10.7 The Calculus of Vector Functions

A vector function $\vec{r}(t)$ is a rule that assigns to each real number $t$ in its domain a unique vector $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ in $\mathbb{R}^3$. The domain of $\vec{r}(t)$ is the collection of all values of $t$ such that $f(t), g(t),$ and $h(t)$ are defined. We call $f(t), g(t)$ and $h(t)$ the components of $\vec{r}(t)$.

Example 10.7.1
Find the domain of the vector function $\vec{r}(t) = \ln t\vec{i} + \frac{1}{t-1}\vec{j} + e^{3t}\vec{k}$.

Solution.
The domain is the set of real numbers where all three components are defined. The component $f(t) = \ln t$ is defined for all $t > 0$. The component $g(t) = \frac{1}{t-1}$ is defined for all $t \neq 1$ and the component $h(t) = e^{3t}$ is defined for all real numbers. Hence, the domain of $\vec{r}(t)$ is the set $(0, 1) \cup (1, \infty)$.

Limit of a Vector Function
For a vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ we define the limit

$$\lim_{t \to t_0} \vec{r}(t) = \left( \lim_{t \to t_0} f(t) \right) \vec{i} + \left( \lim_{t \to t_0} g(t) \right) \vec{j} + \left( \lim_{t \to t_0} h(t) \right) \vec{k}$$

provided that the limits of the three components exist. Limits of vector functions obey the same rules as limits of real-valued functions.

Example 10.7.2
Find $\lim_{t \to 0} \vec{r}(t)$ where $\vec{r}(t) = \frac{\sin t}{t}\vec{i} + te^{2t}\vec{j} + (2t - 3)\vec{k}$.

Solution.
We have

$$\lim_{t \to 0} \vec{r}(t) = \left[ \lim_{t \to 0} \frac{\sin t}{t} \right] \vec{i} + \left[ \lim_{t \to 0} te^{2t} \right] \vec{j} + \left[ \lim_{t \to 0} (2t - 3) \right] \vec{k}
= -3\vec{k}$$

Continuity
We say that a vector function $\vec{r}(t)$ is continuous at $t_0$ if and only if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0).$$

Equivalently,

$$\lim_{t \to t_0} \vec{r}(t) = f(t_0)\vec{i} + g(t_0)\vec{j} + h(t_0)\vec{k}.$$
Example 10.7.3
Show that \( \vec{r}(t) = \cos t \vec{i} + e^{-t^2} \vec{j} + (3t + 1) \vec{k} \) is continuous at \( t = 0 \).

Solution.
We have
\[
\lim_{t \to 0} \vec{r}(t) = \left[ \lim_{t \to 0} \cos t \right] \vec{i} + \left[ \lim_{t \to 0} e^{-t^2} \right] \vec{j} + \left[ \lim_{t \to 0} (3t + 1) \right] \vec{k} = \vec{i} + \vec{j} + \vec{k} = \vec{r}(0)
\]

Space Curves and Continuity
A space curve \( C \) is the collection of all points \((x, y, z)\) in \( \mathbb{R}^3 \) where
\[
\begin{align*}
x &= x(t), \\
y &= y(t), \\z &= z(t)
\end{align*}
\] (10.7.1)
and \( t \) varies through an interval \( I \). The equations (10.7.1) are called \textbf{parametric equations} and \( t \) is called the \textbf{parameter}. We can think of \( C \) as being traced out by a moving particle whose position at time \( t \) is \((f(t), g(t), h(t))\). There is a close connection between continuous vector functions and space curves. Let \( \vec{r}(t) = <f(t), g(t), h(t)> \) be a position vector with tip the point \((f(t), g(t), h(t))\) on \( C \). If \( \vec{r}(t) \) is continuous then \( \vec{r}(t) \) defines a space curve \( C \) that is traced out by the tip of the moving vector \( \vec{r}(t) \), as shown in Figure 10.7.1.

![Figure 10.7.1](image)

Example 10.7.4
Sketch the curve whose vector function is \( \vec{r}(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k} \).

Solution.
The parametric equations are
\[
\begin{align*}
x(t) &= \cos t, \\
y(t) &= \sin t, \\
z(t) &= t.
\end{align*}
\]
Since $x^2 + y^2 = 1$ the curve lies on the right cylinder $x^2 + y^2 = 1$. Since $z = t$, the curve spirals upward as $t$ increases. This curve is called a helix. See Figure 10.7.2.

Example 10.7.5
Find a vector function that represents the curve of the intersection of the right cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Solution.
Since the plane $y + z = 2$ is not parallel to the $xy-$plane, the curve of intersection is an ellipse. This ellipse lies on the cylinder so that $x = \cos t$ and $y = \sin t$. Also, the curve lies in the plane so that $z = 2 - y = 2 - \sin t$. Hence, the vector function for the curve of the intersection is

$$\vec{r}(t) = \cos t \vec{i} + \sin t \vec{j} + (2 - \sin t) \vec{k}, \ 0 \leq t \leq 2\pi.$$ 

Derivatives
Suppose that $\vec{r}(t) = < f(t), g(t), h(t) >$ and $f(t), g(t), h(t)$ are differentiable functions. Proceeding as in the case of the derivative of one variable, we find

$$\vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \lim_{h \to 0} \frac{< f(t+h), g(t+h), h(t+h) > - < f(t), g(t), h(t) >}{h} = \lim_{h \to 0} < \frac{f(t+h) - f(t)}{h}, \frac{g(t+h) - g(t)}{h}, \frac{h(t+h) - g(t)}{h} > = < f'(t), g'(t), h'(t) >.$$

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Geometrically, $\vec{r}'(t)$ is the tangent vector to the space curve whose parametric equations are: $x = f(t)$, $y = g(t)$, $z = h(t)$. See Figure 10.7.3.

![Figure 10.7.3](image)

**Example 10.7.6**
(a) Find the derivative of $\vec{r}(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k}$.
(b) Find the unit tangent vector at the point where $t = 0$.

**Solution.**
(a) We have
\[ \vec{r}'(t) = -\sin t \vec{i} + \cos t \vec{j} + \vec{k}. \]
(b) The unit tangent vector at the point where $t = 0$ is
\[ \vec{T}(0) = \frac{\vec{r}'(0)}{||\vec{r}'(0)||} = \frac{\vec{j} + \vec{k}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \vec{j} + \frac{1}{\sqrt{2}} \vec{k} \]

**Example 10.7.7**
Find parametric equations for the tangent line to the helix $\vec{r}(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k}$ at the point $(0, 1, \pi/2)$.

**Solution.**
The parameter value corresponding to the point $(0, 1, \pi/2)$ is $t = \frac{\pi}{2}$ so the tangent vector is $\vec{r}' \left( \frac{\pi}{2} \right) = <-1, 0, 1>$.
\[
\text{The tangent line is the line passing through } (0, 1, \pi/2) \text{ and parallel to the vector } <-1, 0, 1> \text{ whose parametric equations are}
\]
\[
x = -t, \ y = 1, \ z = \frac{\pi}{2} + t
\]
Remark 10.7.1
Just as for single variable functions, one can take higher order derivatives of a real valued vector function \( \vec{r}(t) \).

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

**Theorem 10.7.1**
Suppose \( \vec{u}(t) \) and \( \vec{v}(t) \) are differentiable vector functions, \( c \) is a scalar, and \( f(t) \) is a real-valued function. Then

1. \[ [\vec{u} \pm \vec{v}]' = \vec{u}' \pm \vec{v}' \]
2. \[ [cu]' = cu' \]
3. \[ [f(t)\vec{u}]' = f'(t)\vec{u} + f(t)\vec{u}' \]
4. \[ [\vec{u} \cdot \vec{v}]' = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}' \]
5. \[ [\vec{u} \times \vec{v}]' = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}' \]
6. **Chain Rule:** \[ [\vec{u}(f(t))]' = f'(t)\vec{u}'(f(t)) \]

**Example 10.7.8**
Let \( \vec{r}(t) \) be a vector function with a constant length, say \( c > 0 \). Show that \( \vec{r}(t) \cdot \vec{r}'(t) = 0 \).

**Solution.**
We have

\[
0 = [\vec{r}(t) \cdot \vec{r}(t)]' = \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 2\vec{r}(t) \cdot \vec{r}'(t).
\]

Thus, \( \vec{r}(t) \cdot \vec{r}'(t) = 0 \) and the two vectors are orthogonal.

**Integrals**
The definite integral of a continuous vector function \( \vec{r}(t) \) can be defined in much the same way as for real-valued functions except that the integral is a vector. But then we can express the integral of \( \vec{r}(t) \) in terms of the integrals of its component functions \( f(t), g(t), \) and \( h(t) \) as follows

\[
\int_a^b \vec{r}(t)dt = \lim_{n \to \infty} \sum_{i=1}^n \vec{r}(t_i^*) \Delta t
\]

\[
= \left( \lim_{n \to \infty} \sum_{i=1}^n f(t_i^*) \Delta t \right) \vec{i} + \left( \lim_{n \to \infty} \sum_{i=1}^n g(t_i^*) \Delta t \right) \vec{j} + \left( \lim_{n \to \infty} \sum_{i=1}^n h(t_i^*) \Delta t \right) \vec{k}
\]

\[
= \left( \int_a^b f(t)dt \right) \vec{i} + \left( \int_a^b g(t)dt \right) \vec{j} + \left( \int_a^b h(t)dt \right) \vec{k}.
\]
We say that $\vec{R}(t)$ is an antiderivative of $\vec{r}(t)$ if $\vec{R}'(t) = \vec{r}(t)$. All antiderivatives of $\vec{r}$ are then given by the indefinite integral

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$$

where $\vec{C}$ is a vector constant of integration.

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_{a}^{b} \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a).$$

**Example 10.7.9**

Find $\int \vec{r}(t)\,dt$ where $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$.

**Solution.**

We have

$$\int \vec{r}(t)\,dt = \left(\int \cos t\,dt\right)\vec{i} + \left(\int \sin t\,dt\right)\vec{j} + \left(\int t\,dt\right)\vec{k} = \sin t\vec{i} - \cos t\vec{j} + \frac{t^2}{2}\vec{k} + \vec{C} \quad \blacksquare$$