8 Symmetry Groups

In this section, we are interested in the symmetries of planar figures. We can identify a symmetry as a transformation of the plane that moves the figure so that it falls back on itself. The only transformations that we’ll consider are those that preserve distance, called isometries. There are four kinds of planar isometries: translations, rotations, reflections, and glide reflections. In this section we will just consider rotations and reflections.

Rotations: A rotation fixes one point in the plane and turns the rest of it some angle around that point.
Translations: A translation is a mapping that sends all points the same distance in the same direction.
Reflections: A reflection fixes one line in the plane, called the axis of reflection, and exchanges points on one side of the axis with points on the other side of the axis at the same distance from the axis.
Glide-reflection: Any product of a translation and a reflection.

Next, we will associate to each figure a group which characterizes the symmetry of the figure. Let $P$ denote the set of all points in the plane then $\text{Sym}(P)$ is the set of all permutations from $P$ to $P$. Let $M$ be the set of all isometries.

Theorem 8.1
$M$ is a subgroup of $\text{Sym}(P)$.

Proof.
We will show that $M$ satisfies the conditions of Theorem 7.5. Indeed, since $\text{dist}(\iota_P(p), \iota_P(q)) = \text{dist}(p, q)$ for any points $p, q \in P$ then $\iota_P$ is an isometry and therefore belongs to $M$. Thus, $M \neq \emptyset$.
Next, let $\alpha, \beta \in M$. We will show that $\alpha \circ \beta^{-1} \in M$. Since $\alpha$ and $\beta$ are permutations on $P$ then $\alpha \circ \beta^{-1}$ is also a permutation on $P$. Moreover, if
\[ p, q \in P \text{ then} \]
\[
\text{dist}((\alpha \circ \beta^{-1})(p), (\alpha \circ \beta^{-1})(q)) = \text{dist}(\alpha(\beta^{-1}(p)), \alpha(\beta^{-1}(q)))
\]
\[
= \text{dist}(\beta^{-1}(p), \beta^{-1}(q)) \quad (\text{since } \alpha \in M)
\]
\[
= \text{dist}(\beta(\beta^{-1}(p)), \beta(\beta^{-1}(q))) \quad (\text{since } \beta \in M)
\]
\[
= \text{dist}(\iota_P(p), \iota_P(q))
\]
\[
= \text{dist}(p, q)
\]
Thus, \( \alpha \circ \beta^{-1} \in M \). By Theorem 7.5, \( M \) is a subgroup of \( \text{Sym}(P) \).

Now, let \( T \) be a subset of \( P \). Define the set

\[
M(T) = \{ \alpha \in M : \alpha(T) = T \}.
\]

By Theorem 7.6(a), \( M(T) \) is a subgroup of \( M \).

**Definition 8.1**

\( M(T) \) is the group of all symmetries leaving \( T \) invariant. We call this group the symmetry group of \( T \).

**Example 8.1**

In this example we describe the symmetry group of a square with vertices \( \{a, b, c, d\} \) consisting of rotations and reflections. The eight symmetries of the square are (See Figure 8.1)

\[
\mu_1 = \text{identity permutation} = (a \ b \ c \ d)
\]
\[
\mu_2 = \text{Rotation clockwise } 90^\circ \text{ around } p = (a \ b \ c \ d)
\]
\[
\mu_3 = \text{Rotation clockwise } 180^\circ \text{ around } p = (a \ b \ c \ d)
\]
\[
\mu_4 = \text{Rotation clockwise } 270^\circ \text{ around } p = (a \ b \ c \ d)
\]
\[
\mu_5 = \text{Reflection through } H = (a \ b \ c \ d)
\]
\[
\mu_6 = \text{Reflection through } V = (a \ b \ c \ d)
\]
\[
\mu_7 = \text{Reflection through } D_1 = (a \ b \ c \ d)
\]
\[
\mu_8 = \text{Reflection through } D_2 = (a \ b \ c \ d)
\]
Thus,

$$M(T) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8\}$$

\[\text{Figure 8.1}\]

**Example 8.2**

Below is the Cayley table for $M(T)$ where $T$ is the square in the previous example.

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