Solutions to Practice Problems

Exercise 20.7
Find \( \lim_{x \to \infty} \left( \frac{\ln x}{x} \cdot \sin \left( \frac{x\pi + 2}{2x} \right) \right) \).

Solution.
By L'Hôpital's rule we have
\[
\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0.
\]
By the continuity of the sine function we have
\[
\lim_{x \to \infty} \sin \left( \frac{x\pi + 2}{2x} \right) = \sin \frac{\pi}{2} = 1.
\]
Thus,
\[
\lim_{x \to \infty} \left( \frac{\ln x}{x} \cdot \sin \left( \frac{x\pi + 2}{2x} \right) \right) = 0. \quad \square
\]

Exercise 20.8
Let \( f, g : [a, b] \to \mathbb{R} \) be continuous on \( [a, b] \) and differentiable in \( a < x < b \) with \( g'(x) \neq 0 \) for all \( a < x < b \). Suppose that \( \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \infty \) for some \( a \leq c \leq b \). Also, suppose that
\[
\lim_{x \to c} \frac{f'(x)}{g'(x)} = A.
\]
Prove that
\[
\lim_{x \to c} \frac{f(x)}{g(x)} = A.
\]

Solution.
We have
\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{1}{g(x)} \cdot \frac{g(x)}{f(x)} = \lim_{x \to c} \frac{g'(x)}{f'(x)} \cdot \left( \lim_{x \to c} \frac{f(x)}{g(x)} \right)^2.
\]
Thus, we have
\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} = A. \quad \square
\]
Exercise 20.9
Use L’Hôpital’s rule to evaluate \( \lim_{x \to 0^+} x^x \). Note that \( 0^0 \) is an undeterminate form. Hint: \( x^x = e^{x \ln x} \).

Solution.
We have
\[
\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x}.
\]
But
\[
\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{x}{x}} = \lim_{x \to 0^+} -x = 0.
\]
Thus,
\[
\lim_{x \to 0^+} x^x = e^{\lim_{x \to 0^+} x \ln x} = e^0 = 1 \quad \blacksquare
\]

Exercise 20.10
Let \( f \) and \( g \) be invertible differentiable functions such that
\( f(1) = 2; g(2) = 1; f'(1) = g'(2) = 3 \).

Find the derivative \( (f^{-1} \circ g^{-1})'(1) \).

Solution.
We have
\[
(f^{-1} \circ g^{-1})'(1) = [f^{-1}]'(g^{-1}(1)) \cdot [g^{-1}]'(1) = \frac{1}{f'(f^{-1}(g^{-1}(1)))} \cdot \frac{1}{g'(g^{-1}(1))} = \frac{1}{f'(f^{-1}(2))} \cdot \frac{1}{g'(2)} = \frac{1}{f'(1)} \cdot \frac{1}{g'(2)} = \frac{1}{9} \quad \blacksquare
\]

Exercise 20.11
Let \( f(x) = x \tan^2 x \) for \( x \in (0, \frac{\pi}{2}) \). Calculate \( (f^{-1})'(\pi) \). Note that \( f(\frac{\pi}{3}) = \pi \).

Solution.
By the inverse function theorem we have
\[
(f^{-1})'(\pi) = \frac{1}{f'(f^{-1}(\pi))} = \frac{1}{f'(\frac{\pi}{3})} = \left( 3 + \frac{8}{\sqrt{3}} \pi \right)^{-1} \quad \blacksquare
\]