4.2: Trigonometric Functions: The Unit Circle

The word *trigonometry* means measurement of triangles. Trigonometric functions, with domains consisting of sets of angles, were first defined using right triangles. Most other functions we work with use domains consisting of real numbers. Looking at trig from a unit circle approach relates the two concepts. The unit circle (radius 1) centered at the origin has equation $x^2 + y^2 = 1$. Every point $(x, y)$ on the circle must satisfy this equation. Let $P = (x, y)$ denote the point on the unit circle that has arc length $t$ units from $(1, 0)$. Since we are on the unit circle, the angle that intercepts that arc has measure $t$ radians. We define the *x*-coordinate of the point to be cosine $t$, or $\cos t$. We define the *y*-coordinate of the point to be sine $t$, or $\sin t$.

Consider the point $(0, 1)$ on the unit circle. The distance from the point $(1, 0)$ to the point $(0, 1)$ along the circle is $\pi/2$ units. The angle that intercepts that arc is $\pi/2$ radians. Therefore the cosine of $\pi/2$ is 0 and the sine of $\pi/2$ is 1. (Or $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$.)

In addition to cosine and sine, there are four other trig functions, all related to cosine and sine. They are tangent ($\tan$), secant ($\sec$), cosecant ($\csc$), and cotangent ($\cot$), and are defined as follows:

\[
\begin{align*}
\cos t &= x \\
\sin t &= y \\
\tan t &= \frac{y}{x} = \frac{\sin t}{\cos t} \\
\sec t &= \frac{1}{\cos t} \\
\csc t &= \frac{1}{\sin t} \\
\cot t &= \frac{\cos t}{\sin t}
\end{align*}
\]

In the figure below, the point $P(5/13, -12/13)$ is shown on the unit circle corresponding to a real number $t$. Find the values of the six trigonometric functions at $t$.

\[
\begin{align*}
\cos t &= \frac{-5}{13} & \sec t &= \frac{-13}{5} \\
\sin t &= \frac{-12}{13} & \csc t &= \frac{-13}{12} \\
\tan t &= \frac{-12}{5} = \frac{12}{-5} & \cot t &= \frac{5}{12}
\end{align*}
\]
The increments into which we divided the unit circle in the last section all correspond to "special angles," which have special values for the trig functions. If you learn the values of sine and cosine for the reference angles (the special angles in Quadrant I), it will make it easy to get their values for any multiple of a special angle.

First special angle: $\frac{\pi}{4}$

Next special angle: $\frac{\pi}{3}$

Next special angle: $\frac{\pi}{6}$
Now we should be able to label the points on the circle below with the proper coordinates, keeping in mind in which quadrants values are positive and negative.
Recall that the equation of the unit circle centered at the origin is \( x^2 + y^2 = 1 \). Since \( \cos t = x \) and \( \sin t = y \), this gives us the equation \( (\cos t)^2 + (\sin t)^2 = 1 \) or \( \cos^2 t + \sin^2 t = 1 \). This is the Pythagorean Identity, and will help us find values of the trig functions if we are only given one coordinate or one trig function value.

**Ex:** Suppose \( \sin t = \frac{7}{8} \) and \( t \) is in Quadrant I. Find the values of the other five trig functions. (What other quadrant could \( t \) be in?)

\[
\begin{align*}
\cos^2 t + \left(\frac{7}{8}\right)^2 &= 1 \\
\cos^2 t + \frac{49}{64} &= 1 \\
\cos^2 t &= \frac{15}{64} \\
\cos t &= \sqrt{\frac{15}{64}} = \frac{\sqrt{15}}{8} \\
\csc t &= \frac{8}{\frac{\sqrt{15}}{8}} = \frac{8\sqrt{15}}{15} \\
\tan t &= \frac{\sin t}{\cos t} = \frac{\frac{7}{8}}{\frac{\sqrt{15}}{8}} = \frac{7}{\sqrt{15}} \\
\cot t &= \frac{\sqrt{15}}{7}
\end{align*}
\]

What do you notice about the cosine of \( t \) and the cosine of \(-t\)?

**Ex:** \( \cos \left(\frac{\pi}{4}\right) \quad \cos \left(-\frac{\pi}{4}\right) \)

\( \frac{\sqrt{2}}{2} \quad \frac{\sqrt{2}}{2} \)

How about the sine of \( t \) and the sine of \(-t\)?

**Ex:** \( \sin \left(\frac{\pi}{4}\right) \quad \sin \left(-\frac{\pi}{4}\right) \)

\( \frac{\sqrt{2}}{2} \quad -\frac{\sqrt{2}}{2} \)

Recall that **even functions** have the property that \( f(-x) = f(x) \). Therefore \( y = \cos t \) must be an even function!

Recall that **odd functions** have the property that \( f(-x) = -f(x) \). Therefore \( y = \sin t \) must be an odd function!

Use even and odd properties of trig functions to find each of the following. (Hint: First find \( \cos \left(\frac{2\pi}{3}\right) \) and \( \sin \left(\frac{3\pi}{4}\right) \).)

1) \( \cos \left(-\frac{2\pi}{3}\right) \)

\( \cos \left(-\frac{2\pi}{3}\right) = -\frac{1}{2} \)

2) \( \sin \left(-\frac{3\pi}{4}\right) \)

\( \sin \left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \)

\[
\begin{align*}
\cos \theta &= -\frac{4}{5} \\
\sin \theta &= -\frac{3}{5} \\
\cos(-\theta) &= -\frac{4}{5} \\
\sin(-\theta) &= \frac{3}{5}
\end{align*}
\]
Recall that to find an angle which was coterminal with a given angle, we would add $2\pi$ (or a multiple of $2\pi$) to the given radian measure. This suggests that $\cos (t + 2\pi) = \cos t$, and that $\sin(t + 2\pi) = \sin t$. This means that $\cos t$ and $\sin t$ are periodic functions, with period $2\pi$. i.e. the time (or distance) that it takes to complete one full cycle is $2\pi$.

Use the periodic properties of sine and cosine to find each value.

1) $\cos \left( \frac{13\pi}{6} \right) = \frac{\sqrt{3}}{2} \\
\cos \left( \frac{13\pi}{6} \right) = \cos \left( \frac{10\pi}{2} + \frac{3\pi}{6} \right) = \cos \left( \frac{3\pi}{6} \right) = \frac{\sqrt{3}}{2}$

2) $\sin \left( \frac{21\pi}{2} \right) = \sin \left( \frac{2\pi}{2} + \frac{19\pi}{2} \right) = \sin \left( \frac{19\pi}{2} \right) = 1 \\
\sin \left( \frac{19\pi}{2} \right) = \sin \left( \frac{\pi}{2} + \frac{18\pi}{2} \right) = \sin \left( \frac{\pi}{2} \right) = 1$

The number of hours of daylight, $H$, on day $t$ of any given year (on January 1, $t = 1$) in San Diego, California, can be modeled by the function below.

$$H(t) = 12 + 2.4 \sin \left[ \frac{2\pi}{365} (t - 80) \right]$$

March 21, the 80th day of the year, is the spring equinox. Find the number of hours of daylight in San Diego on this day.

$$H(80) = 12 + 2.4 \sin \left[ \frac{2\pi}{365} (80 - 80) \right] = 12 \text{ hrs}.$$ 

December 21, the 355th day of the year, is the winter solstice, the day with the minimum number of hours of daylight. To the nearest tenth of an hour, find the number of hours of daylight in San Diego on this day.

$$H(355) = 12 + 2.4 \sin \left[ \frac{2\pi}{365} (355 - 80) \right] \\
\approx 9.6 \text{ hrs}.$$