1.8 Inverse Functions

Two functions which "undo" each other are said to be inverse functions.

When a function is composed with its inverse, we get back the original input.

\[ (f \circ f^{-1})(x) = x, \text{ where } f^{-1}(x) \text{ is the inverse of } f(x). \]

Also, \( (f^{-1} \circ f)(x) = x \).

Ex: Consider \( f(x) = x - 3 \) and \( g(x) = x + 3 \).

Consider \( f(x) = \frac{x}{7} \) and \( g(x) = 7x \).

Ex: Use composition to determine whether or not \( f(x) \) and \( g(x) \) are inverses.

\[ f(x) = (5x - 6)^3; \quad g(x) = \frac{\sqrt[3]{x} + 6}{5} \]

\[ f(x) = \frac{3}{2} x + 4; \quad g(x) = \frac{3x - 4}{2} \]
Just as some actions cannot be undone, (e.g. once you cook an egg, you can't uncook it), not every function has an inverse function. In order for a function to have an inverse, the function must be one-to-one, which means that every output (y-value) must be paired with exactly one input (x-value). The easiest way to check this is by using the horizontal line test.

Determine whether or not each function below is one-to-one.

\[ f(x) = |x + 2| \quad f'(x) = 3x - 5 \quad f(x) = \sqrt[4]{x + 1} \]
If a function is one-to-one, we can find its inverse by doing the following steps:

1) Write \( f(x) \) as \( y \) and switch \( x \) and \( y \) in the equation.
2) Solve for \( y \).
3) Write \( y \) as \( f^{-1}(x) \).

**Ex:** Given a one-to-one function \( f(x) \), find \( f^{-1}(x) \).

\[
f(x) = \sqrt[3]{8x - 3}
\]

\[
f(x) = 2x + 7
\]
The domain of a function is the range of its inverse, and vice-versa.

Find the inverse function for each of the functions below, and then state the domain and range of \( f(x) \) and \( f^{-1}(x) \).

\[
f(x) = \frac{3x - 7}{2x + 5}
\]

\[
f(x) = \sqrt{2x - 8}
\]
Some functions are not one-to-one, but if we restrict their domains, we can force them to be one-to-one and hence have an inverse function. A good example of this is a quadratic function.

\[ f(x) = x^2 - 4, \ x \geq 0 \]
The graph of a function and the graph of its inverse are mirror images of each other across the line $y = x$.

Find the inverse of each function $f(x)$ below and then graph both $f(x)$ and $f^{-1}(x)$ on the same grid.

\[ f(x) = (x + 2)^3 \quad \quad f(x) = 2x - 4 \]
\[ f(x) = \sqrt{x - 3} + 4 \]