2.4: Dividing Polynomials; Remainder and Factor Theorem

We can divide a polynomial by another polynomial using long division, in much the same way that we divide numbers.

(The procedure is outlined on p. 336.)

Ex: Divide using long division. State the quotient, \( q(x) \), and the remainder, \( r(x) \).

\[
(x^3 - 7x^2 + 6x + 5) \div (x - 1)
\]

\[
\begin{array}{c|ccccc}
\multicolumn{2}{c|}{} & x^2 & + 5 & & \\
\hline
x^3 & - 7x^2 & + 6x & + 5 & \\
\hline
& - x^2 & + 5 & & \\
\hline
& - x^4 & + 5x^2 & - 8x & + 1 & \\
\hline
& 5x^2 & - 8x & + 1 & \\
\hline
& & - 5x^2 & + 25 & & \\
\hline
& & & -8x & + 26 & < R
\end{array}
\]

\[
P(x) = D(x) \cdot Q(x) + R(x)
\]

\[
x - 8x + 1 = (x^2 - 5)(x^2 + 5) + -8x + 26
\]

\[
= x^4 - 25 - 8x + 26
\]
If the divisor is a linear factor, then we can use synthetic division to divide rather than long division. This procedure is outlined on p. 339.

Ex: Divide using synthetic division. State the quotient, \( q(x) \), and the remainder, \( r(x) \).

\[
\begin{align*}
(x^3 - 7x^2 + 6x + 5) \div (x - 1) \\
\end{align*}
\]

\[
\begin{array}{c|ccccc}
\hline
& 1 & -7 & 6 & 5 \\
\hline
-1 & 1 & -6 & 0 & 5 \\
\hline
\end{array}
\]

\[
\frac{x^4 - 3x + 1}{x + 4}
\]

\[-4 \bigg| \begin{array}{cccc}
1 & 0 & 0 & -3 \\
\end{array}
\]

\[
\begin{array}{cccc}
\hline
-4 & 16 & -64 & 256 \\
\hline
1 & -4 & 16 & -67 & 261 \\
\hline
\end{array}
\]

\[
\frac{x^3 - 4x^2 + 16x - 67}{x^2 - 6x}
\]

Quotient

\]

Remainder
The Division Algorithm: If \( f(x) \) and \( d(x) \) are polynomials, with \( d(x) \neq 0 \), and the degree of \( d(x) \) is less than or equal to the degree of \( f(x) \), then there exist unique polynomials \( q(x) \) and \( r(x) \) such that 
\[
f(x) = d(x) \cdot q(x) + r(x).
\]
The remainder, \( r(x) \), equals 0 or it is of degree less than the degree of \( d(x) \). If \( r(x) = 0 \), we say that \( d(x) \) divides evenly into \( f(x) \) and that \( d(x) \) and \( q(x) \) are factors of \( f(x) \).

Ex: Divide \( 7 - 11x - 3x^2 + 2x^3 \) by \( x - 3 \). State your result in the form 
\[
f(x) = d(x) \cdot q(x) + r(x).
\]

\[
\begin{array}{c|cccc}
& 2 & -3 & -11 & 7 \\
\hline
3 & & & & \\
\end{array}
\]

\[
\begin{array}{c|cccc}
& 2 & -3 & -11 & 7 \\
\hline
& & 6 & 9 & -6 \\
3 & 6 & 27 & 81 & 108 \\
\hline
& 2 & 3 & -2 & 1 \\
\end{array}
\]

\[f(x) = (x-3)(2x^2 + 3x - 2) + 1\]

\[
2x^3 + 3x^2 - 2x - 6x^2 - 7x + 6 + 1 =
\]

\[
2x^3 - 3x^2 - 11x + 7
\]
The Remainder Theorem: If the polynomial $f(x)$ is divided by $x - c$, then the remainder is $f(c)$.

A consequence of the remainder theorem is that if $x - c$ divides evenly into $f(x)$, the remainder is 0. (i.e. $c$ is a zero of the function.)

Ex: Use synthetic division and the remainder theorem to find the indicated function value.

$$f(x) = x^4 - 5x^3 + 5x^2 + 5x - 6; \quad f(2)$$

\[
\begin{array}{c|ccccc}
2 & 1 & -5 & 5 & 5 & -6 \\
 & 2 & -6 & -2 & 6 \\
\hline
1 & -3 & -1 & 3 & 0
\end{array}
\]

$f(2) = 0$
Solve the equation $2x^3 - 3x^2 - 29x + 60 = 0$, given that 3 is a zero of $f(x) = 2x^3 - 3x^2 - 29x + 60$.

\[
\begin{array}{c|cccc}
3 & 2 & -3 & -29 & 60 \\
& & 6 & 9 & -60 \\
\hline
2 & 3 & -20 & 0 \\
\end{array}
\]

$2x^2 + 3x - 20 = 0$

\[x = \frac{-3 \pm \sqrt{9 - 4(2)(-20)}}{2 \cdot 2} \]

\[= \frac{-3 \pm \sqrt{169}}{4} = \frac{-3 \pm 13}{4} \]

$3, \frac{5}{2}, -4$
Use the graph of $f(x) = x^3 + 4x^2 - 25x - 28$ below to determine a solution of $x^3 + 4x^2 - 25x - 28 = 0$. Use synthetic division to verify this solution. Then solve the polynomial equation algebraically.

\[
\begin{array}{c|cccc}
4 & 1 & 4 & -25 & -28 \\
 & 4 & 32 & 28 \\
\hline
1 & 8 & 7 & 0 \\
\end{array}
\]

$x^2 + 8x + 7 = 0$

$(x + 7)(x + 1) = 0$

$x = -7, x = -1$

\[
\begin{array}{c|cccc}
-1 & 1 & 8 & 7 \\
 & -1 & -7 \\
\hline
1 & 7 & 0 \\
\end{array}
\]

$x + 7 = 0$

$x = -7$