beta(\alpha(a)) = \beta(b) = c \). Hence, \( a \sim c \).

An important fact about an equivalence relation on a set \( A \) is that it induces a partition of \( A \) into disjoint sets as indicated in the next theorem.
Theorem 9.1
Let \( A \) be a nonempty set. Let \( \{A_i\}_{i \in \mathbb{N}} \) be a partition of \( A \). That is, \( \{A_i\}_{i \in \mathbb{N}} \) is a family of subsets of \( A \) that satisfies the two conditions:

(i) \( A = \bigcup_{i \in \mathbb{N}} A_i \);
(ii) For \( i \neq j \), \( A_i \cap A_j = \emptyset \).

The relation \( a \sim b \iff a, b \in A_i \), for some \( i \)

is an equivalence relation on \( A \).

Proof.
We need to show that \( \sim \) is reflexive, symmetric and transitive.

\( \sim \) is reflexive: If \( a \in A \) then by (i), \( a \in A_i \) for some \( i \in \mathbb{N} \). From the definition of \( \sim \) with \( b = a \) we have \( a \sim a \).

\( \sim \) is symmetric: Let \( a, b \in A \) such that \( a \sim b \). Then \( a, b \in A_i \) for some \( i \in \mathbb{N} \). But then \( b, a \in A_i \) so that \( b \sim a \).

\( \sim \) is transitive: Let \( a, b, c \in A \) such that \( a \sim b \) and \( b \sim c \). Then \( a, b \in A_i \) for some \( i \in \mathbb{N} \) and \( b, c \in A_j \) for some \( j \in \mathbb{N} \). By (ii), we must have \( i = j \). Thus, \( a, c \in A_i \) for some \( i \in \mathbb{N} \). Hence, \( a \sim c \). ■

The converse of the above theorem is also true. Before proving this claim we introduce the following concept.

Definition 9.4
If \( \sim \) is an equivalence relation on a nonempty set \( A \) and \( a \sim b \) for some \( a, b \in A \) then we say that \( a \) and \( b \) are equivalent. For a fixed \( a \in A \) the set of all elements in \( S \) equivalent to \( a \) is called an equivalence class with representative \( a \). We will write \([a]\). In set-builder notation

\[ [a] = \{ x \in A : x \sim a \} \text{.} \]

The subset of \( A \) containing exactly one element from each equivalent class is called a complete set of equivalence class representatives.

Example 9.5
In the rectangular coordinate system we define the relation

\[ (x_1, y_1) \sim (x_2, y_2) \iff y_1 = y_2 \text{.} \]
(i) Show that $\sim$ is an equivalence relation on the set of points in the plane.
(ii) Describe the equivalence classes geometrically.
(iii) Give a complete set of equivalence class representatives.

**Solution.**
(i) For any $(x, y) \in \mathbb{R}^2$, $(x, y) \sim (x, y)$ so that $\sim$ is reflexive. Now, if $(x_1, y_1) \sim (x_2, y_2)$ then $y_1 = y_2$. But equality in $\mathbb{R}$ is symmetric so that $y_2 = y_1$. Thus, $(x_2, y_2) \sim (x_1, y_1)$ and hence $\sim$ is symmetric. To show that $\sim$ is transitive, suppose that $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$. Then $y_1 = y_2$ and $y_2 = y_3$. Since '=' is transitive in $\mathbb{R}$ then $y_1 = y_3$. Hence $(x_1, y_1) \sim (x_3, y_3)$.
(ii) For a fixed $(a, b) \in \mathbb{R}^2$, the equivalence class of $(a, b)$ is the horizontal line going through the point $(a, b)$.
(iii) The set of points on a line not parallel to the x-axis.

**Theorem 9.2**
If $\sim$ is an equivalence relation on a nonempty set $A$ and $a, b \in A$ are such that $a \sim b$ then $[a] = [b]$.

**Proof.**
The proof is by double inclusions. Let $x \in [a]$. Then $x \sim a$. Since $a \sim b$ and $\sim$ is transitive then $x \sim b$ which means that $x \in [b]$. Thus, $[a] \subseteq [b]$. Now interchange the letters $a$ and $b$ to show that $[b] \subseteq [a]$. Hence, $[a] = [b]$. ■

**Theorem 9.3**
Let $A$ be a nonempty set and $\sim$ be an equivalence relation on $A$. Then the equivalence classes of $A$ define a partition of $A$. That is,

(i) $A = \bigcup_{a \in A} [a]$;
(ii) If $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$.

**Proof.**
By the definition of $[a]$ we have that $[a] \subseteq A$. Hence, $\bigcup_{a \in A} [a] \subseteq A$. We next show that $A \subseteq \bigcup_{a \in A} [a]$. Indeed, let $b \in A$. Since $\sim$ is reflexive then $b \in [b]$ and consequently $b \in \bigcup_{a \in A} [a]$. Hence, $A \subseteq \bigcup_{a \in A} [b]$. It follows that $A = \bigcup_{a \in A} [a]$. This establishes (i).

It remains to show that if $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$ for $a, b \in A$. Equivalently, we must show that if $[a] \cap [b] \neq \emptyset$ then $[a] = [b]$. Since $[a] \cap [b] \neq \emptyset$ then there is an element $c \in [a] \cap [b]$. This means that $c \in [a]$ and $c \in [b]$. Hence, $a \sim c$ and $b \sim c$. Since $\sim$ is symmetric and transitive then $a \sim b$. Now, by Theorem 9.2, $[a] = [b]$. ■