Solution to Assignment 10.9

1. $\mathbf{r}(t) = \langle -\frac{1}{2}t^2, t \rangle$ \Rightarrow \hspace{1cm} \text{At } t = 2:
   \begin{align*}
   \mathbf{v}(t) &= \mathbf{r}'(t) = \langle -t, 1 \rangle \quad \Rightarrow \quad \mathbf{v}(2) = \langle -2, 1 \rangle \\
   \mathbf{a}(t) &= \mathbf{r}''(t) = \langle -1, 0 \rangle \quad \Rightarrow \quad \mathbf{a}(2) = \langle -1, 0 \rangle \\
   |\mathbf{v}(t)| &= \sqrt{t^2 + 1}
   \end{align*}

3. $\mathbf{r}(t) = 3 \cos t \mathbf{i} + 2 \sin t \mathbf{j}$ \Rightarrow \hspace{1cm} \text{At } t = \pi/3:
   \begin{align*}
   \mathbf{v}(t) &= -3 \sin t \mathbf{i} + 2 \cos t \mathbf{j} \quad \Rightarrow \quad \mathbf{v}(\frac{\pi}{3}) = \langle -\frac{3\sqrt{3}}{2}, \frac{1}{2} \rangle \\
   \mathbf{a}(t) &= -3 \cos t \mathbf{i} - 2 \sin t \mathbf{j} \quad \Rightarrow \quad \mathbf{a}(\frac{\pi}{3}) = \langle -\frac{3}{2}, -\sqrt{3} \rangle \\
   |\mathbf{v}(t)| &= \sqrt{9 \sin^2 t + 4 \cos^2 t} = \sqrt{4 + 5 \sin^2 t}
   \end{align*}
   Notice that $x^2/9 + y^2/4 = \sin^2 t + \cos^2 t = 1$, so the path is an ellipse.

5. $\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + 2 \mathbf{k}$ \Rightarrow \hspace{1cm} \text{At } t = 1:
   \begin{align*}
   \mathbf{v}(t) &= \mathbf{i} + 2t \mathbf{j} \quad \Rightarrow \quad \mathbf{v}(1) = \mathbf{i} + 2 \mathbf{j} \\
   \mathbf{a}(t) &= 2 \mathbf{j} \quad \Rightarrow \quad \mathbf{a}(1) = 2 \mathbf{j} \\
   |\mathbf{v}(t)| &= \sqrt{1 + 4t^2}
   \end{align*}
   Here $x = t$, $y = t^2$ \Rightarrow $y = x^2$ and $z = 2$, so the path of the particle is a parabola in the plane $z = 2$.

7. $\mathbf{r}(t) = \langle t^2 + t^2 - t, t^3 \rangle$ \Rightarrow $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t + 1, 2t - 1, 3t^2 \rangle$, $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 2, 6t \rangle$,
   \begin{align*}
   |\mathbf{v}(t)| &= \sqrt{(2t + 1)^2 + (2t - 1)^2 + (3t^2)^2} = \sqrt{9t^4 + 8t^2 + 2}.
   \end{align*}
9. \( \mathbf{r}(t) = \sqrt{2} \mathbf{i} + e^t \mathbf{j} + e^{-t} \mathbf{k} \Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = \sqrt{2} \mathbf{i} + e^t \mathbf{j} - e^{-t} \mathbf{k}, \quad \mathbf{a}(t) = \mathbf{v}'(t) = e^t \mathbf{j} + e^{-t} \mathbf{k}, \quad |\mathbf{v}(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}. \)

11. \( \mathbf{a}(t) = \mathbf{i} + 2 \mathbf{j} \Rightarrow \mathbf{v}(t) = \int \mathbf{a}(t) \, dt = \int (\mathbf{i} + 2 \mathbf{j}) \, dt = \mathbf{t} + 2t \mathbf{j} + \mathbf{C} \) and \( \mathbf{k} = \mathbf{v}(0) = \mathbf{C}. \)
   
   so \( \mathbf{C} = \mathbf{k} \) and \( \mathbf{v}(t) = \mathbf{t} + 2t \mathbf{j} + \mathbf{k}. \quad \mathbf{r}(t) = \int \mathbf{v}(t) \, dt = \int (\mathbf{t} + 2t \mathbf{j} + \mathbf{k}) \, dt = \frac{1}{2} t^2 \mathbf{i} + t^2 \mathbf{j} + t \mathbf{k} + \mathbf{D}. \)

   But \( \mathbf{i} = \mathbf{r}(0) = \mathbf{D}, \) so \( \mathbf{D} = \mathbf{i} \) and \( \mathbf{r}(t) = \left( \frac{1}{2} t^2 + 1 \right) \mathbf{i} + t^2 \mathbf{j} + t \mathbf{k}. \)

13. (a) \( \mathbf{a}(t) = 2t \mathbf{i} + \sin t \mathbf{j} + \cos 2t \mathbf{k} \Rightarrow \)

   \( \mathbf{v}(t) = \int (2t \mathbf{i} + \sin t \mathbf{j} + \cos 2t \mathbf{k}) \, dt = t^2 \mathbf{i} - \cos t \mathbf{j} + \frac{1}{2} \sin 2t \mathbf{k} + \mathbf{C} \)

   and \( \mathbf{i} = \mathbf{v}(0) = -\mathbf{j} + \mathbf{C}, \) so \( \mathbf{C} = \mathbf{i} + \mathbf{j} \)

   and \( \mathbf{v}(t) = (t^2 + 1) \mathbf{i} + (1 - \cos t) \mathbf{j} + \frac{1}{2} \sin 2t \mathbf{k}. \)

   \( \mathbf{r}(t) = \int [(t^2 + 1) \mathbf{i} + (1 - \cos t) \mathbf{j} + \frac{1}{2} \sin 2t \mathbf{k}] \, dt \)

   \( = \left( \frac{1}{3} t^3 + t \right) \mathbf{i} + (t - \sin t) \mathbf{j} - \frac{1}{4} \cos 2t \mathbf{k} + \mathbf{D} \)

   But \( \mathbf{j} = \mathbf{r}(0) = -\frac{1}{4} \mathbf{k} + \mathbf{D}, \) so \( \mathbf{D} = \mathbf{j} + \frac{1}{4} \mathbf{k} \) and \( \mathbf{r}(t) = \left( \frac{1}{3} t^3 + t \right) \mathbf{i} + (t - \sin t + 1) \mathbf{j} + \left( \frac{1}{4} - \frac{1}{4} \cos 2t \right) \mathbf{k}. \)

15. \( \mathbf{r}(t) = (t^2, 5t, t^2 - 16t) \Rightarrow \mathbf{v}(t) = (2t, 5, 2t - 16), \quad |\mathbf{v}(t)| = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281} \)

   and \( \frac{d}{dt} |\mathbf{v}(t)| = \frac{1}{2} (8t^2 - 64t + 281)^{-1/2} (16t - 64). \) This is zero if and only if the numerator is zero, that is,

   \[ 16t - 64 = 0 \quad \text{or} \quad t = 4. \] Since \( \frac{d}{dt} |\mathbf{v}(t)| < 0 \) for \( t < 4 \) and \( \frac{d}{dt} |\mathbf{v}(t)| > 0 \) for \( t > 4, \) the minimum speed of \( \sqrt{153} \) is attained at \( t = 4 \) units of time.

17. \( |\mathbf{F}(t)| = 20 \text{ N} \) in the direction of the positive z-axis, so \( \mathbf{F}(t) = 20 \mathbf{k}. \) Also \( m = 4 \text{ kg}, \mathbf{r}(0) = 0 \) and \( \mathbf{v}(0) = \mathbf{i} - \mathbf{j}. \)

   Since \( 20 \mathbf{k} = \mathbf{F}(t) = 4 \mathbf{a}(t), \mathbf{a}(t) = 5 \mathbf{k}. \) Then \( \mathbf{v}(t) = 5t \mathbf{k} + \mathbf{c}_1 \) where \( \mathbf{c}_1 = i - j \) so \( \mathbf{v}(t) = \mathbf{i} - \mathbf{j} + 5t \mathbf{k} \) and the speed is \( |\mathbf{v}(t)| = \sqrt{1 + 1 + 25t^2} = \sqrt{25t^2 + 2}. \) Also \( \mathbf{r}(t) = t \mathbf{i} - t \mathbf{j} + \frac{5}{2} t^2 \mathbf{k} + \mathbf{c}_2 \) and \( \mathbf{0} = \mathbf{r}(0), \) so \( \mathbf{c}_2 = 0 \)

   and \( \mathbf{r}(t) = t \mathbf{i} - t \mathbf{j} + \frac{5}{2} t^2 \mathbf{k}. \)
19. \( |v(0)| = 200 \text{ m/s} \) and, since the angle of elevation is 60°, a unit vector in the direction of the velocity is

\[(\cos 60^\circ)i + (\sin 60^\circ)j = \frac{1}{2}i + \frac{\sqrt{3}}{2}j.\]

Thus \( v(0) = 200 \left( \frac{1}{2}i + \frac{\sqrt{3}}{2}j \right) = 100i + 100\sqrt{3}j \) and if we set up the axes so that the projectile starts at the origin, then \( r(0) = 0 \). Ignoring air resistance, the only force is that due to gravity, so \( F(t) = ma(t) = -mgj \) where \( g \approx 9.8 \text{ m/s}^2 \). Thus \( a(t) = -9.8j \) and, integrating, we have \( v(t) = -9.8tj + C \). But

\[100i + 100\sqrt{3}j = v(0) = C, \]

so \( v(t) = 100i + (100\sqrt{3} - 9.8t)j \) and then (integrating again)

\[r(t) = 100t i + (100\sqrt{3}t - 4.9t^2)j + D \text{ where } 0 = r(0) = D.\]

Thus the position function of the projectile is

\[r(t) = 100t i + (100\sqrt{3}t - 4.9t^2)j.\]

(a) Parametric equations for the projectile are \( x(t) = 100t, y(t) = 100\sqrt{3}t - 4.9t^2 \). The projectile reaches the ground when

\[y(t) = 0 \text{ and } t > 0 \rightarrow 100\sqrt{3}t - 4.9t^2 = t(100\sqrt{3} - 4.9t) = 0 \rightarrow t = \frac{100\sqrt{3}}{4.9} \approx 35.3 \text{ s}. \]

So the range is

\[x \left( \frac{100\sqrt{3}}{4.9} \right) = 100 \left( \frac{100\sqrt{3}}{4.9} \right) \approx 3535 \text{ m}.\]

(b) The maximum height is reached when \( y(t) \) has a critical number (or equivalently, when the vertical component of velocity is 0):

\[y'(t) = 0 \Rightarrow 100\sqrt{3} - 9.8t = 0 \Rightarrow t = \frac{100\sqrt{3}}{9.8} \approx 17.7 \text{ s}. \]

Thus the maximum height is

\[y \left( \frac{100\sqrt{3}}{9.8} \right) = 100\sqrt{3} \left( \frac{100\sqrt{3}}{9.8} \right) - 4.9 \left( \frac{100\sqrt{3}}{9.8} \right)^2 \approx 1531 \text{ m}.\]

(c) From part (a), impact occurs at \( t = \frac{100\sqrt{3}}{4.9} \text{ s}. \) Thus, the velocity at impact is

\[v \left( \frac{100\sqrt{3}}{4.9} \right) = 100i + \left[ 100\sqrt{3} - 9.8 \left( \frac{100\sqrt{3}}{4.9} \right) \right]j = 100i - 100\sqrt{3}j \text{ and the speed is} \]

\[\sqrt{10,000 + 30,000} = 200 \text{ m/s}. \]

21. As in Example 5, \( r(t) = (v_0 \cos 45^\circ)t i + [(v_0 \sin 45^\circ)t - \frac{1}{2}gt^2]j \) and \( t = \frac{v_0\sqrt{2}}{g} \text{ s}. \) Now since it lands 90 m away, \( 90 = x = \frac{1}{2}v_0 \sqrt{2} \left\{ \frac{v_0 \sqrt{2}}{g} \right\}^2 \) or \( v_0 = 90g \) and the initial velocity is \( v_0 = \sqrt{90g} \approx 30 \text{ m/s}. \)
23. Let $\alpha$ be the angle of elevation. Then $v_0 = 150 \text{ m/s}$ and from Example 5, the horizontal distance traveled by the projectile is

$$d = \frac{v_0^2 \sin 2\alpha}{g}. \quad \text{Thus} \quad \frac{150^2 \sin 2\alpha}{g} = 800 \quad \Rightarrow \quad \sin 2\alpha = \frac{800g}{150^2} \approx 0.3484 \quad \Rightarrow \quad 2\alpha \approx 20.4^\circ \text{ or } 180 - 20.4 = 159.6^\circ.$$

Two angles of elevation then are $\alpha \approx 10.2^\circ$ and $\alpha \approx 79.8^\circ$.

31. $r(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k} \quad \Rightarrow \quad r'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}, \quad |r'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2},$

$r''(t) = -\cos t \mathbf{i} - \sin t \mathbf{j}, \quad r'(t) \times r''(t) = \sin t \mathbf{i} - \cos t \mathbf{j} + \mathbf{k}.$

Then $a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|} = \frac{\sin t \cos t - \sin t \cos t}{\sqrt{2}} = 0$ and $a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|} = \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$

33. $r(t) = (3t - t^3) \mathbf{i} + 3t^2 \mathbf{j} \Rightarrow r'(t) = (3 - 3t^2) \mathbf{i} + 6t \mathbf{j},$

$|r'(t)| = \sqrt{(3 - 3t^2)^2 + (6t)^2} = \sqrt{9 + 18t^2 + 9t^4} = \sqrt{(3 - 3t^2)^2} = 3 + 3t^2, r''(t) = -6t \mathbf{i} + 6 \mathbf{j}.$

$r'(t) \times r''(t) = (18 + 18t^2) \mathbf{k}.$ Then Equation 9 gives

$$a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|} = \frac{(3 - 3t^2)(-6t) + (6t)(6)}{3 + 3t^2} = \frac{18t + 18t^3}{3 + 3t^2} = \frac{18t(1 + t^2)}{3(1 + t^2)} = 6t \quad \text{[or by Equation 8,]}$$

$$a_T = v = \frac{d}{dt} [3 + 3t^2] = 6t \quad \text{and Equation 10 gives} \quad a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|} = \frac{18 + 18t^2}{3 + 3t^2} = \frac{18(1 + t^2)}{3(1 + t^2)} = 6.$$