6.1 The Method of Integration by Parts

The integration by parts formula is an antidifferentiation method which reverses the product rule of differentiation. To see this, let \( u \) and \( v \) be two differentiable functions of \( x \). Then the product rule asserts that the product function \( uv \) is also differentiable and its derivative is given by

\[
(uv)' = uv' + u'v.
\]

This says that an antiderivative of \( uv' + u'v \) is the function \( uv \). In terms of indefinite integrals we have

\[
\int (uv)' \, dx = \int uv' \, dx + \int u'v \, dx \tag{6.1.1}
\]

or equivalently

\[
\int uv' \, dx = uv - \int u'v \, dx. \tag{6.1.2}
\]

**Remark 6.1.1**

Since the differential of \( u \) is \( du = u' \, dx \) and that of \( v \) is \( dv = v' \, dx \), (6.1.2) becomes

\[
\int udv = uv - \int vdu. \tag{6.1.3}
\]

Either formula (6.1.2) or (6.1.3) is known as the integration by parts formula.

The corresponding integration by parts formula for definite integrals is given by

\[
\int_u^b uv' \, dx = uv \bigg|_a^b - \int_a^b u'v \, dx. \tag{6.1.4}
\]

**Example 6.1.1**

Find \( \int xe^x \, dx \).
Solution.
Let \( u = x \) and \( v' = e^x \). Then \( u' = 1 \) and \( v = \int e^x \, dx = e^x \). Note that in finding \( v \) we did not include the constant of integration. We will write the constant \( C \) in the answer of \( \int uv' \, dx \). Now, substituting in formula (6.1.2) to obtain

\[
\int x e^x \, dx = xe^x - \int e^x \, dx
\]

\[
= xe^x - e^x + C
\]

Remark 6.1.2
If we chose \( u = e^x \) and \( v' = x \) then we would have \( u' = e^x \) and \( v = \frac{x^2}{2} \). In this case, formula (6.1.2) yields

\[
\int x e^x \, dx = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x \, dx
\]

and the second integral is definitely worse than the one to the left of the formula we had in the previous example, i.e. \( \int xe^x \, dx \). It follows that for the method to be useful it is important to choose \( u \) and \( dv \) in such a way to make the integral on the right easier to find than the integral on the left.

Example 6.1.2
Find \( \int x \sin x \, dx \).

Solution.
Let \( u = x \) and \( v' = \sin x \). Then \( u' = 1 \) and \( v = -\cos x \). Substituting in formula (6.1.2) to obtain

\[
\int x \sin x \, dx = -x \cos x - \int (-\cos x) \, dx
\]

\[
= -x \cos x + \int \cos x \, dx
\]

\[
= -x \cos x + \sin x + C
\]

There are examples which don’t look like good candidates for integration by parts because they don’t appear to involve products, but for which the method works well. Such examples often involves \( \ln x \) or the inverse trigonometric functions.
Example 6.1.3
Calculate \( \int_1^5 \ln x \, dx \).

Solution.
Let \( u = \ln x \) and \( v' = 1 \). Then \( u' = \frac{1}{x} \) and \( v = x \). Substituting in formula \((6.1.4)\) we obtain

\[
\int_1^5 \ln x \, dx = x \ln x \bigg|_1^5 - \int_1^5 \frac{1}{x} \, dx \\
= x \ln x \bigg|_1^5 - \int_1^5 dx \\
= [x \ln x - x]_1^5 \\
= 5 \ln 5 - 5 - (\ln 1 - 1) \\
= 5 \ln 5 - 4 \]

Example 6.1.4
Find \( \int x^3 \ln x \, dx \).

Solution.
Let \( u = \ln x \) and \( v' = x^3 \). Then \( u' = \frac{1}{x} \) and \( v = \frac{x^4}{4} \). Substituting in formula \((6.1.2)\) to obtain

\[
\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \cdot \frac{1}{x} \, dx \\
= \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3 \, dx \\
= \frac{x^4}{4} \ln x - \frac{1}{4} \cdot \frac{x^4}{4} + C \\
= \frac{x^4}{4} \ln x - \frac{1}{16} x^4 + C \]

Sometimes evaluating an integral might require integration by parts more than once as the following example shows.

Example 6.1.5
Find \( \int x^2 \cos x \, dx \).

Solution.
Let \( u = x^2 \) and \( v' = \cos x \). Then \( u' = 2x \) and \( v = \sin x \). Substituting in formula \((6.1.2)\) to obtain
\[ \int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx \]
\[ = x^2 \sin x - 2 \int x \sin x \, dx \]
\[ = x^2 \sin x - 2(-x \cos x + \sin x) + C \]
\[ = x^2 \sin x + 2x \cos x - 2 \sin x + C \]

where we have used Example 6.1.2 to evaluate \( \int x \sin x \, dx \). 

The following example illustrates a very useful technique: Use integration by parts to transform the integral into an expression containing another copy of the same integral, possibly multiplied by a constant, then solve for the original integral.

**Example 6.1.6**
Find \( \int \sin^2 x \, dx \).

**Solution.**

*Method I:* Using a half-angle formula to write \( \sin^2 x = \frac{1 - \cos 2x}{2} \). In this case, the problem reduces to integrating \( \frac{1 - \cos 2x}{2} \). That is,

\[ \int \sin^2 x \, dx = \frac{1}{2} \int (1 - \cos 2x) \, dx \]
\[ = \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x \, dx \]
\[ = \frac{1}{2} x - \frac{1}{4} \int \cos u \, du \]
\[ = \frac{1}{2} x - \frac{1}{4} \sin u + C \]
\[ = \frac{1}{2} x - \frac{1}{4} \sin 2x + C \]
where the substitution \( u = 2x \) is used to evaluate the integral \( \int \cos 2x \, dx \).

**Method II:**
We now use integration by parts formula to evaluate the given integral. Let \( u = \sin x \) and \( v' = \sin x \). Then \( u' = \cos x \) and \( v = -\cos x \). Substituting in formula (6.1.2) to obtain

\[
\int \sin^2 x \, dx = -\sin x \cos x - \int -\cos^2 x \, dx = -\sin x \cos x + \int \cos^2 x \, dx.
\]

Using the trigonometric identities \( \sin 2x = 2 \sin x \cos x \) and \( \cos^2 x + \sin^2 x = 1 \) we can rewrite the right-hand side of the above integral as

\[
\int \sin^2 x \, dx = -\frac{1}{2} \sin 2x + \int (1 - \sin^2 x) \, dx
\]
\[
= -\frac{1}{2} \sin 2x + \int dx - \int \sin^2 x \, dx
\]
\[
= -\frac{1}{2} \sin 2x + x - \int \sin^2 x \, dx.
\]

Moving the right integral to the left side to obtain

\[
2 \int \sin^2 x \, dx = -\frac{1}{2} \sin 2x + x + C
\]

and finally dividing both sides by 2 to obtain

\[
\int \sin^2 x \, dx = -\frac{1}{4} \sin 2x + \frac{x}{2} + C \quad \blacksquare
\]

Integration by parts can be used to derive reduction formulas, i.e., formulas that reduce the integral of a power of a function to an integral with lesser power progressively simplifying the integral until it can be evaluated. We illustrate this method in the next example.

**Example 6.1.7**
(a) Derive the reduction formula: \( \int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx \).
(b) Use (a) to find \( \int x^3 e^x \, dx \).
Solution.

(a) Letting \( u(x) = x^n \), \( v'(x) = e^x \), we find \( u'(x) = nx^{n-1} \) and \( v(x) = e^x \).

Hence,

\[
\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx.
\]

(b) We have

\[
\int x^3 e^x \, dx = x^3 e^x - 3 \int x^2 e^x \, dx
\]

\[
= x^3 e^x - 3 \left[ x^2 e^x - 2 \int xe^x \, dx \right]
\]

\[
= x^3 e^x - 3x^2 e^x + 6 \left[ xe^x - \int e^x \, dx \right]
\]

\[
= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C \quad \blacksquare
\]