Solutions to Practice Problems

Exercise 7.9
Prove that the sequence \( \{a_n\}_{n=1}^{\infty} \) where \( a_n = \cos \frac{n\pi}{2} \) is divergent.

Solution.
Consider the two subsequences \( \{a_{2n+1}\}_{n=1}^{\infty} \) that converges to 0 and \( \{a_{4n}\}_{n=1}^{\infty} \) that converges to 1. Thus, the original sequence must be divergent.

Exercise 7.10
Prove that the sequence \( \{a_n\}_{n=1}^{\infty} \) where
\[
a_n = \frac{(n^2 + 20n + 35) \sin n^3}{n^2 + n + 1}
\]
has a convergent subsequence. Hint: Show that \( \{a_n\}_{n=1}^{\infty} \) is bounded.

Solution.
We have
\[
|a_n| = \left| \frac{(n^2 + 20n + 35) \sin n^3}{n^2 + n + 1} \right|
\]
\[
= \frac{(n^2 + 20n + 35) |\sin n^3|}{n^2 + n + 1}
\]
\[
\leq \frac{n^2 + 20n + 35}{n^2 + n + 1}
\]
\[
\leq \frac{n^2 + 20n + 35}{n^2} = 1 + \frac{20}{n} + \frac{35}{n^2} \leq 56
\]
Hence, by the Bolzano-Weierstrass theorem the given sequence has a convergent subsequence.

Exercise 7.11
Show that the sequence defined by \( a_n = 2 \cos n - \sin n \) has a convergent subsequence.

Solution.
We have \(-2 \leq 2 \cos n \leq 2\) and \(-1 \leq -\sin n \leq 1\). Adding we obtain \(-3 \leq a_n \leq 3\) so that the given sequence is bounded. By the Bolzano-Weierstrass theorem, the given sequence has a convergent subsequence.
Exercise 7.12
True or false: There is a sequence that converges to 6 but contains a subsequence converging to 0. Justify your answer.

Solution.
By Exercise 7.4, this cannot happen □

Exercise 7.13
Give an example of an unbounded sequence with a bounded subsequence.

Solution.
Let 
\[
a_n = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
n & \text{if } n \text{ is even}
\end{cases}
\]
Then \( \{a_n\}_{n=1}^{\infty} \) is unbounded. However, the subsequence \( \{a_{2n+1}\}_{n=1}^{\infty} = \{0, 0, 0, \cdots\} \) is bounded □

Exercise 7.14
Show that the sequence \( \{(-1)^n\}_{n=1}^{\infty} \) is divergent by using subsequences.

Solution.
This sequence has two subsequences \( \{a_{2n}\}_{n=1}^{\infty} \) that converges to 1 and \( \{a_{2n+1}\}_{n=1}^{\infty} \) that converges to \(-1\). Thus, the original sequence cannot be convergent according to Exercise 7.4 □

Exercise 7.15
Suppose that a sequence \( \{a_n\}_{n=1}^{\infty} \) has only one convergent subsequence, say with limit \( a \). Does it follow that \( \lim_{n \to \infty} a_n = a \)?

Solution.
Consider the sequence defined by \( a_{2n-1} = 0 \) for \( n = 1, 2, \cdots \) and \( a_{2n} = n \) for \( n = 1, 2, \cdots \). That is, \[
\{0, 1, 0, 2, 0, 3, 0, 4, \cdots\}
\]
The only convergent subsequence is \( \{a_{2n-1}\}_{n=1}^{\infty} \) with limit 0. However, the original sequence is divergent □

Exercise 7.16
Consider the unbounded sequence defined by \( a_n = n + (-1)^n n \). Find a subsequence that converges to 0.

Solution.
The subsequence that converges to 0 is defined by \( a_{2n-1} = 0 \) for all \( n \in \mathbb{N} \) □