### 4.3 The Graphs of Polynomial Functions

**Definition: Polynomial Function**

The function \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) is a polynomial function of degree \( n \), where \( n \) is a nonnegative integer. The numbers \( a_0, a_1, \ldots, a_n \) are called the coefficients of the polynomial function. The number \( a_n \) is called the leading coefficient, and \( a_0 \) is called the constant coefficient.

Determine which functions are polynomial functions. If the function is a polynomial function, identify the degree, the leading coefficient, and the constant coefficient.

\[
\begin{align*}
\text{a. } f(x) &= \sqrt[3]{x^3} - 2x^2 - \frac{1}{6} \\
&= x^{\frac{1}{3}} - 2x^2 - \frac{1}{6} \quad \text{degree } 3; \text{ leading coeff.: } \sqrt{3} \text{, constant } -\frac{1}{6} \\
\text{b. } g(x) &= 4x^3 - 3x^3 + x^2 + \frac{7}{9} \\
&= x^3 + x^2 + \frac{7}{9} \quad \text{No} \\
\text{c. } h(x) &= \frac{3x}{9} - \frac{x^2}{9} + \frac{7x^4}{9} \\
&= \frac{1}{3}x - \frac{1}{9}x^2 + \frac{7}{9}x^4 \\
&= \frac{1}{3}x - \frac{1}{9}x^2 + \frac{7}{9}x^4 \quad \text{Yes; degree } 4, \text{ leading coeff.: } \frac{7}{9} \text{, constant: } 0
\end{align*}
\]

**Objective 2: Sketching the Graphs of Power Functions**

The most basic polynomial functions are monomial functions of the form \( f(x) = ax^n \). Polynomial functions of this form are called power functions. Figure 11 shows the graphs of power functions for \( a = 1 \) and for \( n = 1, 2, 3, 4, \) and \( 5 \).

![Graphs of f(x) = x^n, where n = 1, 2, 3, 4, and 5](Figure 11)
Use Transformations to Sketch Polynomial Functions

a. \( f(x) = -x^4 \)

b. \( f(x) = (x + 1)^3 + 2 \) vs. \( f(x) = x^5 \)

\[ \text{Shifts left 1 and up 2} \]

\[ \frac{y}{x} \]

c. \( f(x) = 2(x - 3)^4 \) vs. \( y = x^4 \)

\[ \text{Vertical stretched \hspace{0.5cm} and shifted right 3} \]

OBJECTIVE 3 Determining the End Behavior of Polynomial Functions

The right-hand ends of both graphs approach infinity as the \( x \)-values approach infinity.

The left-hand end of both graphs approach negative infinity as the \( x \)-values approach negative infinity.

(a) \( f(x) = -2x^4 - 6x^3 + 8x^2 + 5x - 6 \)

Figure 13

(b) \( y = x^5 \)

To determine the end behavior of a polynomial function \( f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0 \) we look at the leading term \( a_nx^n \) and follow a two-step process.
Two-Step Process for Determining the End Behavior of a Polynomial Function \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \)

**Step 1.** Determine the sign of the leading coefficient \( a_n \).

- If \( a_n > 0 \), the right-hand behavior "finishes up."
- If \( a_n < 0 \), the right-hand behavior "finishes down."

**Step 2.** Determine the degree.

- If the degree \( n \) is odd, the graph has opposite left-hand and right-hand end behavior; that is, the graph "starts" and "finishes" in opposite directions.

  - \( a_n > 0 \), odd degree
  - \( a_n < 0 \), odd degree

- If the degree \( n \) is even, the graph has the same left-hand and right-hand end behavior; that is, the graph "starts" and "finishes" in the same direction.

  - \( a_n > 0 \), even degree
  - \( a_n < 0 \), even degree

Use the end behavior of each graph to determine whether the degree is even or odd and whether the leading coefficient is positive or negative.

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\[ y = -x^3 \]

\[ y = x^4 \]

\[ y = -x^2 \]

\[ y = x^2 \]

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\[ y = x^{10} + 3x^9 - 8x^5 + 1000 \]

**Even L.C. +**

**Odd L.C. neg**

Leading coeff. neg
Now that we know how to determine the end behavior of a polynomial function, it is time to find out what happens between the ends. We start by trying to locate the intercepts of the graph. Every polynomial function, \( y = f(x) \), has a \( y \)-intercept that can be found by evaluating \( f(0) \). Locating the \( x \)-intercepts is not that easy of a task. Recall that the number \( x = c \) is called a zero of a function \( f \) if \( f(c) = 0 \). If \( c \) is a real number, then \( c \) is an \( x \)-intercept (see Section 3.2). Therefore, to find the \( x \)-intercepts of a polynomial function \( y = f(x) \), we must find the real solutions of the equation \( f(x) = 0 \).

Find the intercepts of the polynomial function \( f(x) = x^3 - x^2 - 4x + 4 \).

\[ \text{y-inter: } 4 \]
\[ 0 = \frac{x^3}{x^2} - \frac{x^2}{x^2} - 4 \frac{x}{x} + 4 \]
\[ x^2(x-1) - 4(x-1) \]
\[ 0 = (x-1)(x^2-4) \rightarrow x = 1, \ x = 2, \ x = -2 \]

Find the intercepts of the polynomial function \( f(x) = (x^2 - 9)(2-x) \).

\[ \text{y-inter: } -18 \]
\[ f(0) = (-9)(2) = -18 \]
\[ (x-3)(x+3) \rightarrow x^2 - 9 = 0 \]
\[ 2 - x = 0 \]
\[ 2 = x \]
\[ x = \pm 3 \]

\[ \text{inter: } 3, -3, 2 \]

**OBJECTIVE 3**

Determining the Real Zeros of Polynomial Functions and Their Multiplicities (how many times the zero occurs)

**Figure 15**

**Shape of the Graph of a Polynomial Function Near a Zero of Multiplicity \( k \)**

Suppose \( c \) is a real zero of a polynomial function \( f \) of multiplicity \( k \) (where \( k \) is a positive integer), that is, \((x - c)^k \) is a factor of \( f \). Then the shape of the graph of \( f \) near \( c \) is as follows:

If \( k \) is even, then the graph touches the \( x \)-axis at \( c \).

If \( k \) is odd, then the graph crosses the \( x \)-axis at \( c \).
Find all real zeros of \( f(x) = x(x^2 - 1)(x - 1) \). Determine the multiplicities of each zero, and decide whether the graph touches or crosses at each zero.

\[
\begin{align*}
0 &= x(x^2 - 1)(x - 1) \\
0 &= x(x - 1)(x + 1)(x - 1) \\
x &= 0 & \text{mult. 1} & \text{cross} \\
x &= 1 & \text{mult. 2} & \text{touch} \\
x &= -1 & \text{mult. 1} & \text{cross} \\
x &= 1
\end{align*}
\]

Find all real zeros of \( f(x) = x(x - 3)(x - 4)^4 \). Determine the multiplicities of each zero, and decide whether the graph touches or crosses at each zero.

\[
\begin{align*}
0 &= x(x - 3)(x - 4)^4 \\
x &= 0 & \text{mult. 1} & \text{cross} \\
x &= 3 & \text{mult. 1} & \text{cross} \\
x &= 4 & \text{mult. 4} & \text{touch}
\end{align*}
\]

**Objective:** Sketching the Graph of a Polynomial Function

**Four-Step Process for Sketching the Graph of a Polynomial Function**

1. **Step 1.** Determine the end behavior.
2. **Step 2.** Plot the y-intercept \( f(0) = a_0 \).
3. **Step 3.** Completely factor \( f \) to find all real zeros and their multiplicities.
4. **Step 4.** Choose a test value between each real zero and sketch the graph.

Use the four-step process to sketch the graphs of the following polynomial functions:

**a.** \( f(x) = -2(x + 2)^2(x - 1) = -2x^3 + \ldots \)

- Down on right up on left
- \( y \)-int: \(-2(0+2)^2(0-1) = -2(4)(-1) = 8 \)
- \( x \)-int: \( 0 = -2(x+2)^2(x-1) \)
- \( x = -2 \) \( \text{mult. 2} \)
- \( x = 1 \) \( \text{mult. 1} \)

**b.** \( f(x) = x^3 - 2x^2 - 3x^2 \)

- Up on right up on left
- \( y \)-int: \( 0 \)
- \( x \)-int: \( O = x^4 - 2x^3 - 3x^2 \)
- \( x = 0, x = 3, x = -1 \)
- \( \text{mult. 2} \) \( \text{mult. 1} \) \( \text{mult. 1} \)
Analyze the graph to address the following about the polynomial function it represents.

a. Is the degree of the polynomial function even or odd? **Odd** (end behavior is opposite)

b. Is the leading coefficient positive or negative? **Down on right**

c. What is the value of the constant coefficient? **2 (y-intercept)**

d. Identify the real zeros, and state the multiplicity of each.

\[ y = a(x + 3)(x + 2)(x - 1)^2(x - 4) \]