10.2 Introduction to Vectors

In the previous calculus classes we have seen that the study of motion involves the introduction of a variety of quantities which are used to describe the physical world. Examples of these quantities include distance, displacement, speed, velocity, acceleration, force and mass. Some of these are characterized by a single number and others require a number and a direction. In general, objects in the physical world are divided into two categories: scalars and vectors.

Scalars are objects that can be modeled or characterized using a single number. Examples of scalars in the study of motion are distance, speed, and mass.

Vectors are physical objects or quantities that require both a “distance” and a “direction” for their specification. In the study of motion, the quantities displacement, velocity, acceleration, and force are examples of vectors. The purpose of this section is to understand some fundamentals about vectors.

Notation and Terminology

By a displacement vector from a point $A$ to a point $B$ we mean an arrow with its tail at $A$ and its tip at $B$ as shown in Figure 10.2.1.

![Figure 10.2.1](image_url)

The direction of the vector is the direction of the arrow. The distance between $A$ and $B$ is known as the magnitude or length of the vector. We will represent this vector by $\mathbf{AB}$ and its magnitude by $||\mathbf{AB}||$. In many instances, the endpoints of a vector are ignored and the vector is just denoted by $\mathbf{v}$. In this case, the magnitude of the vector will be denoted by $||\mathbf{v}||$.

We define the zero vector to be a displacement vector with zero length. We will denote it by $\mathbf{0}$. The zero vector has no direction.

Algebra of Vectors: Addition and Difference of Two Vectors

Starting from the bookstore, suppose that along a certain direction one can reach a classroom after traveling a distance of 500 ft and then 300 ft in another direction to reach the pool. By drawing the two displacement vectors one can draw a third vector representing the walk from the bookstore straight to the pool as shown in Figure 10.2.2.
If we let \( \vec{v} \) and \( \vec{w} \) represent the vectors from the bookstore to the classroom and the classroom to the pool, respectively, then the vector from the bookstore to the pool is just the sum \( \vec{v} + \vec{w} \) as shown in Figure 10.2.3. This clearly makes sense, because a person can get to the pool from the bookstore by walking there directly or by first walking to the classroom and then to the pool. Either way he or she eventually reaches the pool.

Now, suppose the displacement vectors \( \vec{u} \) and \( \vec{v} \) from the bookstore to both the classroom and the pool are known. What is the displacement vector \( \vec{x} \) from the classroom to the pool? Since \( \vec{u} + \vec{x} = \vec{v} \), we define \( \vec{x} \) to be the difference \( \vec{x} = \vec{u} - \vec{v} \). See Figure 10.2.4.

**Multiplication of a Vector by a Scalar**

If a vector \( \vec{u} \) and a scalar \( \alpha \) are given we would like to know what does \( \alpha \vec{u} \) stand for? The result is a vector of magnitude \( ||\alpha \vec{u}|| = |\alpha||\vec{u}|| \) and in the same direction of \( \vec{u} \) if \( \alpha > 0 \) and opposite direction if \( \alpha < 0 \). Figure 10.2.5 shows three vectors, \( \vec{u} \), \( \frac{1}{2} \vec{u} \), and \( -2 \vec{u} \).
For example, the difference $\vec{u} - \vec{v}$ is just the sum $\vec{u} + (-1)\vec{v}$ as shown in Figure 10.2.6.

**Parallel Vectors**

The operation of multiplying a vector by a scalar leads to the following definition: We say that two vectors $\vec{u}$ and $\vec{v}$ are **parallel** if and only if $\vec{u} = \alpha \vec{v}$ for some non-zero scalar $\alpha$. When $\alpha = 1$, i.e., $\vec{u} = \vec{v}$, we say that $\vec{u}$ and $\vec{v}$ are **equivalent**.

**The Components of Vectors**

Consider the three dimensional Cartesian coordinate system. We associate a direction with each of the three axes of this system. In particular we define three vectors $\hat{i}, \hat{j},$ and $\hat{k}$, each of length 1, to point along the direction of the positive numbers along each of our three axes as shown in Figure 10.2.7.
Now, let $\vec{v}$ be a vector in the three dimensional system with tail $A = (a, b, c)$ and tip $B = (a', b', c')$.

According to Figure 10.2.8 we can write

$$\vec{v} = \overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}.$$ 

But $\overrightarrow{AC} = (a' - a)i + (b' - b)j$ and $\overrightarrow{CB} = (c' - c)k$. Hence

$$\overrightarrow{AB} = (a' - a)i + (b' - b)j + (c' - c)k.$$ 

We call $<a' - a, b' - b, c' - c>$ the components of $\vec{v}$. 

Now, if the tail of a vector is the origin of the coordinate system then we call the vector a position vector. In this case, if $A = (a, b, c)$ are the coordinates of the tip point of the vector then we can write

$$\vec{v} = \overrightarrow{OA} = ai + bj + ck.$$ 

or alternatively, $\vec{v} = <a, b, c>$. 
Example 10.2.1
Find the components of the vectors \( \vec{i}, \vec{j}, \vec{k} \).

Solution.
We have \( \vec{i} = <1, 0, 0>, \vec{j} = <0, 1, 0>, \) and \( \vec{k} = <0, 0, 1> \).

Expressing vectors in terms of their components allows us to manipulate them algebraically. In particular, we can define the operations of addition, subtraction, and scalar multiplication in terms of the vector components. If \( \vec{u} = <a, b, c> = a\vec{i} + b\vec{j} + c\vec{k} \) and \( \vec{v} = <a', b', c'> = a'\vec{i} + b'\vec{j} + c'\vec{k} \) and \( \alpha \) is a scalar then

\[
\vec{u} + \vec{v} = (a + a')\vec{i} + (b + b')\vec{j} + (c + c')\vec{k} = <a + a', b + b', c + c'>.
\]

\[
\vec{u} - \vec{v} = (a - a')\vec{i} + (b - b')\vec{j} + (c - c')\vec{k} = <a - a', b - b', c - c'>.
\]

\[
\alpha\vec{u} = \alpha a\vec{i} + \alpha b\vec{j} + \alpha c\vec{k} = <\alpha a, \alpha b, \alpha c>.
\]

Also, the magnitude of a vector can be expressed in terms of its components. Consider a vector \( \vec{v} = <a, b, c> = a\vec{i} + b\vec{j} + c\vec{k} \). Let \( \overrightarrow{OP} \) be a position vector with the same direction as \( \vec{v} \) and the same magnitude. This implies that \( P = (a, b, c) \). But then the magnitude of \( \vec{v} \) is just the distance between the origin and the point \( P \). That is,

\[
||\vec{v}|| = ||\overrightarrow{OP}|| = \sqrt{a^2 + b^2 + c^2}.
\]

Remark 10.2.1
All the above apply as well for the 2-D case by just letting \( z = 0 \).

Example 10.2.2
Perform the operation \((4\vec{i} + 2\vec{j}) - (3\vec{i} - \vec{j})\).

Solution.
We have

\[
(4\vec{i} + 2\vec{j}) - (3\vec{i} - \vec{j}) = (4 - 3)\vec{i} + (2 - (-1))\vec{j} = \vec{i} + 3\vec{j} \]

Example 10.2.3
Find the length of the vector \( \vec{v} = \vec{i} - \vec{j} + 2\vec{k} \).

Solution.
The length is

\[
||\vec{v}|| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6} \]

Example 10.2.4
Show that the vectors \( \vec{u} = \vec{i} - \vec{j} + 3\vec{k} \) and \( \vec{v} = 4\vec{i} - 4\vec{j} + 12\vec{k} \) are parallel.

Solution.
Since \( \vec{v} = 4\vec{u} \), we conclude that the two vectors are parallel.
Unit Vectors

By a **unit vector** we mean any vector of magnitude equals to 1. From any non-zero vector \( \vec{v} \), we can obtain a unit vector by setting

\[
\vec{u} = \frac{\vec{v}}{||\vec{v}||}.
\]

To see this, write \( \vec{v} = a\vec{i} + b\vec{j} + c\vec{k} \). Then

\[
||\vec{v}|| = \sqrt{a^2 + b^2 + c^2}.
\]

Moreover,

\[
\vec{u} = \frac{a}{\sqrt{a^2 + b^2 + c^2}} \vec{i} + \frac{b}{\sqrt{a^2 + b^2 + c^2}} \vec{j} + \frac{c}{\sqrt{a^2 + b^2 + c^2}} \vec{k}.
\]

Therefore,

\[
||\vec{u}|| = \sqrt{\frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2}} = \sqrt{\frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2}} = 1.
\]

**Example 10.2.5**

Find a unit vector from the point \( P(1,2) \) and toward the point \( Q(4,6) \).

**Solution.**

Let \( \vec{v} = \vec{PQ} = (4 - 1)\vec{i} + (6 - 2)\vec{j} = 3\vec{i} + 4\vec{j} \). Then

\[
||\vec{v}|| = \sqrt{3^2 + 4^2} = 5.
\]

Hence, a unit vector is

\[
\vec{u} = \frac{\vec{v}}{||\vec{v}||} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}.
\]

**Properties of Vector Arithmetic**

The arithmetic operations on vectors introduced earlier in this section satisfy the following properties. These properties are valid for any number of components.

**Theorem 10.2.1**

If \( \vec{u}, \vec{v}, \) and \( \vec{w} \) are three vectors (with the same number of components) and \( a \) and \( b \) are two scalars then we have the following properties:

\[
\vec{u} + \vec{v} = \vec{v} + \vec{u}
\]

\[
\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}
\]

\[
\vec{u} + \vec{0} = \vec{u}
\]

\[
1\vec{u} = \vec{u}
\]

\[
(a + b)\vec{u} = a\vec{u} + b\vec{v}
\]

\[
a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}.
\]

**Proof.**

These properties can be easily proved using components. We will prove the first
one leaving the rest for the reader. Write $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$ and $\vec{v} = v_1 \vec{i} + v_2 \vec{j}$. Then

$$\vec{u} + \vec{v} = u_1 \vec{i} + u_2 \vec{j} + v_1 \vec{i} + v_2 \vec{j}$$
$$= u_1 \vec{i} + v_1 \vec{i} + u_2 \vec{j} + v_2 \vec{j}$$
$$= v_1 \vec{i} + u_1 \vec{i} + v_2 \vec{j} + u_2 \vec{j}$$
$$= v_1 \vec{i} + v_2 \vec{j} + u_1 \vec{i} + u_2 \vec{j}$$
$$= \vec{v} + \vec{u}$$

**Example 10.2.6**
Find the angle between the vector $8\vec{i} + 6\vec{j}$ and the positive $x-$axis.

**Solution.**
Let $\theta$ be the angle between the given vector and the positive $x-$axis as shown in Figure 10.2.9

![Figure 10.2.9](image)

We have: $\tan \theta = \frac{6}{8} = \frac{3}{4} \Rightarrow \theta = \tan^{-1} \left( \frac{3}{4} \right) \approx 36.9^\circ$

**Applications**
A clothesline is tied between two poles, 8 m apart. The line is quite taut and has negligible sag. When a wet shirt with a mass of 0.8 kg is hung at the middle of the line, the mid-point is pulled down 8 cm. Find the tension in each half of the clothesline.

![Figure 10.2.10](image)
Let $\vec{T}_1$ and $\vec{T}_2$ be the tension vectors as shown on Figure 10.2.10. Finding the horizontal and vertical components of these vectors, we get

\[
\vec{T}_1 = -|\vec{T}_1| \cos \theta \hat{i} + |\vec{T}_1| \sin \theta \hat{j}
\]
\[
\vec{T}_2 = |\vec{T}_2| \cos \theta \hat{i} + |\vec{T}_2| \sin \theta \hat{j}
\]
\[
= |\vec{T}_1| \cos \theta \hat{i} + |\vec{T}_1| \sin \theta \hat{j}.
\]

But
\[
\vec{T}_1 + \vec{T}_2 = -\vec{w} = 0.8(9.8)\hat{j}
\]

Hence,
\[
|\vec{T}_1| = \frac{7.84}{2 \sin \theta}.
\]

Thus,
\[
\vec{T}_1 = -\frac{7.84}{2} \cot \theta \hat{i} + \frac{7.84}{2} \hat{j}
\]
\[
= -\frac{7.84}{2} \left( \frac{4}{0.08} \right) \hat{i} + 3.92 \hat{j}
\]
\[
= -196 \hat{i} + 3.92 \hat{j}
\]
\[
\vec{T}_2 = 196 \hat{i} + 3.92 \hat{j}
\]