2.1 Complex Numbers

Here's another math problem. I can't figure out. What's $9 + 4$?

Ooh, that's a tricky one. You have to use calculus and imaginary numbers for this.

You know, eleven, eighteen, thirty-two, and all those. It's a little confusing at first.

How did you learn all this? You've never even gone to school!

Instinct. Tigers are born with it.
The imaginary number \( i \) is \( \sqrt{-1} \). We know that -1 does not have a real square root, since no real number can be squared to give a negative result. So whenever we have an expression involving the square root of a negative, that is an imaginary number.

Find each of the following:

\[
\begin{align*}
\sqrt{-25} & = 5i \\
\sqrt{-1.25} & = 5i \\
\sqrt{-36} & = 6i \\
\sqrt{-27} & = 3\sqrt{3} \\
\sqrt{-7} & = i\sqrt{7} \\
\sqrt{-1.93} & = i\sqrt{7}
\end{align*}
\]
A complex number is a number of the form $a + bi$, where $a$ and $b$ are real numbers, and $i$ is the imaginary unit $\sqrt{-1}$.

To add or subtract complex numbers, simply combine the real parts and combine the imaginary parts.

Ex: Find each of the following:

$$(5 + 8i) + (3 - 2i)$$

$$= 8 + 6i$$

$$(7 - i) - (-3 - 4i)$$

$$= 10 + 3i$$

$$(6 + \sqrt{-49}) + (-1 - \sqrt{-16}) - (8 + \sqrt{-1})$$

$$= -3 + 2i$$
To multiply two complex numbers, just use FOIL (or distributive property). Note that \( i^2 = (\sqrt{-1})^2 = -1 \).

Ex: Find each of the following:

\[(5 + 8i)(3 - 2i) \quad (6 - i)(4 + 5i)\]

\[15 - 10i + 24i - 16i^2 \quad 24 + 30i - 4i - 5i^2\]

\[15 + 14i - 16(-1) \quad -29 + 26i + 5\]

\[31 + 14i \]

\[(1 + 8i)^2 = (1 + 8i)(1 + 8i) \quad (3 - \sqrt{25})(4 + \sqrt{-9})\]

\[1 + 8i + 8i + 64i^2 \quad (3 - 5i)(4 + 3i)\]

\[= -63 + 16i \quad 12 + 9i - 20i - 15i^2\]

\[= -63 + 16i \quad 27 - 11i + 15\]

\[27 - 11i\]

Two complex numbers with the same real part and opposite imaginary parts are called conjugates \((a + bi)(a - bi)\).

Ex: \((7 + 2i)\) and \((7 - 2i)\) \quad \((-5 - i)\) and \((-5 + i)\)

\[(a + bi)(a - bi) = \]

\[a^2 + b^2\]

Conjugate of \(25 + 6i\) is \(25 - 6i\)

Conjugate of \(-33 - 4i\) is \(-33 + 4i\)
To divide one complex number by another, multiply the numerator and the denominator by the conjugate of the denominator, using FOIL. This will create a real number in the denominator, which becomes the denominator of both the real and the imaginary parts of the complex number.

\[
\frac{12 + 2i}{4 - i} \div \frac{4 + i}{4 + i} = \frac{4i}{1 + 3i} \\
\frac{48 + 12i + 8i + 2i^2}{17} = \frac{46 + 20i}{17} = \frac{12}{17} + \frac{4i}{17} = \frac{2}{5} + \frac{2}{5}i
\]

Perform the indicated operations and express the result in standard form \((a + bi)\)

\[
\sqrt{-12} = \sqrt{-4 \cdot 3} = 2i\sqrt{3} \left(2i - i\sqrt{2}\right) = 4i^2\sqrt{3} - 2i^2\sqrt{6} = -4\sqrt{3} + 2\sqrt{6}
\]

\[
\frac{-5 - \sqrt{-9}}{-81} = \frac{-5 - 3i}{9i} \cdot \frac{-9i}{-9i} = \frac{45i + 27i^2}{81} = \frac{-27 + 45i}{81} = \frac{-1}{3} + \frac{5}{9}i
\]