8.4 Series with Positive and Negative Terms

In the previous section, we looked at convergence tests that apply only to series with positive terms. In this section, we consider series whose terms are not necessarily positive.

**Alternating Series**

By an alternating series we mean a series of the form \( \sum_{n=1}^{\infty} (-1)^{n-1}a_n \) where \( a_n > 0 \). For instance, the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \). Here \( a_n = \frac{1}{n} \). The following theorem provides a way for testing alternating series for convergence.

**Theorem 8.4.1 (Alternating Series Test)**

An alternating series \( \sum_{n=1}^{\infty} (-1)^{n-1}a_n \) is convergent if and only if:

(i) The sequence \( \{a_n\}_{n=1}^{\infty} \) is decreasing, i.e. \( a_{n+1} < a_n \) for all \( n \);

(ii) \( \lim_{n \to \infty} a_n = 0 \).

**Proof.**

Let \( \{S_n\}_{n=1}^{\infty} \) be the sequence of partial sums of the series \( \sum_{n=1}^{\infty} (-1)^{n-1}a_n \).

Notice the following

\[
S_4 - S_2 = a_3 - a_4 \geq 0 \\
S_6 - S_4 = a_5 - a_6 \geq 0 \\
S_8 - S_6 = a_7 - a_8 \geq 0 \\
\vdots
\]

Thus,

\[
S_2 \leq S_4 \leq S_6 \leq S_8 \leq \cdots
\]

It follows that the sequence \( \{S_{2n}\}_{n=1}^{\infty} \) is increasing. Moreover, for all \( n \geq 1 \) we have

\[
S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1.
\]

Hence, the sequence \( \{S_{2n}\}_{n=1}^{\infty} \) is bounded from above. By Theorem 8.1.4, there is an \( S > 0 \) such that \( \lim_{n \to \infty} S_{2n} = S \).
Next, we consider the terms of \( \{S_n\}_{n=1}^{\infty} \) with odd subscripts:

\[
S_1 - S_3 = a_2 - a_3 \geq 0 \\
S_3 - S_5 = a_4 - a_5 \geq 0 \\
S_5 - S_7 = a_6 - a_7 \geq 0 \\
\vdots
\]

Thus,

\[ S_1 \geq S_3 \geq S_5 \geq S_7 \geq \cdots \]

It follows that the sequence \( \{S_{2n+1}\}_{n=1}^{\infty} \) is decreasing. Moreover, \( S_{2n+1} = S_{2n} + a_{2n+1} \). Thus, \( \lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} a_{2n+1} = S + 0 = S \). It follows that

\[ \lim_{n \to \infty} S_n = S. \]

Figure 8.4.1 shows how the terms of \( \{S_n\}_{n=1}^{\infty} \) and \( S \) are ordered on a line.

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**Example 8.4.1**

Show that the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \) is convergent.

**Solution.**

To see this, let \( a_n = \frac{1}{n} \). Since \( n < n + 1 \), \( \frac{1}{n+1} < \frac{1}{n} \), that is, \( a_{n+1} < a_n \). Also, \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0 \). Hence, by the previous theorem the given series is convergent.

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**Example 8.4.2**

Test the convergence of the series \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1} \).

**Solution.**

Since \( \lim_{n \to \infty} \frac{n}{n+1} = 1 \neq 0 \)

the alternating series test can not be applied for this series. However, the \( n^{\text{th}} \) term test can be used which shows that the series is divergent.
Example 8.4.3

Test the convergence of the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2 + 1}$.

Solution.

Let $f(x) = \frac{x^2}{x^2 + 1}$ for $x \geq 2$. Then $f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0$ for $x \geq 2$. Thus, the sequence $\left\{ \frac{n}{n^2 + 1} \right\}_{n=1}^{\infty}$ satisfies the conditions of the alternating series test so that the given series is convergent □

Remark 8.4.1

(a) It follows from the above theorem that if $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges then

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \leq a_1.$$ (b) From Figure 8.4.1, for each $n \geq 1$, we have $S$ is between $S_n$ and $S_{n+1}$ so that the distance between $S$ and $S_n$ is less than the distance between $S_n$ and $S_{n+1}$. Thus, we have an upper bound for the error:

$$|S_n - S| < |S_{n+1} - S_n| = a_{n+1}.$$ (c) Keep in mind that the tests used in this section are basically used to test for convergence. However, when a series is convergent these tests do not provide a value for the sum.

Example 8.4.4

Find the sum of the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(n-1)!}$ correct to three decimal places.

Solution.

We first show that the given series is convergent. Let $a_n = \frac{1}{(n-1)!}$. Then $a_{n+1} = \frac{1}{n!} \leq \frac{1}{(n-1)!} = a_n$ so that the series $\{a_n\}_{n=1}^{\infty}$ is decreasing. Since $0 < \frac{1}{(n-1)!} < \frac{1}{n-1} \to 0$, by the squeeze rule, we obtain $\lim_{n \to \infty} a_n = 0$. Hence, by the alternating series test, the given series is convergent. Now, notice that $a_8 = \frac{1}{7!} < 0.0002$ and $S_7 \approx 0.368056$. By Remark 8.4.1(b), we know that $|S - S_7| \leq a_8 < 0.0002$. Hence, $S \approx 0.368$ □
Absolute Convergence

Consider a series \( \sum_{n=1}^{\infty} a_n \) which has both positive and negative terms. We say that this series is **absolutely convergent** if the series of absolute values \( \sum_{n=1}^{\infty} |a_n| \) is convergent. The following theorem provides a test of convergence for series of the above type.

**Theorem 8.4.2**

If \( \sum_{n=1}^{\infty} |a_n| \) is convergent then \( \sum_{n=1}^{\infty} a_n \) is convergent. That is, absolute convergence implies convergence.

**Proof.**

The proof of this result is quite simple. Let \( b_n = |a_n| \geq 0 \). By assumption, the series \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} |a_n| \) is convergent. But \( a_n \leq b_n \) for all \( n \). Now, part (i) of the comparison test asserts that the series \( \sum_{n=1}^{\infty} a_n \) must be convergent.

**Example 8.4.5**

Show that the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \) is convergent.

**Solution.**

Indeed, the series of absolute values \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent (p-series with \( p = 2 \)) so by the above theorem, the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \) is also convergent.

**Example 8.4.6**

Show that the series \( 1 - x + x^2 - x^3 + \cdots \) is absolutely convergent for \( |x| < 1 \).

**Solution.**

Since the geometric series \( 1 + x + x^2 + x^3 + \cdots \) converges for \( |x| < 1 \), the given series is absolutely convergent.

**Remark 8.4.2**

It is very important to be very careful with the statement of the above theorem. The theorem says that if we know that the series \( \sum_{n=1}^{\infty} |a_n| \) is convergent
then the series $\sum_{n=1}^{\infty} a_n$ is definitely convergent. The converse is not true in general. That is, it is possible that $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent. The following example illustrates this situation.

**Example 8.4.7**

Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent but the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

**Solution.**

The alternating series test asserts that the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent. However, the series $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (Harmonic series).

**Conditional Convergence**

When a series is such that $\sum_{n=1}^{\infty} |a_n|$ is divergent but $\sum_{n=1}^{\infty} a_n$ is convergent then we say that the series $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent**.

**The Ratio Test**

The integral test is hard to apply when the integrand involves factorials or complicated expressions. We shall now introduce a test that can be used to help determine convergence or divergence of series when other tests are not applicable. This is also a test for absolute convergence.

**Theorem 8.4.3**

Let $\sum_{n=1}^{\infty} a_n$ be a series and suppose that $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \geq 0$.

(a) If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.

(c) If $L = 1$ then the test fails, that is the test does not tell us anything about the convergence of the series. Thus, another test should be considered.
Proof.

(a) Suppose that $0 \leq L < 1$. Since $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$, we can find a positive integer $N$ such that

$$
\left| \frac{a_{n+1}}{a_n} \right| - L \leq \left| \frac{a_{n+1}}{a_n} \right| - L < \frac{1 - L}{2}, \text{ for } n \geq N.
$$

This is equivalent to

$$
\left| \frac{a_{n+1}}{a_n} \right| < \frac{L + 1}{2}, \text{ for } n \geq N.
$$

Let $r = \frac{L + 1}{2}$. Clearly, $L < r < 1$. Thus,

$$
|a_{n+1}| < r|a_n|, \text{ for } n \geq N.
$$

Hence,

$$
|a_{N+1}| < r|a_N|,
|a_{N+2}| < r^2|a_{N+1}| < r^2|a_N|,
|a_{N+3}| < r^3|a_{N+2}| < r^3|a_N|,
\vdots
$$

Since the series $\sum_{n=1}^{\infty} r^n|a_N|$ is a geometric series with $r < 1$, it is convergent.

By the comparison test the series $\sum_{n=1}^{\infty} |a_{N+n}|$ is also convergent. The series $\sum_{n=1}^{\infty} a_{N+n}$ is convergent. Since the convergence or divergence is unaffected by deleting a finite number of terms, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) Suppose now that $L > 1$. Then there is a positive integer $N$ such that for $n \geq N$

$$
\left| \frac{a_{n+1}}{a_n} \right| - L < \frac{L - 1}{2}
$$

or

$$
-\frac{L - 1}{2} < \left| \frac{a_{n+1}}{a_n} \right| - L < \frac{L - 1}{2}
$$

i.e.,

$$
\left| \frac{a_{n+1}}{a_n} \right| > \frac{L + 1}{2}, \text{ for } n \geq N.
$$
Let \( r = \frac{L+1}{2} \). Then \( L > r > 1 \). Thus, for \( n \geq N \) we have \( \left| \frac{a_{n+1}}{a_n} \right| > 1 \) or \( |a_{n+1}| > |a_n| \). This implies that \( \lim_{n \to \infty} a_n \neq 0 \) and by \( n^{th} \) term test, the series \( \sum_{n=1}^{\infty} a_n \) is divergent.

(c) Consider the series \( \sum_{n=1}^{\infty} \frac{1}{n} \). Then this series is divergent and \( \lim_{n \to \infty} \frac{1}{n} = 1 \).

On the other hand, the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent with \( \lim_{n \to \infty} \frac{1}{n(n+1)} = 1 \).

Thus, when \( L = 1 \) the test is inconclusive.

**Example 8.4.8**

1. The series \( \sum_{n=1}^{\infty} (-1)^{n-1} n \) is convergent by the alternating series test. Note that \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{n}{n+1} = 1 \) i.e., \( L = 1 \) in the previous theorem.

2. The series \( \sum_{n=1}^{\infty} (-1)^n - 1 \) is divergent. Also, note that \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \), i.e. \( L = 1 \).

3. The series \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n!}{n} \) is convergent since \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{n!}{(n+1)} = \lim_{n \to \infty} \frac{1}{n+1} = 0 \) so \( L = 0 < 1 \) in the above theorem.

4. The series \( \sum_{n=1}^{\infty} (-2)^{n-1} \) is divergent since \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 2 > 1 \).

**Remark 8.4.3**

When testing a series for convergence, normally concentrate on the \( n^{th} \) term test and the ratio test. Use the comparison test only when both tests fail.

The following test is convenient to apply when \( n^{th} \) powers occur.

**Theorem 8.4.4 (The root test)**

Consider a series \( \sum_{n=1}^{\infty} a_n \) and let \( L = \lim_{n \to \infty} \sqrt[n]{|a_n|} \).

(i) If \( L < 1 \) then the given series is absolutely convergent (and hence convergent).

(ii) If \( L > 1 \) then the series is divergent.

(iii) If \( L = 1 \) the test is inconclusive and a different test must be considered.
Example 8.4.9
Use the root test for each of the following series.

(a) \( \sum_{n=1}^{\infty} \frac{n^n}{3^{1+n}} \).

(b) \( \sum_{n=1}^{\infty} \left( \frac{n + 1}{2n + 3} \right)^n \).

(c) \( \sum_{n=1}^{\infty} \frac{1}{n^p} \).

Solution.
(a) We have
\[
L = \lim_{n \to \infty} \frac{n^n}{3^{1+n}} = \lim_{n \to \infty} \frac{n}{3} = \infty > 1
\]
so the given series is divergent.

(b) We have
\[
L = \lim_{n \to \infty} \left( \frac{n + 1}{2n + 3} \right)^n = \lim_{n \to \infty} \frac{n + 1}{2n + 3} = \frac{1}{2} < 1
\]
so the given series is convergent.

(c) We have
\[
L = \lim_{n \to \infty} \frac{1}{n^p} = \lim_{n \to \infty} \frac{1}{(n^{\frac{1}{p}})^p} = 1
\]
so that the test is inconclusive \( \blacksquare \).