11.1 Functions of Two/Three Variables

In many real world problems one encounters a quantity that depends on more than one input. For example, if an amount of money $A$ is invested at a simple annual interest rate $r$ for a period of $t$ years then the balance at the end of $t$ years is given by the formula $B = A(1 + r)^t$. Thus, $B$ can be regarded as a function of three variables $A, r,$ and $t$. In function notation we will write

$$B = f(A, r, t).$$

**Multivariable calculus** is the study of functions of more than one variable. In this course we will mainly focus on functions having two or three variables. However, functions of four, five, or more variables do occur in models of the physical world and the results presented in the course also apply to such functions.

First, we introduce the definition of a function of two variables: A scalar-valued function of two real variables $x$ and $y$ is a rule, $f$, that associates with each choice of $x$ and $y$ a single real number $f(x, y)$ called the value of $f$ at $(x, y)$. In function notation, we write

$$z = f(x, y).$$

We call $z$ the dependent variable and $x$ and $y$ the independent variables. The set $\text{Dom}(f) = \{(x, y) : f(x, y) \text{ exists}\}$ is called the domain of $f$. The range of $f$ is the set of all possible values of $f(x, y)$ for each $(x, y)$ in the domain of $f$.

**Example 11.1.1**
Consider the function $f(x, y) = \sqrt{1 - x^2 - y^2}$.
(a) Find $f\left(\frac{1}{2}, \frac{1}{2}\right)$ and $f(2, 1)$.
(b) Find the domain and the range of the function $f$.

**Solution.**
(a) We have $f\left(\frac{1}{2}, \frac{1}{2}\right) = \sqrt{1 - \frac{1}{4} - \frac{1}{4}} = \sqrt{\frac{1}{2}}$. But $f(2, 1) = \sqrt{1 - 4 - 1} = \sqrt{-4}$ which is an imaginary number. This means that $(2, 1)$ is not in the domain of $f$.
(b) The domain consists of all points $(x, y)$ that satisfy the inequality $x^2 + y^2 \leq 1$. That is, the domain is the disk centered at the origin and with radius 1. The range is the closed interval $0 \leq z \leq 1$.

**Example 11.1.2**
Find the domain and range of the function $f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$.

**Solution.**
The domain is $\{(x, y) | y < x\}$ because of the square root in the denominator and the range is $\{z | z > 0\}$. 

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Example 11.1.3
Find the domain and range of the function \( f(x, y) = \sqrt{\frac{x}{y}} \).

Solution.
The domain is \( \{(x, y) | xy \geq 0 \text{ and } y \neq 0\} \) and the range is \( \{z | z \geq 0\} \).

Graph of \( z = f(x, y) \)
If you recall that the graph of a function \( f \) of one variable \( x \) is the set of all points \( (x, y) \) in the two dimensional plane such that \( y = f(x) \) and \( x \) in the domain of \( f \). That is, the graph is a curve in the 2-D system. In a similar way, the graph of a function \( f \) of two variables \( x \) and \( y \) is the set of all ordered triples \( (x, y, z) \) such that \( z = f(x, y) \) and \( (x, y) \) is in the domain of \( f \). The graph is a surface in the 3-D space.

Example 11.1.4
Sketch the graph of \( z = f(x, y) = 12 - 3x - 4y \).

Solution.
The graph is a plane shown in Figure 11.1.1. A function of the form \( z = f(x, y) = c + ax + by \) is called a linear function. The graph is a plane in the 3-D space. To graph the plane one usually finds the intersection points with the three axes and then graphs the triangle that connects those three points. This triangle will be a portion of the plane and it will give us a fairly decent idea on what the plane itself should look like.

![Figure 11.1.1](image)

Example 11.1.5
Sketch the graph of \( z = f(x, y) = \sqrt{1 - x^2 - y^2} \).

Solution.
From \( z = \sqrt{1 - x^2 - y^2} \) we obtain \( x^2 + y^2 + z^2 = 1 \). Thus, the graph of \( f \) is the upper half of the sphere centered at the origin and with radius 1 as shown in...
Example 11.1.6
Sketch the graph of \( z = f(x, y) = x^2 + y^2 \).

Solution.
The graph is the paraboloid shown in Figure 11.1.3.

Example 11.1.7
Sketch the graph of \( z = f(x, y) = 6 - x^2 - y^2 \).
Solution.
First note that \( z = 6 - (x^2 + y^2) \). The graph of this function is a reflection of the previous paraboloid about the \( xy \)-plane followed by a vertical translation 6 units up along the \( z \)-axis as shown in Figure 11.1.4.

![Figure 11.1.4](image)

As you can see, graphing functions in space manually is quite difficult and is not an easy matter. You will not need to do this. We usually graph function of two variables using a graphical device such as a computer or a graphing calculator.

Example 11.1.8
Use a computer or a graphing calculator to graph the cylinder \( z = f(x, y) = x^2 \).

Solution.
Note that the missing variable is \( y \). So the cylinder has axis of symmetry the \( y \)-axis. The graph is shown in Figure 11.1.5.

![Figure 11.1.5](image)

**Level Curves and Contour Diagrams**
Graphs provide one way of visualizing functions of two variables. Another important way of visualizing such functions is by drawing their contour diagrams. Given a function of two variables \( z = f(x, y) \). The cross-section between the
surface and a horizontal plane projected onto the $xy$–plane is called a **level curve** or a **contour curve**. Thus, level curves have algebraic equations of the form $f(x, y) = k$ for all possible values of $k$. A **contour diagram** or **contour map** of a function $f(x, y)$ is a 2-dimensional graph showing several level curves in the $xy$–plane corresponding to several values of $k$.

**Example 11.1.9**

Draw a contour diagram of $z = f(x, y) = \sqrt{x^2 + y^2}$ showing several level curves.

**Solution.**

The surface representing the given function is a cone centered at the origin as shown in Figure 11.1.6(a). Horizontal planes crossing this surface trace circles in these planes. Thus, the level curves are circles centered at the origin in the $xy$–plane. Figure 11.1.6(b) shows a contour map consisting of the level curves $k = 0, 1, 2, 3, 4, 5$.

![Figure 11.1.6](image)

**Remark 11.1.1**

One can create a contour diagram from a surface and vice versa. If the surface is given, then we create contour diagrams by joining all the points at the same height on the surface and dropping the curve into the $xy$–plane. On the other hand, if the contour diagram is given with a constant value assigned to each contour curve, we obtain the surface by lifting each contour curve to a height equal to its assigned value.

**Example 11.1.10**

Find equations for the level curves of $f(x, y) = x^2 - y^2$ and draw a contour diagram for $f$.

**Solution.**

The level curves are the curves of the form $x^2 - y^2 = k$, for all possible values of $k$. For $k = 0$ we find the two perpendicular lines $y = -x$ and $y = x$. For $k \neq 0$
the graphs of \( x^2 - y^2 = k \) are hyperbolas with asymptotes consisting of the lines \( y = \pm x \). Figure 11.1.7 shows the contour map. Note that the \( x^2 - y^2 > 0 \) is a hyperbola that crosses the \( x \)-axis whereas \( x^2 - y^2 < 0 \) is a hyperbola crossing the \( y \)-axis.

**Example 11.1.11**

Draw a contour diagram of the function \( f(x, y) = 2x + 3y + 6 \) showing several level curves.

**Solution.**

The contour diagram is a set of evenly spaced parallel lines of common slope \(-\frac{2}{3}\) as shown in Figure 11.1.8.

**Functions of Three Variables**

Functions of three variables appear in many applications. For instance, the temperature \( T \) at a point on the surface of the earth depends on the longitude \( x \) and the latitude \( y \) of the point and on the time \( t \), so we could write \( T = f(x, y, t) \) so that \( T \) is a function of three variables.

Functions of three variables are defined in the same way as functions of two variables. We say that \( f \) is a function of the variables \( x, y, z \) if \( f \) is a rule that assigns to every ordered triples \( (x, y, z) \) a unique number \( w = f(x, y, z) \). The **domain** of \( f \) is the set of all ordered triples \( (x, y, z) \) such that \( f(x, y, z) \) exists.
Thus, the domain is a subset of 3-D. The range of \( f \) is the collection of all numbers \( f(x, y, z) \) where \((x, y, z)\) is in the domain of \( f \).

**Example 11.1.12**

Find the domain and range of the function \( w = f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2} \).

**Solution.**
The domain consists of all ordered triples \((x, y, z)\) satisfying the inequality \( x^2 + y^2 + z^2 \leq 1 \). Thus, the domain is the unit ball centered at the origin and with radius 1. The range is the interval \( 0 \leq w \leq 1 \).

The graph of \( f \) is the set of the 4-D space consisting of all points of the form \((x, y, z, f(x, y, z))\).

We have seen that a function of a single variable has a graph that represents a curve in the \( xy \)-plane, i.e., a two dimensional space or 2-D space. Likewise, the graph of a function of two variables is a surface in 3-D. Going to a function of three variables gives us a surface in 4-D space which can’t be drawn. This makes visualizing functions with three or more variables much more difficult. For functions of three variables however, their contours are surfaces in 3-D space, so at least we can visualize them by graphing these surfaces.

By a level surface of a function \( w = f(x, y, z) \) we mean a 3-D surface of the form \( f(x, y, z) = c \), where \( c \) is a constant. The function \( f \) can be represented by the family of level surfaces obtained by allowing \( c \) to vary. Keep in mind that the value of the function is constant on each level surface.

**Example 11.1.13**

What do the level surfaces of \( f(x, y, z) = x^2 + y^2 + z^2 \) look like?

**Solution.**
The level surfaces are given by the equations \( x^2 + y^2 + z^2 = c \) where \( c \geq 0 \). The set of level surfaces for this function are just a concentric set of spheres of different radii. Figure 11.1.9 shows the inside of the lower half spheres.
Example 11.1.14
What do the level surfaces of \( f(x, y, z) = x^2 + y^2 \) look like?

Solution.
Level surfaces have equations of the form
\[
x^2 + y^2 = c
\]
where \( c \geq 0 \). Each level surface is a circular cylinder of radius \( \sqrt{c} \) around the \( z \)-axis. The level surfaces are concentric cylinders as shown in Figure 11.1.10. Note that \( f \) has smaller values on the narrow cylinders near the \( z \)-axis and larger values on the wider ones.

Figure 11.1.10

Example 11.1.15
What do the level surfaces of \( f(x, y, z) = x^2 + y^2 - z^2 \) look like?

Solution.
Level surfaces are given by the equation \( x^2 + y^2 - z^2 = c \). For \( c < 0 \) the level surface is a surface known as a hyperboloid with two sheets. If \( c = 0 \) we obtain a cone. If \( c > 0 \) the surface is called a hyperboloid with one sheet. See Figure 11.1.11.

Finally we note that the graph of a two-variable function \( z = f(x, y) \) can be thought of as one member in a family of level surfaces representing the three-variable function:
\[
g(x, y, z) = f(x, y) - z
\]
Indeed, the graph of \( z = f(x, y) \) is just the level surface \( g(x, y, z) = 0 \). Conversely, a single level surface \( g(x, y, z) = c \) can be regarded as the graph of
a function $f(x, y)$ if it is possible to solve for $z$ in terms of $x$ and $y$. For example, the level surface $x^2 + y^2 - z + 3 = 0$ is just the graph of the function $z = f(x, y) = x^2 + y^2 + 3$. 

Figure 11.1.11