Solutions to Practice Problems

Exercise 8.8
(a) Show that if \( \{a_n\}_{n=1}^{\infty} \) is Cauchy then \( \{a_n^2\}_{n=1}^{\infty} \) is also Cauchy.
(b) Give an example of a Cauchy sequence \( \{a_n^2\}_{n=1}^{\infty} \) such that \( \{a_n\}_{n=1}^{\infty} \) is not Cauchy.

Solution.
(a) Since \( \{a_n\}_{n=1}^{\infty} \) is Cauchy, it is convergent. Since the product of two convergent sequences is convergent the sequence \( \{a_n^2\}_{n=1}^{\infty} \) is convergent and therefore is Cauchy.
(b) Let \( a_n = (-1)^n \) for all \( n \in \mathbb{N} \). The sequence \( \{a_n\}_{n=1}^{\infty} \) is not Cauchy since it is divergent. However, the sequence \( \{a_n^2\}_{n=1}^{\infty} = \{1, 1, \ldots\} \) converges to 1 so it is Cauchy ■

Exercise 8.9
Let \( \{a_n\}_{n=1}^{\infty} \) be a Cauchy sequence such that \( a_n \) is an integer for all \( n \in \mathbb{N} \). Show that there is a positive integer \( N \) such that \( a_n = C \) for all \( n \geq N \), where \( C \) is a constant.

Solution.
Let \( \epsilon = \frac{1}{2} \). Since \( \{a_n\}_{n=1}^{\infty} \) is Cauchy, there is a positive integer \( N \) such that if \( m, n \geq N \) we have \( |a_m - a_n| < \frac{1}{2} \). But \( a_m - a_n \) is an integer so we must have \( a_n = a_N \) for all \( n \geq N \) ■

Exercise 8.10
Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence that satisfies
\[ |a_{n+2} - a_{n+1}| < c^2|a_{n+1} - a_n| \]
for all \( n \in \mathbb{N} \), where \( 0 < c < 1 \).
(a) Show that \( |a_{n+1} - a_n| < c^n|a_2 - a_1| \) for all \( n \geq 2 \).
(b) Show that \( \{a_n\}_{n=1}^{\infty} \) is a Cauchy sequence.

Solution.
(a) See Exercise 1.10.
(b) Let $\epsilon > 0$ be given. Since $\lim_{n \to \infty} c^n = 0$ we can find a positive integer $N$ such that if $n \geq N$ then $|c|^n < (1 - c)\epsilon$. Thus, for $n > m \geq N$ we have

$$|a_n - a_m| \leq |a_{m+1} - a_m| + |a_{m+2} - a_{m+1}| + \cdots + |a_n - a_{n-1}|$$

$$< c^m |a_2 - a_1| + c^{m+1} |a_2 - a_1| + \cdots + c^{n-1} |a_2 - a_1|$$

$$< c^m (1 + c + c^2 + \cdots) |a_2 - a_1|$$

$$= \frac{c^m}{1 - c} |a_2 - a_1| < \epsilon$$

It follows that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

**Exercise 8.11**

What does it mean for a sequence $\{a_n\}_{n=1}^{\infty}$ to not be Cauchy?

**Solution.**

A sequence $\{a_n\}_{n=1}^{\infty}$ is not a Cauchy sequence if there is a real number $\epsilon > 0$ such that for all positive integers $N$ there exist $n, m \in \mathbb{N}$ such that $n, m \geq N$ and $|a_n - a_m| \geq \epsilon$.

**Exercise 8.12**

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two Cauchy sequences. Define $c_n = |a_n - b_n|$. Show that $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

**Solution.**

Let $\epsilon > 0$ be given. There exist positive integers $N_1$ and $N_2$ such that if $n, m \geq N_1$, and $n, m \geq N_2$ we have $|a_n - a_m| < \frac{\epsilon}{2}$ and $|b_n - b_m| < \frac{\epsilon}{2}$. Let $N = N_1 + N_2$. If $n, m \geq N$ then $|c_n - c_m| = ||a_n - b_n| - |a_m - b_m|| \leq |(a_n - b_n) + (a_m - b_m)| \leq |a_n - a_m| + |b_n - b_m| < \epsilon$. Hence, $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

**Exercise 8.13**

Explain why the sequence defined by $a_n = (-1)^n$ is not a Cauchy sequence.

**Solution.**

We know that every Cauchy sequence is convergent. We also know that the given sequence is divergent. Thus, it can not be Cauchy.

**Exercise 8.14**

Show that every subsequence of a Cauchy sequence is itself a Cauchy sequence.
Solution.
Let \( \{a_n\}_{n=1}^{\infty} \) be a Cauchy sequence. Let \( \{a_{n_k}\}_{k=1}^{\infty} \) be a subsequence of \( \{a_n\}_{n=1}^{\infty} \).
By Exercise 8.7, the sequence \( \{a_n\}_{n=1}^{\infty} \) is convergent and hence Cauchy. 

Exercise 8.15
Prove that if a subsequence of a Cauchy sequence converges to \( L \), then the full sequence also converges to \( L \).

Solution.
Let \( \{a_n\}_{n=1}^{\infty} \) be a Cauchy sequence. Let \( \{a_{n_k}\}_{k=1}^{\infty} \) be a subsequence of \( \{a_n\}_{n=1}^{\infty} \) converging to \( L \). By Exercise ??, the sequence \( \{a_n\}_{n=1}^{\infty} \) is convergent say to a limit \( L' \). By Exercise ??, we must have \( L = L' \).

Exercise 8.16
Prove directly from the definition that the sequence
\[
a_n = \frac{n + 3}{2n + 1}, \quad n \in \mathbb{N}
\]
is a Cauchy sequence.

Solution.
Let \( \epsilon > 0 \) be given. Let \( N \) be a positive integer to be chosen. Suppose that \( n, m \geq N \). We have
\[
|a_n - a_m| = \left| \frac{n + 3}{2n + 1} - \frac{m + 3}{2m + 1} \right| = 3 \frac{|m - n|}{(2n + 1)(2m + 1)} \leq \frac{2m + 2n}{(2n + 1)(2m + 1)} = \frac{(2n + 1) + (2m + 1) - 2}{(2n + 1)(2m + 1)} \leq \frac{1}{2m + 1} + \frac{1}{2n + 1} - \frac{2}{(2n + 1)(2m + 1)} \leq \frac{2}{2N + 1}
\]
Choose \( N \) so that \( \frac{2}{2N + 1} < \epsilon \). That is \( N > \frac{2 - \epsilon}{2\epsilon} \). In this case,
\[
|a_n - a_m| < \epsilon
\]
for all \( n, m \geq N \). That is, \( \{\frac{n + 3}{2n + 1}\}_{n=1}^{\infty} \) is Cauchy.
Exercise 8.17
Consider a sequence defined recursively by $a_1 = 1$ and $a_{n+1} = a_n + (-1)^n n^3$ for all $n \in \mathbb{N}$. Show that such a sequence is not a Cauchy sequence. Does this sequence converge?

Solution.
We will show that there is an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exist $m$ and $n$ such that $m, n \geq N$ but $|a_m - a_n| \geq \epsilon$. Note that $|a_{n+1} - a_n| = n^3 \geq 1$. Let $\epsilon = 1$. Let $N \in \mathbb{N}$. Choose $m = N + 1$ and $n = N$. In this case, $|a_m - a_n| = N^3 \geq 1 = \epsilon$. Hence, the given sequence is not a Cauchy sequence. Since every convergent sequence must be Cauchy, the given sequence is divergent.