5.4 Exponential and Logarithmic Equations

OBJECTIVE 1 Solving Exponential Equations

We have already solved exponential equations using the method of relating the bases. For example, we can solve \(4^{x+3} = \frac{1}{2}\) using that method.

\[
\begin{align*}
(2^2)^{x+3} &= 2^{-1} \\
2^{2(x+3)} &= 2^{-1} \\
2^{2x+6} &= 2^{-1} \rightarrow 2x + 6 &= -1 \\
\frac{2x}{2} &= -\frac{7}{2} \\
x &= -\frac{7}{2}
\end{align*}
\]

But suppose we are given an exponential equation in which the bases cannot be related, such as \(2^{x+1} = 3\). We can solve this equation using logarithms.
To solve an exponential equation, do the following steps:
1) Isolate the exponential expression \( b^x \).
2) Take the \( \log \) or \( \ln \) of both sides and use the power property to bring the exponent to the front.
3) Solve the resulting equation.

Examples:

a) \( 2^x - 7 = 4 \)
\[
\ln 2^x = \ln 11
\]
\[
\frac{x \ln 2}{\ln 2} = \frac{\ln 11}{\ln 2}
\]
\[
x = \frac{\ln 11}{\ln 2} \approx 3.459
\]

b) \( 5e^{2x} = 20 \)
\[
\frac{\ln 11}{\ln 2}
\]
\[
\frac{\ln e^{2x}}{\ln 2} = \ln 4
\]
\[
\frac{2x}{\ln 2} = \ln 4
\]
\[
x = \frac{\ln 4}{2} \approx 0.693
\]

c) \( 2^{x+1} = 3 \)
\[
\ln 2^{x+1} = \ln 3
\]
\[
\frac{(x+1) \ln 2}{\ln 2} = \frac{\ln 3}{\ln 2}
\]
\[
x + 1 = \frac{\ln 3}{\ln 2}
\]
\[
x = \frac{\ln 3}{\ln 2} - 1 \approx 0.585
\] **exact answer**
\[
nm{decimal approx.}
\]

d) \( 7^{x+3} = 4^{2-x} \)
\[
\ln 7^{x+3} = \ln 4^{2-x}
\]
\[
(x+3) \ln 7 = (2-x) \ln 4
\]
\[
x \ln 7 + 3 \ln 7 = 2 \ln 4 - x \ln 4
\]
\[
+x \ln 4 - 3x \ln 7 - 3 \ln 7 + x \ln 4
\]
\[
x \ln 7 + x \ln 4 = 2 \ln 4 - 3 \ln 7
\]
\[
x \left( \ln 7 + \ln 4 \right) = (2 \ln 4 - 3 \ln 7)
\]
\[
\frac{\ln 7 + \ln 4}{\ln 7 + \ln 4}
\]
\[
x \approx -0.920
\]
Solve each equation. Round to four decimal places.

a. \( \frac{26e^{x-5}}{2^{\frac{3}{4}}} = \frac{17}{2^{\frac{3}{5}}} \)

\[
\ln e^{x-5} = \ln \left( \frac{17}{2^{\frac{3}{5}}} \right)
\]

\[
x - \frac{5}{4} = \ln \left( \frac{17}{2^{\frac{3}{5}}} \right) + \frac{5}{4}
\]

\[
x \approx 4.614
\]

d. \( e^{2x-1} \cdot e^{x+4} = 11 \)

\[
e^{2x-1+x+4} = 11
\]

\[
\ln e^{3x+3} = \ln 11
\]

\[
\frac{3x+3}{3} = \ln 11
\]

\[
x \approx -0.201
\]

c. \( 4x^2 - 2x = 64 \)

\[
\ln 4 \cdot (x^2 - 2x) = \ln 64
\]

\[
\frac{(x^2 - 2x) \ln 4}{\ln 4} = \ln 64
\]

\[
x^2 - 2x = \frac{\ln 64}{\ln 4} = 3
\]

\[
x^2 - 2x - 3 = 0
\]

\[
(x - 3)(x + 1) = 0
\]

\[
x = 3, \ x = -1
\]

d. \( 8e^{-\frac{2}{3}x} \cdot e^x = 1 \)

\[
8e^{-\frac{2}{3}x + x} = 1
\]

\[
8e^{-\frac{2}{3}x + \frac{1}{3}x} = 1
\]

\[
8e^{-\frac{1}{3}x} = 1
\]

\[
8e^{-\frac{1}{3}x} = 1
\]

\[
\frac{8e}{8} \cdot e^{\frac{2}{3}x} = 1
\]

\[
\frac{8e}{8} \cdot e^{\frac{2}{3}x} = 1
\]

\[
\ln e^{\frac{2}{3}x} = \ln \frac{8}{8}
\]

\[
\frac{2}{3}x = \frac{3}{2} \ln \frac{8}{8}
\]

\[
x \approx -3.119
\]
OBJECTIVE 2 Solving Logarithmic Equations

Solve a Logarithmic Equation Using the Logarithm Property of Equality

\[ \log_5(x - 1) = \log_5 64 \]

\[ (x - 1)^2 = 64 \]

\[ (x - 1) = \pm 8 \]

\[ x - 1 = 8 \quad x - 1 = -8 \]

\[ x = 9 \quad x = -7 \]

\[ \frac{5x - 3}{-2} = \frac{2x + 7}{-2x} \]

\[ 3x - 3 = 7 \quad \frac{3x}{3} = 10 \quad x = \frac{10}{3} \]

\[ \log_5(9 - 1) = \log_5 64 \]

\[ 2 \log_5(8) = \log_5 64 \]

\[ \log_5 8^2 = \log_5 64 \]
We have already seen that to solve a logarithmic equation, we can **convert it to exponential form**. If an equation involves more than one logarithm, we can use the product or quotient property to condense it into one logarithm and then convert it to exponential form. Before converting to exponential form, be sure the logarithm is isolated! Also, solutions may be extraneous, so they must be checked in the original equation.

**Examples:**

\[
\log_3(2x - 1) = 2
\]

\[
4 = 2x - 1
\]
\[
16 = 2x - 1
\]
\[
\frac{17}{2} = \frac{8x}{2}
\]
\[
x = \frac{17}{2}
\]

\[
\log_4\left(2 \cdot \frac{17}{2} - 1\right) = 2
\]
\[
\log_4(16) = 2
\]

\[
\log_3(5x + 1) = 1
\]
\[
3 = 5x + 1
\]
\[
\frac{3}{5} = 5x + 1
\]
\[
-\frac{2}{5} = \frac{5x}{5}
\]
\[
x = -\frac{2}{5}
\]

\[
\ln(3x + 7) = 0
\]
\[
e^0 = 3x + 7
\]
\[
1 = 3x + 7
\]
\[
-7 = 3x
\]
\[
-x = \frac{3x}{3}
\]
\[
-2 = x
\]
Solve \( \log_2 (x + 10) + \log_2 (x + 6) = 5 \).

\[
\log_2 \left( (x+10)(x+6) \right) = 5
\]

\[
2^5 = x^2 + 16x + 60 \quad \rightarrow \quad \frac{32}{3} = x^2 + 16x + 60
\]

\[
x^2 + 16x + 28 = 0 \quad \rightarrow \quad (x+14)(x+2) = 0 \quad \rightarrow \quad x = -14, x = -2
\]

\[
\log_2 (8) + \log_2 (4) = 5
\]

\[
3 + 2 = 5
\]

Solve \( \ln (x - 4) - \ln (x - 5) = 2 \). Round to four decimal places.

\[
\ln \left( \frac{x-4}{x-5} \right) = 2
\]

\[
(x-3)e^2 = \frac{(x-4)(x-5)}{x-5}
\]

\[
\frac{e^2 x - 5e^2}{-x + 2e^2} = x - 4
\]

\[
e^2 x - x = -4 + 5e^2
\]

\[
x \left( \frac{e^2 + 1}{e^2 - 1} \right) = -4 + 5e^2
\]

\[
x \approx 5.1565
Solve each logarithmic equation.

\[ 2 \ln x - \ln(2x - 3) = \ln(2x) - \ln(x - 1) \]

\[ \log_3(2x - 5) = -2 \quad x = \frac{23}{9} \]

\[ \log_3(3x + 1) - \log_3(x - 2) = 2 \]

\[ \log_6(x - 8) = 2 - \log_6(x + 8) \]

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\[ x^3 - 5x^2 + 6x = 0 \]
\[ x(x^2 - 5x + 6) = 0 \]
\[ x(x-3)(x-2) = 0 \]
\[ x = 0, \quad x = 3, \quad x = 2 \]