4.7 Inverse Trigonometric Functions

When we studied inverse functions in chapter 1, we learned that in order to have an inverse, a function must be one-to-one, which means its graph should pass the horizontal line test. Clearly, the graphs of sine and cosine do not. (Nor do the graphs of tangent, cotangent, secant, or cosecant.) However, if we restrict their domains, we can get them to be. Consider the following:

\[
\begin{align*}
y &= \sin x \\
y &= \cos x \\
y &= \tan x
\end{align*}
\]

We can get one-to-one functions if we restrict the domain of sine to \([-\pi/2, \pi/2]\), the domain of cosine to \([0, \pi]\), and the domain of tangent to \((-\pi/2, \pi/2)\). The range for sine and cosine is \([-1, 1]\), and for tangent the range is \((-\infty, \infty)\).

So when we are taking the inverse trig function, our input will be a real number (between -1 and 1 for sine and cosine), and the output will be an angle in the restricted domain!

**Example:**

\[
\begin{align*}
\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) &= \frac{\pi}{4} & \text{because} & \quad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\
\sin^{-1}\left(\frac{1}{2}\right) &= \frac{\pi}{6} & \text{because} & \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \\
\cos^{-1}\left(\frac{1}{2}\right) &= \frac{\pi}{3} & \text{because} & \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}
\end{align*}
\]
Examples: Find the exact value of each of the following.

1) \( \cos^{-1}(-1/2) = \frac{2\pi}{3} \)
2) \( \sin^{-1}(-1/2) = -\frac{\pi}{6} \)
3) \( \cos^{-1}(-1) = \pi \)

Another thing you should recall from our discussion of inverse functions is that when inverse functions are composed at \( x \), the result is \( x \). This is the case with inverse trig functions as long as \( x \) is in the domain.

Ex:

\[
\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right) = \frac{1}{2} \quad \text{and} \quad \sin\left(\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right) = -\frac{\sqrt{2}}{2}
\]

\[
\cos^{-1}\left(\cos\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4} \quad \text{and} \quad \sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}
\]
Ex: 1) $\sin^{-1}(0.764)$  
2) $\cos^{-1}(-0.25)$  
3) $\tan^{-1}(35)$

We can find the exact values of some expressions by creating a right triangle.

Ex: 1) $\tan(\sin^{-1}(3/5))$  
2) $\sin(\cos^{-1}(7/10))$

What is the tangent of an angle whose sine is $\frac{3}{5}$?

\[
\begin{align*}
\triangle & \quad \sin \theta = \frac{3}{5} \\
\quad & \quad a = 4, \\
\quad & \quad b = 3, \\
\quad & \quad c = \sqrt{362}, \\
\quad & \quad a^2 + b^2 = c^2 \\
& \quad a^2 + 9 = 25 \\
& \quad a^2 = 16 \\
& \quad a = 4
\end{align*}
\]

3) $\sin(\tan^{-1}(19))$
3) $\cos(\tan^{-1}(7/4))$

\[\frac{4 \cdot \sqrt{65}}{\sqrt{65} \cdot \sqrt{65}} = \frac{4 \sqrt{65}}{65}\]

$\quad 4) \tan(\sin^{-1}(1/5))$

\[\frac{1 \cdot \sqrt{6}}{2\sqrt{6} \cdot \sqrt{6}} = \frac{\sqrt{6}}{12}\]

\[\sqrt{24} = \sqrt{4 \cdot 6} = 2\sqrt{6}\]

\[\frac{5}{1} = \frac{1}{\sqrt{a^2 - 1}}\]

\[a^2 + 1 = 5\]
\[a^2 + 1 = 25\]
\[a^2 = 24\]
\[a = \sqrt{24}\]