2.6: Rational Functions and Their Graphs

A rational function is a function of the form \( f(x) = \frac{p(x)}{q(x)} \) where \( p(x) \) and \( q(x) \) are polynomial functions.

To determine the domain of a rational function, determine values which must be omitted by determining the zeros of \( q(x) \), the denominator.

Ex: Determine the domain of each rational function below.

\[
\begin{align*}
 f(x) &= \frac{x^2 - 4}{x^2 - 9} & f(x) &= \frac{5x}{(x - 1)(x + 4)} & f(x) &= \frac{x^3 - 16x}{x^2 - 3x - 10} \\
 x^2 - 9 &\neq 0 & (x - 1)(x + 4) &\neq 0 & x^2 - 3x - 10 &\neq 0 \\
 x^2 &\neq 9 & x \neq 1, & x \neq -4 & (x - 5)(x + 2) &\neq 0 \\
 x &\neq \pm 3 & x \neq 5, & x \neq -2 \\
 D: & (-\infty, -3) \cup (-3, 3) \cup (3, \infty) \\
 f(x) &= \frac{5}{x} & f(x) &= \frac{x - 1}{x^2 + 9} & f(x) &= \frac{x^2 - 5x + 6}{x^2 + 6x + 5} \\
 x &\neq 0 & D: & (-\infty, \infty) & x^2 + 6x + 5 &\neq 0 \\
 & & No \ real \ values \ make \ x^2 + 9 = 0 & (x + 1)(x + 5) &\neq 0 \\
 & & & x \neq -1, & x \neq -5
\end{align*}
\]
The graph of a rational function will usually have asymptotes, which are lines that the graph approaches as either \( x \) or \( y \) approaches \( \infty \) or \( -\infty \).

To determine the **vertical asymptotes**, determine the values that make the denominator = 0. (i.e. Find the zeros of \( q(x) \).) Note, however, that if there is a corresponding factor in the numerator, then the graph **will have a hole** rather than a vertical asymptote at that \( x \)-value.

Find the vertical asymptotes (or holes) of each of the rational functions below.

\[
f(x) = \frac{x^2 - 4}{x^2 - 9} = \frac{(x-2)(x+2)}{(x-3)(x+3)}
\]

\( \text{v.a. } x = 3, x = -3 \)

\[
f(x) = \frac{5x}{(x-1)(x+4)}
\]

\( \text{v.a. } x = 1 \quad x = -4 \)

\[
f(x) = \frac{x^3 - 16x}{x^2 - 3x - 10} = \frac{x(x-4)(x+4)}{(x-5)(x+2)}
\]

\( \text{v.a. } x = 5, x = -2 \)

\[
f(x) = \frac{5}{x}
\]

\( \text{v.a. } x = 0 \)

\[
f(x) = \frac{x-1}{x^2 - 1} = \frac{x+1}{(x-1)(x+1)} \approx \frac{1}{x+1}
\]

hole at \( (1, \frac{1}{2}) \)

\( \text{v.a. } x = -1 \)

\[
f(x) = \frac{x^2 - 5x + 6}{x^2 - 7x + 12} = \frac{(x-2)(x-3)}{(x-3)(x-4)}
\]

hole at \( (3,-1) \)

\( \text{v.a. } x = 4 \)
The graph of a rational function will have at most 1 horizontal asymptote, which will only occur if the degree of \( p(x) \) is less than or equal to the degree of \( q(x) \). To determine the horizontal asymptote, first examine the ratio of the leading terms of \( p(x) \) and \( q(x) \). The value of this ratio as \( x \) approaches \( \infty \) gives us the horizontal asymptote. If the degree of \( p(x) \) is greater than the degree of \( q(x) \), then there will not be a horizontal asymptote.

Ex: Determine the horizontal asymptote, if any, of each function below.

\[
f(x) = \frac{x^2 - 4}{x^2 - 9} \quad f(x) = \frac{5x}{(x-1)(x+4)} \quad f(x) = \frac{x^3 - 16x}{x^2 - 3x - 10}
\]

\[
\frac{x^2}{x^2} = 1
\]

\[
h.a. \ y = 1
\]

\[
\frac{5x}{x^2 + \cdots} = \frac{5x}{x^2} = \frac{5}{x} \to 0 \\
\text{Let } x \text{ get really BIG!}
\]

\[
\frac{5}{1,000,000,000} \quad \text{h.a. } y = 0
\]

If the degree of \( p(x) \) is smaller than the degree of \( q(x) \), the h.a. is \( y = 0 \).
If the degree of the numerator is exactly one more than the degree of the denominator, there will be a slant asymptote. To find the slant asymptote, use long division to divide \( q(x) \) into \( p(x) \). Disregard the remainder.

Ex: Find the slant asymptote of each rational function below.

\[
f(x) = \frac{x^3 - 16x}{x^2 - 3x - 10} \quad 5a. \quad y = x + 3
\]

\[
f(x) = \frac{x^2 - 6x}{x - 2} \quad 5a. \quad y = x - 4
\]
Finally, to determine the $x$-intercept(s) and $y$-intercept, as always, for the $x$-intercept, let $y = 0$ and solve for $x$. This is equivalent to finding the zeros of $p(x)$, the numerator. For the $y$-intercept, let $x = 0$ and solve for $y$.

Ex: Determine the $x$-intercept(s) and the $y$-intercept of each rational function below.

\[
f(x) = \frac{x^2 - 4}{x^2 - 9} \quad f(x) = \frac{5x}{(x-1)(x+4)} \quad f(x) = \frac{x^3 - 16x}{x^2 - 3x - 10}
\]

\[
\begin{align*}
0 &= \frac{x^2 - 4}{x^2 - 9} \\
0 &= x^2 - 4 \\
x^2 &= 4 \\
x &= \pm 2 \\
x\text{-int: } &x = \pm 2 \\
y\text{-int: } &\frac{-4}{-9} = \frac{4}{9}
\end{align*}
\]

\[
\begin{align*}
0 &= 5x \\
x\text{-int: } &x = 0 \\
y\text{-int: } &0
\end{align*}
\]

\[
\begin{align*}
x^3 - 16x &= 0 \\
x(x^2 - 16) &= 0 \\
x = 0, x^2 &= 16 \\
x &= \pm 4
\end{align*}
\]

\[
y\text{-int: } &0
\]
Once you have found all of the asymptotes and the intercepts, you are ready to graph the rational function! We use dotted lines to graph asymptotes, since they do not actually contain points from the graph. The graph cannot intersect a vertical asymptote, since those values are not in the domain of the function, but it can intersect a horizontal or slant asymptote.

Ex: Determine all asymptotes and intercepts of each rational function below, and then graph each function.

\[ f(x) = \frac{x^2 - 4}{x^2 - 1} = \frac{(x-2)(x+2)}{(x-1)(x+1)} \]

V.a. \( x = \pm 1 \)

h.a. \( y = 1 \)

x-int: \( \pm 2 \)

y-int: \( 4 \)

\[ f(x) = \frac{x^2 - 6x}{x - 2} = \frac{x(x - 6)}{x - 2} \]

V.a. \( x = 2 \)

No h.a.

S.a.: \( y = x - 4 \)

x-int: \( 0, 6 \)

y-int: \( 0 \)
Ex: Determine all asymptotes and intercepts of each rational function below, and then graph each function.

\[ f(x) = \frac{2x + 8}{x^2 - 3x - 10} = \frac{2(x+4)}{(x-5)(x+2)} \]

\[ f(x) = \frac{3x - 9}{x} \]

\[ \text{v.a. } x = 5, \ x = -2 \]
\[ \text{h.a. } y = 0 \]
\[ x\text{-int: } -4; \ y\text{-int: } -\frac{4}{5} \]

\[ \text{v.a. } x = 0 \]
\[ \text{h.a. } y = 3 \]
\[ x\text{-int: } 3 \]
\[ y\text{-int: None} \]
Ex: Determine all asymptotes, holes, and intercepts of each rational function below, and then graph each function.

\[ f(x) = \frac{(x-5)(x+3)}{(x-5)(x-1)} \]

hole at \( (5, 2) \)

\( \frac{5+3}{5-1} \)

\( \frac{8}{4} \)

\( 2 \)

v.a. \( x = 1 \)

h.a. \( y = 1 \)

\[ f(x) = \frac{x^2 - 3x}{x^2 - 9} \]

\( x = -3 \)

\( y = -3 \)
Write the equation of a rational function which satisfies the given conditions:

a) \( f \) has a vertical asymptote at \( x = 5 \), a horizontal asymptote at \( y = 3 \), an \( x \)-intercept at \(-4\), and a \( y \)-intercept at \(-12/5\).

\[
f(x) = \frac{3(x + 4)}{x - 5} = \frac{3x + 12}{x - 5}
\]

b) \( f \) has vertical asymptotes at \( x = 4 \) and \( x = -2 \), a horizontal asymptote at \( y = 0 \), an \( x \)-intercept at \(-7\), and a \( y \)-intercept at \(7/8\).

\[
f(x) = \frac{-1(x + 7)}{(x - 4)(x + 2)}
\]

c) \( f \) has a vertical asymptote at \( x = 8 \), a slant asymptote at \( y = 2x + 28 \), \( x \)-intercepts at \(-4 \) and \(-2\), and a \( y \)-intercept at \(-2\).