2.5: Zeros of Polynomial Functions

You should know that the solutions of \( ax^2 + bx + c = 0 \) are \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \). (The quadratic formula)

There is an analogous formula for polynomials of degree three: The solutions of \( ax^3 + bx^2 + cx + d = 0 \) are

\[
x = \sqrt[3]{\frac{-b^3 + 3bc - d}{27a^3} + \left(\frac{b^2c - d^2 - 2c^2}{9a^2}\right)^{\frac{1}{3}}} + \sqrt[3]{\frac{-b^3 + 3bc - d}{27a^3} - \left(\frac{b^2c - d^2 - 2c^2}{9a^2}\right)^{\frac{1}{3}}} + \frac{c}{3a}.
\]

This is the cubic formula. A formula like this was first published by Cardano in 1545.

The solutions of \( ax^4 + bx^3 + cx^2 + dx + e = 0 \) are

\[
x = \frac{b}{4a} - \frac{1}{2} \pm \sqrt{\left(\frac{b^2}{4a^2} - \frac{2c}{3a} + \frac{2^{1/3} (c^2 - 3bd + 12ae)}{3a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{3a} + \frac{2^{1/3} (c^2 - 3bd + 12ae)}{3a}\right)^2}
\]

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There are several strategies (besides using the cubic and quartic formulas) to determine the zeros of a polynomial function. The first is to determine the possible rational zeros using the Rational Zero Theorem.

**The Rational Zero Theorem:** If \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) has integer coefficients and \( \frac{p}{q} \) is a rational zero of \( f \), then \( p \) is a factor of the constant term, \( a_0 \), and \( q \) is a factor of the leading coefficient, \( a_n \).

Ex: Use the rational zero theorem to list all possible rational zeros of each function below.

1) \( f(x) = 2x^3 + 7x^2 - 32x + 5 \) 
2) \( f(x) = 3x^3 + 7x^2 - 12x - 28 \)

Now determine which of the possible rational zeros are actual zeros. This can be done using synthetic division, or, more quickly, by using the table in the calculator.
Suppose we're working with a cubic polynomial. Once we have determined one rational zero, we can use synthetic division to reduce the cubic polynomial to a quadratic. Then we can use the quadratic formula to finish solving the equation. At this point, it doesn't matter whether the zeros are rational or even real.

Ex: List all possible rational zeros of the function below. Then determine which ones are actual zeros. Finally, find the remaining zeros.

\[ f(x) = 5x^3 - 15x^2 + 2x - 6 \]

Ex: List all possible rational roots of the equation below. Then determine which ones are actual roots. Finally, find the remaining roots, thereby solving the equation.

\[ x^4 - 2x^2 - 16x - 15 = 0 \]
Properties of Roots of Polynomial Equations:

1. If a polynomial equation is of degree \( n \), then counting multiple roots separately, the equation has \( n \) roots.

2. If \( a + bi \) is a root of a polynomial equation with real coefficients \((b \neq 0)\), then \( a - bi \) is also a root. Imaginary roots, if they exist, occur in conjugate pairs.

The Fundamental Theorem of Algebra:

If \( f(x) \) is a polynomial of degree \( n \), where \( n \geq 1 \), then the equation \( f(x) = 0 \) has at least one complex root. Moreover, if roots of multiplicity "m" are counted \( m \) times, the polynomial has precisely \( n \) roots.

Ex: Find an \( n \)th degree polynomial function with real coefficients which satisfies the given conditions.

\[ n = 3; \ 4 \text{ and } 2i \text{ are zeros;} \ f(-1) = -50 \]

Ex: Find an \( n \)th degree polynomial function with real coefficients which satisfies the given conditions.

\[ n = 4; \ -3, \ 2, \ \text{and } 2 + 3i \text{ are zeros;} \ f(1) = 100 \]
Another helpful test that tells us about the number and sign of the real zeros of a function is Descartes’ Rule of Signs.

**Descartes’ Rule of Signs:**

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with real coefficients.

1. The number of **positive real zeros** of $f$ is either
   a) the same as the number of sign changes of $f(x)$
   or
   b) less than the number of sign changes of $f(x)$ by a positive even integer. If $f(x)$ has only one variation in sign, then $f$ has exactly one positive real zero.

2. The number of **negative real zeros** of $f$ is either
   a) the same as the number of sign changes of $f(-x)$
   or
   b) less than the number of sign changes of $f(-x)$ by a positive even integer. If $f(-x)$ has only one variation in sign, then $f$ has exactly one positive real zero.

**Ex:** Use Descartes’ Rule of Signs to determine the number of positive and negative real zeros of $f(x) = 5x^3 - 15x^2 + 2x - 6$.
Ex: Use Descartes’ Rule of Signs to determine the number of positive and negative real zeros of \( f(x) = 2x^4 - 5x^3 - x^2 - 6x + 4 \).

Ex: Find all zeros of the given polynomial function or solve the given polynomial equation. Use the Rational Zero Theorem, Descartes’ Rule of Signs, and synthetic division.

\[ f(x) = x^3 + 12x^2 + 21x + 10 \]

\[ 3x^4 - 11x^3 - 3x^2 - 6x + 8 = 0 \]