10.5 Equations of Lines and Planes

We first start this section by deriving the vector and parametric equations of a straight line in three dimensional space. In 3-D, like in 2-D, a line is uniquely determined when one point on the line and a direction vector are given. To this end, let \( P_0(x_0, y_0, z_0) \) be a fixed point on a straight line \((L)\) and let \( \vec{v} = <a, b, c> \) be a vector parallel to \((L)\), i.e., a direction vector\(^1\) of \((L)\). Let \( P(x, y, z) \) be an arbitrary point of \((L)\). Let \( \vec{r}_0 = \overrightarrow{OP}_0 \) and \( \vec{r} = \overrightarrow{OP} \) be position vectors as shown in Figure 10.5.1.

![Figure 10.5.1](image)

As we can see from Figure 10.5.1, a necessary and sufficient condition for a point \( P \) to be on the line \((L)\) is that \( \overrightarrow{P_0P} = <x - x_0, y - y_0, z - z_0> \) is parallel to \( \vec{v} \). That is, \( \overrightarrow{P_0P} = t\vec{v} \) for some scalar \( t \). From Figure 10.5.1, we can write \( \vec{r}_0 + \overrightarrow{P_0P} = \vec{r} \). Thus,

\[
\vec{r} = \vec{r}_0 + t\vec{v}.
\]  
(10.5.1)

\(^1\)The numbers \( a, b, \) and \( c \) are called **direction numbers**
Equation (10.5.1) is called the vector equation of the line \((L)\). The parameter \(t\) can be any real number. As it varies, the point \(P\) moves along the line. When \(t = 0\), \(P\) is the same as \(P_0\). When \(t > 0\), \(P\) is away from \(P_0\) in the direction of \(\vec{v}\) and when \(t < 0\), \(P\) is away from \(P_0\) in the direction opposite to \(\vec{v}\).

Now, in terms of components, Equation (10.5.1) leads to the vector equality
\[
<x, y, z> = <x_0 + ta, y_0 + tb, z_0 + tc>
\]
or equivalently
\[
x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct. \tag{10.5.2}
\]

Equations (10.5.2) are called the parametric equations of the line \((L)\).

Also, from Equations (10.5.2), we can write
\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{10.5.3}
\]

Equations (10.5.3) are called the symmetric equations of the line \((L)\).

**Example 10.5.1**
Find vector, parametric and symmetric equations of the line passing through the point \(P(-1, 4, 2)\) and parallel to \(\vec{v} = <1, 2, 3>\).

**Solution.**
The vector equation of the line is
\[
\vec{r} = \vec{r}_0 + t\vec{v} = <x_0, y_0, z_0> + t<1, 2, 3> = <-1 + t, 4 + 2t, 2 + 3t>.
\]
The parametric equations are
\[
x = -1 + t, \quad y = 4 + 2t, \quad z = 2 + 3t.
\]
The symmetric equations are
\[
x + 1 = \frac{y - 4}{2} = \frac{z - 2}{3}
\]

**Remark 10.5.1**
Vector, parametric, and symmetric equations of a line are not unique since they depend on a specific given point on the line and a specific vector parallel to the line.
Example 10.5.2
Find the vector equation of the line segment with tail \( P_0 \) and tip \( P_1 \).

Solution.
Let \( P \) a point on the given line segment. Then \( \overrightarrow{P_0P} \) and \( \overrightarrow{P_0P_1} \) are parallel so that \( \overrightarrow{P_0P} = t\overrightarrow{P_0P_1} \) for \( 0 \leq t \leq 1 \). On the other hand, using position vectors, we can write \( \overrightarrow{r_0} + t\overrightarrow{P_0P_1} = \overrightarrow{r} \) or
\[
\overrightarrow{r_0} + t\overrightarrow{P_0P_1} = \overrightarrow{r}.
\]
But \( \overrightarrow{r_0} + \overrightarrow{P_0P_1} = \overrightarrow{r_1} \). Hence, \( \overrightarrow{P_0P_1} = \overrightarrow{r_1} - \overrightarrow{r_0} \). Substituting this into Equation (10.5.4), we find the vector equation of the requested line segment
\[
\overrightarrow{r} = (1 - t)\overrightarrow{r_0} + t\overrightarrow{r_1}, \quad 0 \leq t \leq 1
\]

Example 10.5.3
Find the vector and parametric equations of the line segment with tail \( P_0(2, 4, -3) \) and tip \( P_1(3, -1, 1) \).

Solution.
The vector equation is
\[
\overrightarrow{r} = (1-t)(2\vec{i}+4\vec{j}-3\vec{k}) + t(3\vec{i}-\vec{j}+\vec{k}) = (2+t)\vec{i}+(4-5t)\vec{j}+(4t-3)\vec{k}, \quad 0 \leq t \leq 1.
\]
The parametric equations are
\[
x = 2 + t, \quad y = 4 - 5t, \quad z = 4t - 3, \quad 0 \leq t \leq 1
\]

Example 10.5.4
Two lines are said to be skew if they do not cross and they are not parallel (and hence they do not belong to the same plane). Show that the lines \( (L_1) \) and \( (L_2) \) with parametric equations
\[
\begin{align*}
x &= 1 + t, & y &= -2 + 3t, & z &= 4 - t \end{align*}
\]
and
\[
\begin{align*}
x &= 2s, & y &= 3 + s, & z &= -3 + 4s
\end{align*}
\]
respectively, are skew lines.
Solution.
The vector direction to \((L_1)\) is \( \vec{v}_1 = <1, 3, -1> \). The vector direction to \((L_2)\) is \( \vec{v}_2 = <2, 1, 4> \). Since

\[
\cos(\vec{v}_1, \vec{v}_2) = \frac{\vec{v}_1 \cdot \vec{v}_2}{||\vec{v}_1|| ||\vec{v}_2||} = \frac{1}{\sqrt{231}} \neq 0 \text{ and } -1
\]

so that the two lines are not parallel. If the two lines intersect, we should have

\[
1 + t = 2s, \quad -2 + 3t = 3 + s, \quad 4 - t = -3 + 4s.
\]

Solving the first two equations, we find \( t = \frac{11}{5} \) and \( s = \frac{8}{5} \). But these values do not work for our third equation. This, shows that the two lines do not intersect. Hence, we conclude that the two lines are skew.

Equations of Planes
Now, we will try to find the equation of a plane given a point in the plane and a vector normal to the plane. But first we need to define by what we mean by a normal vector.

A **normal** vector to a plane is a vector that is orthogonal to every vector of the plane. If \( \vec{n} = ai + bj + ck \) is normal to a given plane and \( P_0 = (x_0, y_0, z_0) \) is a fixed point in the plane then for any point \( P = (x, y, z) \) in the plane the vectors \( \vec{n} \) and \( \overrightarrow{P_0P} \) are orthogonal (i.e., perpendicular) (See Figure 10.5.2.)

![Figure 10.5.2](image)

In terms of dot product this means that

\[
\overrightarrow{P_0P} \cdot \vec{n} = 0.
\]

Resolving \( \overrightarrow{P_0P} \) in terms of components we find

\[
\overrightarrow{P_0P} = (x - x_0)i + (y - y_0)j + (z - z_0)k.
\]
In this case, the dot product $\overrightarrow{P_0P} \cdot \vec{n} = 0$ leads to the equation

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

(10.5.5)

which is the equation of our required plane.

**Example 10.5.5**

Write an equation of a plane through $P(1,0,-1)$ with normal vector $\vec{n} = 2\vec{i} + 2\vec{j} - \vec{k}$.

**Solution.**
The equation is

$$2(x-1) + 2(y-0) + (-1)(z+1) = 0 \text{ or } z = 2x + 2y - 3$$

By collecting terms in equation (10.5.5), we can rewrite the equation of the plane in the form

$$ax + by + cz + d = 0 \tag{10.5.6}$$

where $d = -(ax_0 + by_0 + cz_0)$. Equation (10.5.6) is called a **linear equation** in $x, y,$ and $z$.

**Example 10.5.6**

Find a normal vector to the plane $-x + 3y + 2z = 7$.

**Solution.**

Since the coefficients of $\vec{i}, \vec{j},$ and $\vec{k}$ in a normal vector are the coefficients of $x, y,$ and $z$ in the equation of the plane, a normal vector is $\vec{n} = -\vec{i} + 3\vec{j} + 2\vec{k}$.

**Example 10.5.7** (The Equation of a Plane Through Three Points)

Find the equation for the plane through the points $P_0 = (0,1,-7), P_1 = (3,1,-9),$ and $P_2 = (0,-5,-8)$.

**Solution.**

We have $\overrightarrow{P_0P_1} = (3-0)i + (1-1)j + (-9+7)k = 3\vec{i} - 2\vec{k}$. Similarly, $\overrightarrow{P_0P_2} = -6\vec{j} - \vec{k}$. Thus, the vector $\vec{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = -12\vec{i} + 3\vec{j} - 18\vec{k}$ is normal to the plane at $P_0$. Hence, the equation of the plane is

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

where $P = (x,y,z)$. Hence,

$$-12x + 3(y-1) - 18(z+7) = 0$$

or

$$-12x + 3y - 18z - 129 = 0$$
Example 10.5.8 (Parallel planes)
Two planes are parallel if their normal vectors are parallel. Show that the planes $x + 2y - 3z = 4$ and $2x + 4y - 6z = 3$ are parallel.

Solution.
The vector normal to the plane $x + 2y - 3z = 4$ is $\vec{v} = < 1, 2, -3 >$. The vector normal to the plane $2x + 4y - 6z = 3$ is $\vec{w} = < 2, 4, -6 > = 2\vec{v}$. Since $\vec{v}$ and $\vec{w}$ are parallel so are the two planes.

Example 10.5.9 (Intersecting planes)
When two planes intersect their intersection is a straight line. The angle between the two planes is the angle between their normal vectors as shown in Figure 10.5.3.

(a) Find the angle of intersection of the two planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
(b) Find the symmetric equations of the line of intersection.

Solution.
(a) The angle between the two planes is
\[
\theta = \cos^{-1} \left( \frac{<1, 1, 1> \cdot <1, -2, 3>}{||<1, 1, 1>|| ||<1, -2, 3>||} \right) = \cos^{-1} \left( \frac{2}{\sqrt{42}} \right) \approx 72^\circ.
\]

(b) To find a point on the line of intersection, we find a point where the line intersect the $xy$–plane. In this case, we set $z = 0$ and obtain the system of equations $x + y = 1$ and $x - 2y = 1$. Solving this system by elimination, we find $x = 1$ and $y = 0$. Hence, the point $(1, 0, 0)$ is on the line of intersection.

Next, a direction vector is the vector
\[
<1, 1, 1> \times <1, -2, 3> = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\vec{i} - 2\vec{j} - 3\vec{k}.
\]
Hence, the symmetric equations of the line of intersection are

\[
\frac{x - 1}{5} = \frac{y}{-2} = \frac{z}{-3} \quad \blacksquare
\]

We conclude this section, by finding the distance of a point \(P_1(x_1, y_1, z_1)\) to a plane with linear equation \(ax + by + cz + d = 0\). To this end, let \(P_0(x_0, y_0, z_0)\) be any point in the plane. According to Figure 10.5.4, the distance from \(P_1\) to the plane is the absolute value of the scalar projection of \(P_0P_1\) onto the normal vector \(\vec{n} = <a, b, c>\).

![Figure 10.5.4](image)

Hence,

\[
D = \left| \frac{\overrightarrow{P_0P_1} \cdot \vec{n}}{||\vec{n}||} \right| = \left| \frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2 + b^2 + c^2}} \right|
\]

\[
= \left| \frac{(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)}{\sqrt{a^2 + b^2 + c^2}} \right|
\]

\[
= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}
\]

where we used the fact that the point \(P_0\) is in the plane so that \(ax_0 + by_0 + cz_0 = -d\).

**Example 10.5.10**

Find the distance between the parallel planes \(z = x + 2y + 1\) and \(3x + 6y - 3z = 4\).

**Solution.**

The vector normal to the plane \(z = x + 2y + 1\) is \(\vec{v} = <1, 2, -1>\). The vector normal to the plane \(3x + 6y - 3z = 4\) is \(\vec{w} = <3, 6, -3> = 3\vec{v}\). Since \(\vec{v}\) and \(\vec{w}\) are parallel so are the two planes.
A point must be found that lies on one of the planes. When \( x = y = 0 \) in the plane \( z = x + 2y + 1 \), there exists the point \((0, 0, 1)\). To find the distance between the planes we can now use the equation for the distance between a point and a plane.

\[
D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|3(0) + 6(0) - 3(1) - 4|}{\sqrt{3^2 + 6^2 + (-3)^2}} = \frac{7}{3\sqrt{6}} \]

\(\blacksquare\)