1.2 A List of Commonly Occurring Functions

In this section, we discuss the most common functions occurring in calculus.

**Linear Functions**

A linear function $f(x)$ is one that can be written as

$$f(x) = mx + b.$$

The graph of a linear function is a straight line. Since $b = f(0)$, the point $(0, b)$ is the point where the line crosses the vertical line. We call it the **y-intercept** or the vertical intercept. So the $y$-intercept is the output corresponding to the input $x = 0$, sometimes known as the **initial value** of $y$.

Now, a line can be horizontal, vertical, rising to the right or falling to the right. The **slope**, which is defined as the rise over the run, is the parameter that provides information about the steepness of a straight line.

- If $m = 0$ then $f(x) = b$ is a constant function whose graph is a horizontal line at $(0, b)$.
- For a vertical line, the slope is undefined since any two points on the line have the same $x$–value and this leads to a division by zero in the formula for the slope. The equation of a vertical line has the form $x = a$.
- Suppose that the line is neither horizontal nor vertical. If $m > 0$ then for any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the line the ratio $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is positive. This implies that $f(x_2) > f(x_1)$. Hence, $f(x)$ is increasing. That is, the line is rising to the right. Conversely, if $f(x)$ is increasing and $x_1 < x_2$ then $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is positive.
- $f(x)$ is decreasing if and only if $m < 0$.

**Example 1.2.1**

Find the vertical intercept and the slope of the linear function $2x + y = 3$. Graph the line.

**Solution.**

Rewriting this expression in the slope-intercept form we find $y = -2x + 3$. Thus, the slope is $-2$ and the vertical intercept is $3$. To graph the line we
need two points on the line. The first one is the $y$–intercept $(0, 3)$. Since the slope is $-2$, starting from the $y$–intercept we move down 2 units and 1 to the right. This is the location of the second point. Now, we graph the line by connecting this point and the point $(0, 3)$ as shown in Figure 1.2.1.

![Figure 1.2.1](image)

**Polynomial Functions**

Polynomial functions are among the simplest, most important, and most commonly used mathematical functions. These functions consist of one or more terms of variables with whole number exponents. (Whole numbers are positive integers and zero.) All such functions in one variable (usually $x$) can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are all real numbers, called the coefficients of $f(x)$. The number $n$ is a non-negative integer. It is called the degree of the polynomial. A polynomial of degree zero is just a constant function. A polynomial of degree one is a linear function, of degree two a quadratic function, etc. The number $a_n$ is called the leading coefficient and $a_0$ is called the constant term.

Note that the terms in a polynomial are written in descending order of the exponents. Polynomials are defined for all values of $x$.

**Example 1.2.2**

Find the leading coefficient, the constant term and the degree of the polynomial $f(x) = 4x^5 - x^3 + 3x^2 + x + 1$.

**Solution.**

The given polynomial is of degree 5, leading coefficient 4, and constant term 1.
The shape of the graph of a polynomial depends on the degree and leading coefficient:

- If the degree \( n \) is even and the leading coefficient, \( a_n \), of a polynomial is positive, then the right and left hand sides of the graph will rise towards \( +\infty \).
- If the degree \( n \) is even and the leading coefficient, \( a_n \), of a polynomial is negative, then the right and left hand sides of the graph will fall towards \( -\infty \).
- If the degree \( n \) is odd and the leading coefficient, \( a_n \), of a polynomial is positive, then the right side will rise towards \( +\infty \) and the left hand side of the graph will fall towards \( -\infty \).
- If the degree \( n \) is odd and the leading coefficient, \( a_n \), of a polynomial is negative, then the right hand side will fall towards \( -\infty \) and the right hand side of the graph will rise towards \( \infty \).

**Example 1.2.3**
According to the graphs given below, indicate the sign of \( a_n \) and the parity of \( n \) for each curve in Figure 1.2.2.

![Figure 1.2.2](image)

Figure 1.2.2
Solution.
(a) \(a_n < 0\) and \(n\) is odd.
(b) \(a_n > 0\) and \(n\) is odd.
(c) \(a_n > 0\) and \(n\) is even.
(d) \(a_n < 0\) and \(n\) is even

**Power Functions**
A function \(f(x)\) is a **power function** of \(x\) if and only if

\[ f(x) = x^n. \]

The number \(n\) is called the **power** of \(x\). If \(n\) is a non-negative integer then the domain of \(f(x) = x^n\) consists of all real numbers.

When \(n = 0\) then the graph is a horizontal line at \((0, 1)\). When \(n = 1\) then the graph is a straight line through the origin with slope equals to 1. See Figure 1.2.3.

![Figure 1.2.3](image)

The graphs of all power functions with \(n = 2, 4, 6, \ldots\) have the same characteristic \(\cup-\) shape and they satisfy the following properties:

1. Pass through the points \((0, 0), (1, 1),\) and \((-1, 1)\).
2. Decrease for negative values of \(x\) and increase for positive values of \(x\).
3. Are symmetric about the \(y\)-axis because the functions are even.
4. Are concave up. See Figure 1.2.4.
The graphs of power functions with $n = 1, 3, 5, \cdots$ resemble the side view of a chair and satisfy the following properties:

1. Pass through $(0, 0)$ and $(1, 1)$ and $(-1, -1)$.
2. Increase on every interval.
3. Are symmetric about the origin because the functions are odd.
4. Are concave down for negative values of $x$ and concave up for positive values of $x$. See Figure 1.2.5.

Graphs of $y = x^n$ with $n = -1, -3, \cdots$.
1. Passes through $(1, 1)$ and $(-1, -1)$ and does not have a $y$–intercept.
2. Is decreasing everywhere that it is defined.
3. Is symmetric about the origin because the function is odd.
4. Is concave down for negative values of $x$ and concave up for positive values of $x$.

5. Has the $x$–axis as a horizontal asymptote and the $y$–axis as a vertical asymptote. See Figure 1.2.6.

![Graph of $y = x^n$ with $n = -2, -4, \ldots$](image)

Figure 1.2.6

Graphs of $y = x^n$ with $n = -2, -4, \ldots$

1. Pass through $(1, 1)$ and $(-1, 1)$ and do not have a $y$– or $x$–intercept.

2. Are increasing for negative values of $x$ and decreasing for positive values of $x$.

3. Are symmetric about the $y$–axis because the functions are even.

4. Are concave up everywhere that they are defined.

5. Have the $x$–axis as a horizontal asymptote and the $y$–axis as vertical asymptote. See Figure 1.2.7.

![Graphs of $y = x^{\frac{1}{r}}$ where $r = 2, 4, \ldots$](image)

Figure 1.2.7

Graphs of $y = x^{\frac{1}{r}}$ where $r = 2, 4, \ldots$ has the following properties:

1. Domain consists of all non-negative real numbers.
2. Pass through \((0, 0)\) and \((1, 1)\).
3. Are increasing for \(x > 0\).
4. Are concave down for \(x > 0\). See Figure 1.2.8.

![Figure 1.2.8](image)

Graphs of \(y = x^{\frac{1}{r}}\) where \(r = 3, 5, \cdots\) has the following properties:
1. Domain consists for all real numbers.
2. Pass through \((0, 0)\), \((1, 1)\) and \((-1, -1)\).
3. Are increasing.
4. Are concave down for \(x > 0\) and concave up for \(x < 0\). See Figure 1.2.9.

![Figure 1.2.9](image)
Rational Functions

A **rational function** is a function that is the ratio of two polynomial functions \( f(x) \) and \( g(x) \). The domain consists of all numbers such that \( g(x) \neq 0 \). The vertical line \( x = a \) where \( g(a) = 0 \) is called a **vertical asymptote**.

**Example 1.2.4**

Find the domain of the function \( f(x) = \frac{x-2}{x^2-x-6} \).

**Solution.**

The domain consists of all numbers \( x \) such that \( x^2-x-6 \neq 0 \). But this last quadratic expression is \( 0 \) when \( x = -2 \) or \( x = 3 \). Thus, the domain is the set \( (-\infty, -2) \cup (-2, 3) \cup (3, \infty) \).

Trigonometric Functions

By the **unit circle** we mean the circle with center at the point \( O(0,0) \) and radius \( 1 \). Such a circle has the equation \( x^2 + y^2 = 1 \).

Now, let \( t \) be any real number. From the point \( A(1,0) \), walk on the unit circle a distance \( t \) arriving at some point \( P(a,b) \). Then the arc \( AP \) subtends a central angle \( \theta \). See Figure 1.2.10.

![Figure 1.2.10](image)

We define the following trigonometric functions:

\[
\begin{align*}
\sin t &= b \\
\cos t &= a \\
\tan t &= \frac{b}{a} \\
\csc t &= \frac{1}{b} \\
\sec t &= \frac{1}{a} \\
\cot t &= \frac{a}{b}
\end{align*}
\]

where \( a \neq 0 \) and \( b \neq 0 \). If \( a = 0 \) then the functions \( \sec t \) and \( \tan t \) are undefined. If \( b = 0 \) then the functions \( \csc t \) and \( \cot t \) are undefined.
The above trigonometric functions are referred to as **circular functions**. It follows from the above definition that

\[
\sin^2 x + \cos^2 x = 1.
\]

The table below lists the domain, range, and the periodicity of each of the trigonometric functions discussed above.

<table>
<thead>
<tr>
<th>f(x)</th>
<th>Domain</th>
<th>Range</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin x</td>
<td>(−∞, ∞)</td>
<td>[−1, 1]</td>
<td>2π</td>
</tr>
<tr>
<td>cos x</td>
<td>(−∞, ∞)</td>
<td>[−1, 1]</td>
<td>2π</td>
</tr>
<tr>
<td>tan x</td>
<td>x ≠ (2n + 1)\frac{π}{2}</td>
<td>(−∞, ∞)</td>
<td>π</td>
</tr>
<tr>
<td>sec x</td>
<td>x ≠ (2n + 1)\frac{π}{2}</td>
<td>(−∞, −1) ∪ (1, ∞)</td>
<td>2π</td>
</tr>
<tr>
<td>csc x</td>
<td>x ≠ nπ</td>
<td>(−∞, −1) ∪ (1, ∞)</td>
<td>2π</td>
</tr>
<tr>
<td>cot x</td>
<td>x ≠ nπ</td>
<td>(−∞, ∞)</td>
<td>π</td>
</tr>
</tbody>
</table>

Figure 1.2.11 shows the graphs of \( y = \cos x \) and \( y = \sin x \).

Figure 1.2.11

Figure 1.2.12 shows the graphs of \( y = \tan x \) and \( y = \cot x \).

Figure 1.2.12
Figure 1.2.13 shows the graphs of $y = \sec x$ and $y = \csc x$.

Building New Functions from Old Ones

In the remaining of this section, we will discuss some procedures for building new functions from old ones. The first procedure is known as the composition of functions.

Suppose we are given two functions $f$ and $g$ such that the range of $g$ is contained in the domain of $f$ so that the output of $g$ can be used as input for $f$. We define a new function, called the composition of $f$ with $g$, by the formula

$$(f \circ g)(x) = f(g(x)).$$

Using a Venn diagram (See Figure 1.2.14) we have

![Venn Diagram](image)

**Example 1.2.5**

Suppose that $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

(a) Find $f \circ g$ and $g \circ f$. 

10
(b) Calculate \((f \circ g)(5)\) and \((g \circ f)(-3)\).
(c) Are \(f \circ g\) and \(g \circ f\) equal?

**Solution.**
(a) \((f \circ g)(x) = f(g(x)) = f(x^2 - 3) = 2(x^2 - 3) + 1 = 2x^2 - 5\). Similarly, \((g \circ f)(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 3 = 4x^2 + 4x - 2\).
(b) \((f \circ g)(5) = 2(5)^2 - 5 = 45\) and \((g \circ f)(-3) = 4(-3)^2 + 4(-3) - 2 = 22\).
(c) \(f \circ g \neq g \circ f\)

With only one function you can build new functions using composition of the function with itself. Also, there is no limit on the number of functions that can be composed.

**Example 1.2.6**
Suppose that \(f(x) = 2x + 1\) and \(g(x) = x^2 - 3\).
(a) Find \((f \circ f)(x)\).
(b) Find \([f \circ (f \circ g)](x)\).

**Solution.**
(a) \((f \circ f)(x) = f(f(x)) = f(2x + 1) = 2(2x + 1) + 1 = 4x + 3\).
(b) \([f \circ (f \circ g)](x) = f(f(g(x))) = f(f(x^2 - 3)) = f(2x^2 - 5) = 2(2x^2 - 5) + 1 = 4x^2 - 9\)

**Decomposition of Functions**
If a formula for \((f \circ g)(x)\) is given then the process of finding the formulas for \(f\) and \(g\) is called decomposition.

**Example 1.2.7**
Decompose \((f \circ g)(x) = \sqrt{1 - 4x^2}\).

**Solution.**
One possible answer is \(f(x) = \sqrt{x}\) and \(g(x) = 1 - 4x^2\). Another possible answer is \(f(x) = \sqrt{1 - x^2}\) and \(g(x) = 2x\). Thus, decomposition of functions is not unique.

**Combinations of Functions**
Next, we are going to construct new functions from old ones using the operations of addition, subtraction, multiplication, and division.
Let \(f(x)\) and \(g(x)\) be two given functions. Then for all \(x\) in the common domain of these two functions we define new functions as follows:
• **Sum:** 
  \((f + g)(x) = f(x) + g(x)\).

• **Difference:** 
  \((f - g)(x) = f(x) - g(x)\).

• **Product:** 
  \((f \cdot g)(x) = f(x) \cdot g(x)\).

• **Division:** 
  \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\) provided that \(g(x) \neq 0\).

In the following example we see how to construct the four functions discussed above when the individual functions are defined by formulas.

**Example 1.2.8**

Let \(f(x) = x + 1\) and \(g(x) = \sqrt{x + 3}\). Find the common domain and then find a formula for each of the functions \(f + g, f - g, f \cdot g, \frac{f}{g}\).

**Solution.**

The domain of \(f(x)\) consists of all real numbers whereas the domain of \(g(x)\) consists of all numbers \(x \geq 3\). Thus, the common domain is the interval \([-3, \infty)\). For any \(x\) in this domain we have

\[
(f + g)(x) = x + 1 + \sqrt{x + 3} \\
(f - g)(x) = x + 1 - \sqrt{x + 3} \\
(f \cdot g)(x) = x\sqrt{x + 3} + \sqrt{x + 3} \\
\left(\frac{f}{g}\right)(x) = \frac{x + 1}{\sqrt{x + 3}} \text{ provided } x > -3.
\]

**Transformations of Functions**

We close this section by giving a summary of the various transformations obtained when either the input or the output of a function is altered.

**Vertical Shifts:** The graph of \(f(x) + k\) with \(k > 0\) is a vertical shift of the graph of \(f(x)\), \(k\) units upward, whereas for \(k < 0\) it is a shift by \(k\) units downward.

**Horizontal Shifts:** The graph of \(f(x + k)\) with \(k > 0\) is a horizontal shift of the graph of \(f(x)\), \(k\) units to the left, whereas for \(k < 0\) it is a shift by \(k\) units to the right.

**Reflections about the \(x\)-axis:** For a given function \(f(x)\), the graph of \(-f(x)\) is a reflection of the graph of \(f(x)\) about the \(x\)-axis.

**Reflections about the \(y\)-axis:** For a given function \(f(x)\), the graph of \(f(-x)\) is a reflection of the graph of \(f(x)\) about the \(y\)-axis.

**Vertical Stretches and Compressions:** (\(x\) is fixed but \(y\) varies) Let \(f(x)\) be a given function. The graph of \(kf(x)\) is a vertical stretch (away from
$x$–axis) of the graph of $f(x)$ by a factor of $k$ if $k > 1$. The graph of $kf(x)$ is a vertical compression (closer to $x$–axis) of the graph of $f(x)$ by a factor of $k$ is $0 < k < 1$.

**Horizontal Stretches and Compressions:** ($x$ varies but $y$ is fixed) The graph of $f(kx)$ is a horizontal compression (closer to $y$–axis) of the graph of $f(x)$ by a factor of $\frac{1}{k}$ if $k > 1$. The graph of $f(kx)$ is a horizontal stretch (away from $y$–axis) by a factor of $\frac{1}{k}$ if $0 < k < 1$.

**Remark 1.2.1**

Starting from a base function $y = f(x)$, to graph the function $y = cf(\frac{b}{a}x + b) + d = cf\left(\frac{b}{a}\left[x + \frac{b}{a}\right]\right) + d$ we first graph $y = f(ax)$ (a horizontal shift or compression by a factor of $\frac{1}{k}$) then graph $y = f\left(\frac{b}{a}\left[x + \frac{b}{a}\right]\right)$ which is a horizontal stretch by $\frac{b}{a}$, followed by a vertical stretch or compression by a factor of $c$, i.e., the graph of $y = cf\left(\frac{b}{a}\left[x + \frac{b}{a}\right]\right)$, and finally a vertical shift by $d$. 