1 Solving Algebraic Equations

This section illustrates the processes of solving linear and quadratic equations.

The Geometry of Real Numbers

In order to understand the mathematics of solving equations and inequalities a good understanding of the geometry of real numbers is deemed important. The various sets of numbers in increasing order are:

• The set of all positive integers
  \[ \mathbb{N} = \{1, 2, 3, \cdots \}. \]

• The set of all integers
  \[ \mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots \}. \]

• The set of all rational numbers
  \[ \mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ with } b \neq 0\}. \]

• The set \( \mathbb{R} \) of all real numbers.

We will pay close attention to the set of real numbers. Geometrically, \( \mathbb{R} \) can be described by a horizontal axis pointing to the right as shown in Figure 1.1 with numbers being represented by points called coordinates. This line is referred to as the real line.

![Figure 1.1](image)

When the coordinate of a number \( a \) is to the left of that of a number \( b \) we say that \( a \) is less than \( b \) and we write \( a < b \). We can also say that \( b \) is greater than \( a \) and write \( b > a \). The symbols \(< \) and \(>\) are inequality symbols. Two other inequality symbols are \(\leq \) and \(\geq \). Inequality symbols help us represent portions of the real lines by intervals.
The various types of intervals and their graphs are shown in Figure 1.2.

The distance between two numbers \( x \) and \( a \) on the real line is denoted by \( |x - a| \). For example, the distance between the number \( x \) and 0 is \( |x| \). We call \( |x| \) the absolute value of \( x \). Since it is a distance, it is a non-negative number. But \( x \) can be a non-negative or negative. So we define

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0
\end{cases}
\]

Thus, \(|-3| = -(3) = 3\) while \(|3| = 3\). Some of the main properties of the absolute value are:

\[
| -x | = |x| \\
|x + y| \leq |x| + |y| \\
|xy| = |x||y| \\
\left| \frac{x}{y} \right| = \frac{|x|}{|y|}, \ y \neq 0 \\
|x^n| = |x|^n.
\]

**Solving Linear Equations**

By a linear equation we mean an equation of the form

\[
ax + b = 0
\]

where \( a \neq 0 \) and \( b \) are given numbers and \( x \) is the variable to be found, also called the solution or root of the equation. The process of finding \( x \) is referred to as solving the given equation.

To solve a linear equation in one variable, isolate the variable on one side of the equation. This can be done thanks to the following two properties of real numbers:

**Property I:** Adding or subtracting the same number to both sides of an equation does not change the solution to the equation.

**Property II:** Multiplying or dividing both sides of an equation by a nonzero number does not change the solution to the equation.
Remark 1.1
The above two properties apply to any equation and not only for linear equations.

Example 1.1
Solve the equation: \(-3x + 20 = 2\).

Solution.
To isolate \(x\), subtract first 20 from both sides of the given equation to obtain \(-3x = -18\). Now, divide both sides by \(-3\) to obtain \(x = 6\).

Solving Quadratic Equations
The second type of equations that we discuss here is the so-called quadratic equations. By a quadratic equation we mean an equation of the form

\[ ax^2 + bx + c + 0, \quad a \neq 0, \]

where \(a, b,\) and \(c\) are given numbers and \(x\) is the variable to be found. There are two methods for finding \(x\).

- **Solving by Factoring**
  The process of factoring consists of rewriting the equation in the form

  \[ a(x - r)(x - s) = 0. \]

  Now, by the zero product property, which states that if \(u \cdot v = 0\) then either \(u = 0\) or \(v = 0\), we can conclude that either \(x - r = 0\) or \(x - s = 0\). That is, \(x = r\) or \(x = s\).

  To factor \(ax^2 + bx + c\)
  1. find two integers that have a product equal to \(ac\) and a sum equal to \(b\),
  2. replace \(bx\) by two terms using the two new integers as coefficients,
  3. then factor the resulting four-term polynomial by grouping. Thus, obtaining \(a(x - r)(x - s) = 0\).
  4. use the zero product property: \(ab = 0 \implies a = 0\) or \(b = 0\).

Example 1.2
Find the zeros of \(f(x) = 2x^2 + 9x + 4\).

Solution.
We need two integers whose product is \(ac = 8\) and sum equals to \(b = 9\). Such two integers are 1 and 8. Thus,
\[ 2x^2 + 9x + 4 = 2x^2 + x + 8x + 4 \]
\[ = x(2x + 1) + 4(2x + 1) \]
\[ = (2x + 1)(x + 4) \]
\[ = 2(x + \frac{1}{2})(x + 4). \]

Hence, the zeros are \( x = -\frac{1}{2} \) and \( x = -4 \).

**Solving by Using the Method of Completing the Square:**

Many quadratic functions are not easily factored. For example, the function \( f(x) = 3x^2 - 7x - 7 \). However, the zeros can be found by using the method of completing the square:

\[
ax^2 + bx + c = 0 \text{ (subtract c from both sides)} \\
ax^2 + bx = -c \text{ (multiply both sides by 4a)} \\
4a^2x^2 + 4abx = -4ac \text{ (add } b^2 \text{ to both sides)} \\
4a^2x^2 + 4abx + b^2 = b^2 - 4ac \\
(2ax + b)^2 = b^2 - 4ac \\
2ax + b = \pm \sqrt{b^2 - 4ac} \\
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

provided that \( b^2 - 4ac \geq 0 \). This last formula is known as the **quadratic formula**. Note that if \( b^2 - 4ac < 0 \) then the equation \( ax^2 + bx + c = 0 \) has no real solutions ( but has complex solutions if you are familiar with complex numbers.)

**Example 1.3**

Find the zeros of \( f(x) = 3x^2 - 7x - 7 \).

**Solution.**

Letting \( a = 3, b = -7 \) and \( c = -7 \) in the quadratic formula we have

\[ x = \frac{7 \pm \sqrt{133}}{6} \]
Example 1.4
Find the zeros of the function \( f(x) = 6x^2 - 2x + 5 \).

Solution.
Letting \( a = 6, b = -2 \), and \( c = 5 \) in the quadratic formula we obtain
\[
x = \frac{2 \pm \sqrt{-116}}{12}
\]
But \( \sqrt{-116} \) is not a real number. Hence, the function has no zeros.■