2 Problem-Solving Strategies

Strategies are tools that might be used to discover or construct the means to achieve a goal. They are essential parts of the "devising a plan step", the second step of Pólya’s procedure which is considered the most difficult step. Elementary school children now learn strategies that they can use to solve a variety of problems. In this section, we discuss three strategies of problem solving: guessing and checking, using a variable, and drawing a picture or a diagram.

- Problem-Solving Strategy 1: Guess and Check
  The guessing-and-checking strategy requires you to start by making a guess and then checking how close our guess to the answer of the problem. Next, on the basis of this result, you revise your guess and try again. This strategy can be regarded as a form of trial and error, where the information about the error helps us choose what trial to make next.
  This strategy may be appropriate when: there is a limited number of possible answers to test; you want to gain a better understanding of the problem; you can systematically try possible answers.
  This strategy is often used when a student does not know how to solve a problem more efficiently or if the student does not have the tools to solve the problem in a faster way.

Example 2.1
In Figure 2.1 the numbers in the big circles are found by adding the numbers in the two adjacent smaller circles. Complete the second diagram so that the pattern holds.

Solution

![Diagram](image)

Figure 2.1
Understand the problem
In this example, we must find three numbers \( a, b, \) and \( c \) such that

\[
\begin{align*}
a + b &= 16, \\
a + c &= 11, \\
b + c &= 15.
\end{align*}
\]

See Figure 2.2.

Devise a plan
We will try the guess and check strategy.

Carry out the plan
We start by guessing a value for \( a \). Suppose \( a = 10 \). Since \( a + b = 16 \), we find \( b = 6 \). Since \( b + c = 15 \) we find \( c = 9 \). But then \( a + c \) is 19 instead of 11 as it is supposed to be. This does not check.

Since the value of \( a = 10 \) yields a large \( a + c \), we will reduce our guess for \( a \). Take \( a = 5 \). As above, we find \( b = 11 \) and \( c = 4 \). This gives \( a + c = 9 \) which is closer to 11 than 19. So our next guess is \( a = 6 \). This implies that \( b = 10 \) and \( c = 5 \). Now \( a + c = 11 \) as desired. See Figure 2.3.

Figure 2.2

Figure 2.3
Look back
Is there an easier solution? Looking carefully at the initial example and the completed solution to the problem we notice that if we divide the sum of the numbers in the larger circles by 2 we obtain the sum of the numbers in the smaller circle. From this we can devise an easier solution. Looking at Figure 2.1, and the above discussion we have \( a + b + c = 21 \) and \( a + b = 16 \). This gives, \( 16 + c = 21 \) or \( c = 5 \). Since \( a + c = 11 \) we find \( a = 6 \). Finally, since \( b + c = 15 \) we obtain \( b = 10 \).

Example 2.2
Leah has $4.05 in dimes and quarters. If she has 5 more quarters than dimes, how many of each does she have?

Solution.
Understand the problem
What are we asked to determine? We need to find how many dimes and how many quarters Leah has.
What is the total amount of money? $4.05.
What else do we know? There are five more quarters than dimes.

Devise a plan
Pick a number, try it, and adjust the estimate.

Carry out the plan
Try 5 dimes. That would mean 10 quarters. In this case the total is
\[
5 \times \$0.10 + 10 \times \$0.25 = \$3.00.
\]
Increase the number of dimes to 7.
\[
7 \times \$0.10 + 12 \times \$0.25 = \$3.70.
\]
Try again. This time use 8 dimes.
\[
8 \times \$0.10 + 13 \times \$0.25 = \$4.05
\]
Leah has 8 dimes and 13 quarters.
Look back
Did we answer the question asked, and does our answer seem reasonable?
Yes.

Practice Problems

Problem 2.1
Susan made $2.80 at her lemonade stand. She has 18 coins. What combination of coins does she have?

Problem 2.2
A rectangular garden is 4 feet longer than it is wide. How wide is the garden if the perimeter of the garden is 28 feet? (Hint: Draw a diagram and use the guess and check strategy.)

Problem 2.3
There are two two-digit numbers that satisfy the following conditions:

(1) Each number has the same digits,
(2) the sum of digits in each number is 10,
(3) the difference between the two numbers is 54.
What are the two numbers?

(a) Understanding the problem
The numbers 58 and 85 are two-digit numbers which have the same digits, and the sum of the digits is 13. Find two two-digit numbers such that the sum of the digits is 10 and both numbers have the same digits.

(b) Devise a plan
Since there are only nine two-digit numbers whose digits have a sum of 10, the problem can be easily solved by guessing. What is the difference of your two two-digit numbers from part (a)? If this difference is not 54, it can provide information about your next guess.

(c) Carry out the plan
Continue to guess and check. Which numbers has a difference of 54?

(d) Looking back
This problem can be extended by changing the requirement that the sum of the two digits equal 10. Solve the problem for the case in which the digits have a sum of 12.
Problem 2.4
John is thinking of a number. If you divide it by 2 and add 16 to the result, you get 28. What number is John thinking of?

Problem 2.5
Place the digits 1, 2, 3, 4, 5, 6 in the circles in Figure 2.4 so that the sum of the three numbers on each side of the triangle is 12.

Figure 2.4

Problem 2.6
Carmela opened her piggy bank and found she had $15.30. If she had only nickels, dimes, quarters, and half-dollars and an equal number of coins of each kind, how many coins in all did she have?

Problem 2.7
When two numbers are multiplied, their product is 759; but when one is subtracted from the other, their difference is 10. What are those two numbers?

Problem 2.8
Sandy bought 18 pieces of fruit (oranges and grapefruits), which cost $4.62. If an orange costs $0.19 and a grapefruit costs $0.29, how many of each did she buy?

Problem 2.9
A farmer has a daughter who needs more practice in mathematics. One morning, the farmer looks out in the barnyard and sees a number of pigs and chickens. The farmer says to her daughter, ”I count 24 heads and 80 legs. How many pigs and how many chickens are out there?”

Problem 2.10
At a benefit concert 600 tickets were sold and $1,500 was raised. If there were $2 and $5 tickets, how many of each were sold?
Problem 2.11
At a bicycle store, there were a bunch of bicycles and tricycles. If there are 32 seats and 72 wheels, how many bicycles and how many tricycles were there?

Problem 2.12
If you have a bunch of 10 cents and 5 cents stamps, and you know that there are 20 stamps and their total value is $1.50, how many of each do you have?

- **Problem-Solving Strategy 2: Use a variable**
  Often a problem requires that a number be determined. Represent the number by a variable, and use the conditions of the problem to set up an equation that can be solved to ascertain the desired number.
  This strategy is most appropriate when: a problem suggests an equation; there is an unknown quantity related to known quantities; you are trying to develop a general formula.

**Example 2.3**
Find the sum of the whole numbers from 1 to 1000.

**Solution.**

**Understand the problem**
We understand that we are to find the sum of the first 1000 nonzero whole numbers.

**Devise a plan**
We will apply the use of variable strategy. Let $s$ denote the sum, i.e.

$$ s = 1 + 2 + 3 + \cdots + 1000 $$ (1)

**Carry out the plan**
Rewrite the sum in $s$ in reverse order to obtain

$$ s = 1000 + 999 + 998 + \cdots + 1 $$ (2)

Adding (1) - (2) to obtain

$$ 2s = 1001 + 1001 + 1001 + \cdots + 1001 = 1000 \times 1001. $$
Dividing both sides by 2 to obtain

\[ s = \frac{1000 \times 1001}{2} = 500500. \]

**Look back**

Is it true that the process above apply to the sum of the first \( n \) whole integers? The answer is yes.

**Example 2.4**

Lindsey has a total of $82.00, consisting of an equal number of pennies, nickels, dimes and quarters. How many coins does she have in all?

**Solution.**

**Understand the problem**

We want to know how many coins Lindsey has.

How much money does she have in total? $82.00. How many of each coin does she have? We don't know exactly, but we know that she has an equal number of each coin.

**Devise a plan**

We know how much each coin is worth, and we know how much all of her coins are worth in total, so we can write an equation that models the situation.

**Carry out the plan**

Let \( p \) be the number of pennies, \( n \) the number of nickels, \( d \) the number of dimes, and \( q \) the number of quarters. We then have the equation

\[ p + 5n + 10d + 25q = 8200. \]

We know that she has an equal number of each coin, so \( p = n = d = q \). Substituting \( p \) for the other variables gives an equation in just one variable. The equation above becomes \( p + 5p + 10p + 25p = 41p = 8200 \), so \( p = 200 \). Lindsey has 200 pennies. Since she has an equal number of each coin, she also has 200 nickels, 200 dimes and 200 quarters. Therefore, she has 800 coins.

**Look back**

Did we answer the question asked? Yes.
Practice Problems

Problem 2.13
A dog’s weight is 10 kilograms plus half its weight. How much does the dog weigh?

Problem 2.14
The measure of the largest angle of a triangle is nine times the measure of the smallest angle. The measure of the third angle is equal to the difference of the largest and the smallest. What are the measures of the angles? (Recall that the sum of the measures of the angles in a triangle is $180^\circ$)

Problem 2.15
The distance around a tennis court is 228 feet. If the length of the court is 6 feet more than twice the width, find the dimensions of the tennis court.

Problem 2.16
The floor of a square room is covered with square tiles. Walking diagonally across the room from corner to corner, Susan counted a total of 33 tiles on the two diagonals. What is the total number of tiles covering the floor of the room?

Problem 2.17
In three years, Chad will be three times my present age. I will then be half as old as he. How old am I now?

Problem 2.18
A fish is 30 inches long. The head is as long as the tail. If the head was twice as long and the tail was its present, the body would be 18 inches long. How long is each portion of the fish?

Problem 2.19
Two numbers differ by 2 and have a product of 3. What are the two numbers?

Problem 2.20
Jeremy paid for his breakfast with 36 coins consisting of nickels and dimes. If the bill was $3.50, then how many of each type of coin did he use?
Problem 2.21
The sum of three consecutive odd integers is 27. Find the three integers.

Problem 2.22
At an 8% sales tax rate, the sales tax Peter’s new Ford Taurus was $1,200. What was the price of the car?

Problem 2.23
After getting a 20% discount, Robert paid $320 for a Pioneer CD player for his car. What was the original price of the car?

Problem 2.24
The length of a rectangular piece of property is 1 foot less than twice the width. The perimeter of the property is 748 feet. Find the length and the width.

Problem 2.25
Sarah is selling her house through a real estate agent whose commission rate is 7%. What should the selling price be so that Sarah can get the $83,700 she needs to pay off the mortgage?

Problem 2.26
Ralph got a 12% discount when he bought his new 1999 Corvette Coupe. If the amount of his discount was $4,584, then what was the original price?

Problem 2.27
Julia framed an oil painting that her uncle gave her. The painting was 4 inches longer than it was wide, and it took 176 inches of frame molding. What were the dimensions of the picture?

• Problem-Solving Strategy 3: Draw a Picture
It has been said that a picture worth a thousand words. This is particularly true in problem solving. Drawing a picture often provides the insight necessary to solve a problem.
This strategy may be appropriate when: a physical situation is involved; geometric figures or measurements are involved; a visual representation of the problem is possible.

Example 2.5
Can you cut a pizza into 11 pieces with four straight cuts?
Solution.

Understand the problem
The pieces need not have the same size and shape.

Devise a plan
If we try to cut the pizza with four cuts using the usual standard way we will end with a total of 8 equally shaped slices. See Figure 2.5.

![Figure 2.5](image)

Carry out the plan
The cuts are made as shown in Figure 2.6.

![Figure 2.6](image)

Look back
The above is not the only way to cut the pizza. There are many other ways.

Example 2.6
In a stock car race the first five finishers in some order were a Ford, a Pontiac, a Chevrolet, a Buick, and a Dodge.

(a) The Ford finished seven seconds before the Chevrolet.
(b) The Pontiac finished six seconds after the Buick.
(c) The Dodge finished eight seconds after the Buick.
(d) The Chevrolet finished two seconds before the Pontiac.

In what order did the cars finish the race?
Solution.

**Understand the problem**
We are told to determine the order in which the five cars finished the race.

**Devise a plan**
We draw a line to represent the track at the finish of the race and place the cars on it according to the conditions of the problem. Mark the line off in time intervals of one second. We use the first letter of each car’s name to represent the car. So the question is to order the letters B, C, D, F, and P on the line according to the given information.

**Carry the plan**
The finishing position of each of the five cars is given in Figure 2.7.

![Figure 2.7](image)

**Look back**
We see that pictures can help to solve problems.

**Practice Problems**

**Problem 2.28**
Bob can cut through a log in one minute. How long will it take Bob to cut a 20-foot log into 2-foot sections?

**Problem 2.29**
How many posts does it take to support a straight fence 200 feet long if a post is placed every 20 feet?

**Problem 2.30**
Albright, Badgett, Chalmers, Dawkins, and Earl all entered the primary to seek election to the city council. Albright received 2000 more votes than Badgett and 4000 fewer than Chalmers. Earl received 2000 votes fewer than Dawkins and 5000 votes more than Badgett. In what order did each person finish in the balloting?

**Problem 2.31**
A 9-meter by 12-meter rectangular lawn has a concrete walk 1 meter wide all around it outside lawn. What is the area of the walk?
Problem 2.32
An elevator stopped at the middle floor of a building. It then moved up 4 floors and stopped. It then moved down 6 floors, and then moved up 10 floors and stopped. The elevator was now 3 floors from the top floor. How many floors does the building have?

Problem 2.33
In the Falkland Islands, south of Argentina, Armado, a sheepherder’s son, is helping his father build a rectangular pen to keep their sheep from getting lost. The pen will be 24 meters long, 20 meters wide, and have a fence posts 4 meters apart. How many fence posts do they need?

Problem 2.34
Five people enter a racquetball tournament in which each person must play every other person exactly once. Determine the total number of games that will be played.

Problem 2.35
When two pieces of ropes are placed end to end, their combined length is 130 feet. When the two pieces are placed side by side, one is 26 feet longer than the other. What are the lengths of the two pieces?

Problem 2.36
There are 560 third- and fourth-grade students in Russellville elementary school. If there are 80 more third graders than fourth graders, how many third graders are there in the school?

Problem 2.37
A well is 20 feet deep. A snail at the bottom climbs up 4 feet every day and slips back 2 feet each night. How many days will it take the snail to reach the top of the well?

Problem 2.38
Five friends were sitting on one side of a table. Gary set next to Bill. Mike sat next to Tom. Howard sat in the third seat from Bill. Gary sat in the third seat from Mike. Who sat on the other side of Tom?