6.2 The Law of Cosines

Continuing our discussion of dealing with oblique (non-right) triangles, today we will look at the Law of Cosines.

Given a triangle with angles A, B, and C, and sides opposite them a, b, and c, respectively, as shown below, the following relationships hold:

Law of Cosines:

\[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ b^2 = a^2 + c^2 - 2ac \cos B \]
\[ c^2 = a^2 + b^2 - 2ab \cos C \]
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The law of cosines requires that you have either two sides and an included angle or all three sides.

Examples: Given the triangles below, use the Law of Cosines to find the missing sides and angles. Round side lengths to the nearest tenth and angles to the nearest degree.

1) \[ a \approx 8.2 \]
2) \[ 43^\circ = C \]
3) \[ b^2 = (15^2 + 20^2 - 2 \cdot 15 \cdot 20 \cdot \cos 52) \]
\[ b = \sqrt{5} \]

\[ B = 180 - (28 + 100) = 52 \]

4) \[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ 10^2 = 9^2 + 3^2 - 2 \cdot 9 \cdot 3 \cdot \cos A \]
\[ 100 = 81 + 9 - 54 \cos A \]
\[ -90 = 90 - 54 \cos A \]
\[ -1 \approx \cos A \]
\[ 101 \approx A \]

\[ q^2 = 10^2 + 3^2 - 2 \cdot 10 \cdot 3 \cos B \]
\[ q^2 = 100 + 9 - 60 \cos B \]
\[ 81 = 100 + 9 - 60 \cos B \]
\[ -109 = -60 \cos B \]
\[ -109 = -60 \cos B \]
\[ 62^\circ = \cos^{-1} \left( \frac{-28}{-60} \right) = \frac{\cos^{-1} \left( - \frac{28}{60} \right)}{60} \]
Construct an oblique triangle with the given information and then use the Law of Cosines to find the missing values. Round side lengths to the nearest tenth and angles to the nearest degree.

1) a = 20, b = 14, C = 40°

\[ \sin^{-1}\left( \frac{14\sin A}{20} \right) = \sin^{-1}\left( \frac{20\sin 40°}{12.9} \right) \]

\[ A \approx 85° \]

2) a = 5, b = 6, c = 7

\[ \sqrt{c^2 = a^2 + b^2 - 2ab \cos C} \]

\[ c \approx 12.9 \]

\[ 20^2 = 14^2 + 12.9^2 - 2 \cdot 14 \cdot 12.9 \cos A \]
Applications of Law of Cosines

1) Find the distance across the lake from A to B, to the nearest yard, using the measurements shown in the figure.

\[ c \approx 363 \text{ yds.} \]

2) A Major League baseball diamond has four bases forming a square whose sides measure 90 feet each. The pitcher's mound is 60.5 feet from home plate on a line joining home plate and second base. Find the distance from the pitcher's mound to first base. Round to the nearest tenth of a foot.

\[ x \approx 63.7 \text{ ft} \]
3) \[ 7^2 = 5^2 + 6^2 - 2 \cdot 5 \cdot 6 \cdot \cos A \]

\[ 49 = 25 + 36 - 60 \cos A \]

\[ 49 = 61 - 60 \cos A \]

\[ -12 = -60 \cos A \]

\[ \frac{-12}{-60} = \cos^{-1} \left( \frac{12}{60} \right) = \cos^{-1} \left( \cos A \right) \]

\[ 78^\circ \approx A \]

If you are on island A, on what bearing should you navigate to go to island C? 12° East of North

If you are on island B, on what bearing should you navigate to go to island C? 46° West of North

\[ 5^2 = 7^2 + 6^2 - 2 \cdot 7 \cdot 6 \cdot \cos B \]

\[ 25 = 49 + 36 - 84 \cos B \]

\[ 25 = 85 - 84 \cos B \]

\[ -50 = -84 \cos B \]

\[ \frac{-50}{-84} = \cos^{-1} \left( \frac{50}{84} \right) = \cos^{-1} (\cos B) \rightarrow B \approx 44^\circ \]
Another method for finding the area of a triangle (if we do not know the height) is Heron’s Formula, which is based on the perimeter, \( p \), of the triangle, \( p = a + b + c \).

\[
\text{Area} = \frac{1}{4} \sqrt{p(p-2a)(p-2b)(p-2c)}
\]

Determine the area of the following triangles below by using Heron’s formula.

1) \( a = 3' \), \( b = 6' \), \( c = 8' \)

\[
A = \frac{1}{4} \sqrt{17(17-6)(17-12)(17-16)} \approx 7.6 \text{ sq. in.}
\]

2) \( p = 27' \)

\[
A = \frac{1}{4} \sqrt{27(27-12)(27-5)(27-10)} \approx 24.5 \text{ sq. ft.}
\]