4 Composition of Mappings as a Binary Operation

For a nonempty set $S$, let $\mathcal{M}(S)$ be the set of all mappings from $S$ to $S$. We have seen that composition of mappings defines a binary operation in $\mathcal{M}(S)$. In this section, we want to study the properties of this operation.

First, we discuss the question of commutativity and inverse elements. In general, composition is not commutative in $\mathcal{M}(S)$. Also, not every element of $\mathcal{M}(S)$ is invertible.

**Example 4.1**

Let $S$ be a nonempty set.

(a) Show that composition in $\mathcal{M}(S)$ is not, in general, commutative.

(b) Show that not every element of $\mathcal{M}(S)$ is invertible.

**Solution.**

(a) Consider the set $\mathcal{M}(\mathbb{Z})$ of all functions from the set of integers into itself. Then the operation of composition is a binary operation on $\mathcal{M}(\mathbb{Z})$. Consider the two functions $\alpha(n) = 2n$ and

$$\beta(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$$

Then $(\beta \circ \alpha)(n) = n$ for all $n \in \mathbb{Z}$. That is, $\beta \circ \alpha = \iota_{\mathbb{Z}}$. However, since

$$(\alpha \circ \beta)(n) = \begin{cases} n & \text{if } n \text{ is even} \\ 8 & \text{if } n \text{ is odd} \end{cases}$$

then $\beta \circ \alpha \neq \alpha \circ \beta$. Hence, composition is not commutative.

(b) From part (a), we see that $\beta \circ \alpha = \iota_{\mathbb{Z}}$. Thus, $\beta$ is a left inverse of $\alpha$. However, $\alpha \circ \beta \neq \iota_{\mathbb{Z}}$ so that $\beta$ is not a right inverse of $\alpha$. Hence, $\alpha$ is not invertible.

Under what conditions on $S$, composition in $\mathcal{M}(S)$ is commutative?
Theorem 4.1
Let $S$ be a nonempty set such that $|S| < 2$. Then composition of mappings in $\mathcal{M}(S)$ is commutative.

Proof.
We will show that if $|S| \geq 2$ then composition of mappings is not commutative. Indeed, if $|S| \geq 2$ then there exist two distinct elements $a$ and $b$ of $S$. Define $\alpha, \beta : S \rightarrow S$ as follows:

$$
\alpha(a) = a \quad \alpha(b) = a \\
\beta(a) = b \quad \beta(b) = b.
$$

Then $(\alpha \circ \beta)(a) = \alpha(\beta(b)) = \alpha(b) = a$ and $(\beta \circ \alpha)(a) = \beta(\alpha(a)) = \beta(b) = b$. Since $a \neq b$ then $\alpha \circ \beta \neq \beta \circ \alpha$. That is, composition in $\mathcal{M}(S)$ is not commutative.$\blacksquare$

The following theorem summarizes the two properties of composition in $\mathcal{M}(S)$.

Theorem 4.2
Let $S$ be a nonempty set.

(a) Composition in $\mathcal{M}(S)$ is associative.

(b) $\iota_S$ is the identity element in $\mathcal{M}(S)$.

Proof.
(a) Let $x \in S$. Then for any $\alpha, \beta, \gamma \in \mathcal{M}(S)$, we have

$$
[\alpha \circ (\beta \circ \gamma)](x) = \alpha(\beta \circ \gamma(x))
$$

$$
= \alpha(\beta(\gamma(x)))
$$

$$
= (\alpha \circ \beta)(\gamma(x))
$$

$$
= [(\alpha \circ \beta) \circ \gamma](x)
$$

Thus, $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$.

(b) If $x \in S$ and $\alpha \in \mathcal{M}(S)$ then

$$
(\alpha \circ \iota_S)(x) = \alpha(\iota_S(x)) = \alpha(x)
$$

so that $\alpha \circ \iota_S = \alpha$. Similarly, $\iota_S \circ \alpha = \alpha$. Hence, $\iota_S$ is the identity element of $\mathcal{M}(S)$ under the operation of composition.$\blacksquare$
Example 4.2
Is it true that if a nonempty set $S$ is closed under a binary operation $\ast$ then
every subset $A$ of $S$ is also closed under $\ast$?

Solution.
If a set is closed under an operation then this does not mean that every
subset is also closed under this operation. For example, $\mathbb{R}$ is closed under
the operation of subtraction whereas $\mathbb{N}$ is not. Thus, composition is a bi-
nary operation on a subset of $\mathcal{M}(S)$ if and only if the subset is closed under
composition.

Now, let’s look at the set $\mathcal{I}(S)$ of all invertible mappings from $S$ to $S$. Clearly,$\mathcal{I}(S)$ is a subset of $\mathcal{M}(S)$.

Theorem 4.3
(a) $\mathcal{I}(S)$ is closed under composition.
(b) Composition is associative on $\mathcal{I}(S)$.
(c) $\iota_S$ is the identity element of $\mathcal{I}(S)$.
(d) Every element of $\mathcal{I}(S)$ is invertible.

Proof.
(a) Let $\alpha$ and $\beta$ be two invertible mappings from $S$ to $S$. Then by Theorem
2.5(b), $\alpha \circ \beta$ is also invertible. That is, $\alpha \circ \beta \in \mathcal{I}(S)$. Hence, $\mathcal{I}(S)$ is closed
under composition.
(b) Since $\mathcal{I}(S)$ is closed with respect to composition and composition on
$\mathcal{M}(S)$ is associative there then it is certainly associative when restricted on
$\mathcal{I}(S)$.
(c) Since $\iota_S^{-1} = \iota_S$ then $\iota_S \in \mathcal{I}(S)$. Since $\iota_S$ is the identity element of $\mathcal{M}(S)$
then $\iota_S$ is the identity element of $\mathcal{I}(S)$.
(d) Follows from the definition of $\mathcal{I}(S)$.

The following theorem can be used to test whether a mapping is one-to-one.

Theorem 4.4
Let $S$ be a nonempty set and $\alpha \in \mathcal{M}(S)$. Then $\alpha$ is one-to-one if and only if
there exists a $\beta \in \mathcal{M}(S)$ such that $\beta \circ \alpha = \iota_S$.

Proof.
Suppose first that $\alpha$ is one-to-one. Pick an element $x_0 \in S$ and define
\( \beta : S \to S \) as follows

\[
\beta(x) = \begin{cases} 
  y & \text{if } \alpha(y) = x \\
  x_0 & \text{if } \alpha(y) \neq x, \forall y \in S
\end{cases}
\]

We show that \( \beta \in \mathcal{M}(S) \). Indeed, if \( \beta(x) = y \) and \( y' \in S \) is such that \( \beta(x) = y' \) then \( \alpha(y) = \alpha(y') \) and since \( \alpha \) is one-to-one then \( y = y' \). That is, \( y \) is the unique element such that \( \beta(x) = y \). If \( \beta(x) = x_0 \) then \( x_0 \) is the unique element such that \( \beta(x) = x_0 \) since \( x_0 \) is fixed. It remains to show that \( \beta \circ \alpha = \iota_S \). To see this, let \( x \in S \) and \( \alpha(x) = y \). Then \( (\beta \circ \alpha)(x) = \beta(\alpha(x)) = \beta(y) = x = \iota_S(x) \). Thus, \( \beta \circ \alpha = \iota_S \).

Conversely, suppose that \( \beta \in \mathcal{M}(S) \) such that \( \beta \circ \alpha = \iota_S \). Since \( \iota_S \) is one-to-one then \( \beta \circ \alpha \) is one-to-one and therefore by Theorem 2.2(b), \( \alpha \) is one-to-one.

For testing onto mappings we have

**Theorem 4.5**

Let \( S \) be a nonempty set and \( \alpha \in \mathcal{M}(S) \). Then \( \alpha \) is onto if and only if there exists a \( \beta \in \mathcal{M}(S) \) such that \( \alpha \circ \beta = \iota_S \).

**Proof.**

Suppose first that \( \alpha \) is onto. Then for each \( y \in S \) there is an \( x \in S \) such that \( \alpha(x) = y \). Define \( \beta : S \to S \) by \( \beta(y) = x \). Then \((\alpha \circ \beta)(y) = \alpha(\beta(y)) = \alpha(x) = y \) so that \( \alpha \circ \beta = \iota_S \).

Conversely, suppose that \( \beta \in \mathcal{M}(S) \) is such that \( \alpha \circ \beta = \iota_S \). Since \( \iota_S \) is onto then \( \alpha \circ \beta \) is onto. By Theorem 2.1 (b), \( \alpha \) is also onto.

Many important operations involve composition on special sets of invertible mappings. We close this section by giving two examples. The first example exhibits an example of a set of mappings where composition is commutative.

**Example 4.3**

Let \( P \) denote the Cartesian plane. Let \( G_p \) be the set of all rotations about a fixed point \( p \). If two rotations differ by a multiple of 360° then we say that they are equal. If \( \alpha \) and \( \beta \) are two elements of \( G_p \) then \( \alpha \circ \beta \) is the rotation obtained by first applying \( \beta \) and then applying \( \alpha \). Thus, \( G_p \) is closed under composition. By Theorem 4.2, composition is associative. An identity element of \( G_p \) is the rotation of 0°. Each rotation has an inverse: rotation of
the same magnitude in the opposite direction. Finally, as an operation on $G_p$, composition is commutative. \( \blacksquare \)

**Example 4.4**

Let $L$ be the set of all linear mappings $\alpha_{a,b} : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\alpha_{a,b}(x) = ax + b$ where $a$ and $b$ are two real numbers with $a \neq 0$.

(i) $L$ is a set of invertible mappings from $\mathbb{R}$ into $\mathbb{R}$.

Indeed, if $\alpha_{a,b} = ax + b$ with $a \neq 0$ then $\alpha_{a,b}^{-1}(x) = \alpha_{\frac{1}{a},-\frac{b}{a}}(x) = \frac{1}{a}x - \frac{b}{a}$.

(ii) $L$ is closed under composition.

To see this, pick two members $\alpha_{a,b}$, say $\alpha_{a,b} = ax + b$ and $\alpha_{c,d}(x) = cx + d$, where $a \neq 0$ and $c \neq 0$. Note that $ac \neq 0$. Moreover,

$$
(\alpha_{a,b} \circ \alpha_{c,d})(x) = \alpha_{a,b}(\alpha_{c,d}(x)) = \alpha_{a,b}(cx + d) = a(cx + d) + b = acx + ad + b = \alpha_{ac,ad+b}(x)
$$

Thus, $\alpha_{a,b} \circ \alpha_{c,d} \in L$.

(iii) Composition is associative on $L$.

Since $L$ is a subset $\mathcal{M}(\mathbb{R})$ and composition is associative on $\mathcal{M}(\mathbb{R})$ then composition is also associative on $L$.

(iv) $\alpha_{1,0}$ is the identity element of $L$.

To see this, let $\alpha_{a,b} \in L$. Then by (ii)

$$
\alpha_{a,b} \circ \alpha_{1,0} = \alpha_{a,b}
$$

and

$$
\alpha_{1,0} \circ \alpha_{a,b} = \alpha_{a,b} \blacksquare
$$

**Remark 4.1**

The identity element can be found as follows: If $\alpha_{e,f}$ is the identity element of $L$ then for any $\alpha_{a,b} \in L$ we must have

$$
\alpha_{e,f} \circ \alpha_{a,b} = \alpha_{a,b} \circ \alpha_{e,f} = \alpha_{a,b}.
$$

This and (ii) imply that $ea = a$ and $eb + f = b$. Thus, $e = 1$ and $f = 0$. 

5
Remark 4.2
Note that, $\alpha_{a,0}(x) = ax, a > 1$ is magnification since it magnifies the distance of each point from the origin by a factor of $a$. Also, $\alpha_{1,b}(x) = x + b, b > 0$ is a translation of $x$, $b$ units to the right. Finally, note that $\alpha_{a,b} = \alpha_{1,b} \circ \alpha_{a,0}$ so that for $a > 1$ and $b > 0$, $\alpha_{a,b}$ corresponds to a magnification followed by a translation.