Solutions to Practice Problems

Exercise 18.7
Assume $a_0, a_1, \cdots, a_n$ are real numbers such that

$$\frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \cdots + \frac{a_1}{2} + a_0 = 0$$

Show that the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

has at least one root in $(0, 1)$.

Solution.
Let

$$F(x) = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \cdots + \frac{a_1}{2} x^2 + a_0 x.$$ 

Note that $F$ is continuous in $[0, 1]$ and differentiable in $(0, 1)$ with derivative $F'(x) = f(x)$. Moreover, $F(0) = F(1) = 0$. By Rolle’s theorem, there is a $c \in (0, 1)$ such that $F'(c) = 0$. Hence, $f(c) = 0$.

Exercise 18.8
(a) Show that the function $f(x) = x^3 - 4x^2 - 3x + 1$ has a root in $[0, 2]$.
(b) Use Rolle’s theorem to show that there is exactly one root in $[0, 2]$.

Solution.
(a) We have $f(0) = 1 > 0$ and $f(2) = -13 < 0$ so that by IVT there is a root in $[0, 2]$.
(b) Suppose that $x_1$ and $x_2$ are two roots of $f$ in $[0, 2]$. Then by Rolle’s theorem we must have $c \in (0, 2)$ such that $f'(c) = 0$. But the solutions to $f'(x) = 0 = 3x^2 - 8x - 3$ are $x = 3$ and $x = -\frac{1}{3}$ where neither is in $[0, 2]$. Hence, $f$ has exactly one solution in $[0, 2]$.

Exercise 18.9
Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable, and let $a, b \in \mathbb{R}$ be such that $a < b$. Show that there is a $c \in (a, b)$ such that

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Hint: Apply Rolle’s theorem to the function $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$. 

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Solution.
Let \( h : [a, b] \to \mathbb{R} \) be the function defined by
\[
h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).
\]
Then \( h \) is continuous on \([a, b]\) and differentiable in \( a < x < b \) with derivative
\[
h'(x) = [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x).
\]
Moreover
\[
h(a) = h(b) = f(b)g(a) - g(b)f(a).
\]
By Rolle’s theorem there is a \( a < c < b \) such that \( h'(c) = 0 \). That is
\[
[f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0
\]
or
\[
[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c) \quad \blacksquare
\]

Exercise 18.10
Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is twice differentiable with \( f''(x) \neq 0 \) for all \( x \in \mathbb{R} \). Show that for any real number \( L \) the equation \( f(x) = L \) can have at most two solutions.

Solution.
Suppose the contrary. Let \( x_1 < x_2 < x_3 \) be solutions to the equation \( f(x) = L \). Then by Rolle’s theorem applied to \([x_1, x_2]\) and \([x_2, x_3]\) we can find real numbers \( x_1 < x_4 < x_2 \) and \( x_2 < x_5 < x_3 \) such that \( f'(x_4) = f'(x_5) = 0 \). Since \( f' \) is continuous in \([x_4, x_5]\) and \( f' \) is differentiable in \((x_4, x_5)\), we can apply Rolle’s theorem to \( f' \) on the interval \([x_4, x_5]\) to obtain a number \( x_4 < x_6 < x_5 \) such that \( f''(x_6) = 0 \). But this contradicts the fact that \( f''(x) \neq 0 \) for all \( x \in \mathbb{R} \). Thus, the equation \( f(x) = L \) has at most two solutions \( \blacksquare \)

Exercise 18.11
Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function. Let \( a, b, k \in \mathbb{R} \) be such that \( a < b \) and \( f''(a) < k < f'(b) \). Define the function \( g : \mathbb{R} \to \mathbb{R} \) by \( g(x) = kx - f(x) \).
(a) Show that \( g \) has a local maximum in \((a, b)\).
(b) Show that there is a \( c \in (a, b) \) such that \( f'(c) = k \).

Solution.
(a) Since \( f \) is differentiable so is \( g(x) \). Since \( g \) is continuous in \([a, b]\), \( g \) has a maximum at some point \( c \in [a, b] \). Since \( g'(a) = k - f'(a) > 0 \) and \( g'(b) = k - f'(b) < 0 \) we must have \( c \in (a, b) \).
(b) Since \( g \) has a maximum at \( x = c \), we can write \( g'(c) = 0 \). But this implies that \( f'(c) = k \) \( \blacksquare \)