4.1 Quadratic Functions

**Definition** Quadratic Function

A quadratic function is a function of the form \( f(x) = ax^2 + bx + c \), where \( a, b, \) and \( c \) are real numbers with \( a \neq 0 \). Every quadratic function has a “u-shaped” graph called a **parabola**.

![Graphs of quadratic functions](image)

Figure 1
Graphs of \( f(x) = x^2 \) and \( g(x) = -x^2 \)

A parabola either opens up or opens down, depending on the leading coefficient. If \( a > 0 \), as in Figure 1a, the parabola will “open up.” If \( a < 0 \), as in Figure 1b, the parabola will “open down.”

Without graphing, determine whether the graph of each quadratic function opens up or down.

a) \( f(x) = -3x^2 + 6x + 1 \)  
   \[ a = -3 \]  
   **down**

b) \( f(x) = x^2 - x - 12 \)  
   \[ a = 1 \]  
   **up**

c) \( f(x) = 6 + x^2 \)  
   \[ a = -1 \]  
   **down**
Characteristics of a Parabola

1. Vertex - lowest or highest point on the graph \((h, k)\)
2. Axis of symmetry \(x = h\)
3. \(y\)-Intercept \(f(0) = c\)
4. \(x\)-Intercept(s) or real zeros \(2, 1, \text{ or } 0\)
5. Domain and range
   - \(D: (-\infty, \infty)\)
   - \(R: \text{ varies } [k, \infty)\) or \((-\infty, k]\)

Graph each quadratic function as a transformation of \(y = x^2\).

\(y = x^2\)

\(f(x) = -x^2 + 3\) reflects across x-axis and shifts up 3.

\(f(x) = -2(x + 3)^2 - 1\) vertically stretched, shifts left 3, down 1.

\(f(x) = 2(x + 1)^2\) vertically stretched, shifted left 1.
### Definition
**Standard Form of a Quadratic Function**

A quadratic function is in **standard form** if it is written as \( f(x) = a(x - h)^2 + k \).

- The graph is a parabola with vertex \((h, k)\). The parabola "opens up" if \( a > 0 \). The parabola "opens down" if \( a < 0 \).
- Domain: \((-\infty, \infty)\)
- Range: \([k, \infty)\) when \( a > 0 \) and \((-\infty, k]\) when \( a < 0 \).

### Graph a Quadratic Function That Is in Standard Form

Given that the quadratic function \( f(x) = -(x - 2)^2 - 4 \) is in standard form, address the following:

- **a.** What are the coordinates of the vertex? \((2, -4)\)  
  - **b.** Does the graph "open up" or "open down"?  
    - **Down** \( (a \text{ is neg}) \)
- **c.** What is the equation of the axis of symmetry? \( x = 2 \)
- **d.** Find any \( x \)-intercepts. None
- **e.** Find the \( y \)-intercept. 
  \[ f(0) = -(0 - 2)^2 - 4 = -(4) - 4 = -8 \]
- **f.** Sketch the graph.
- **g.** State the domain and range in interval notation.  
  \[ \text{D: } (-\infty, \infty) \]  
  \[ \text{R: } (-\infty, -4] \]
Graph a Quadratic Function That Is in Standard Form

Given that the quadratic function \( f(x) = (x + 3)^2 - 4 \) is in standard form, address the following:

a. What are the coordinates of the vertex?
b. Does the graph "open up" or "open down"?
c. What is the equation of the axis of symmetry?
d. Find any \( x \)-intercepts.
e. Find the \( y \)-intercept.
f. Sketch the graph.
g. State the domain and range in interval notation.

\[
D: (-\infty, \infty) \\
R: [-4, \infty)
\]

Graphing Quadratic Functions Using the Vertex Formula

Formula for the Vertex of a Parabola
Given a quadratic function of the form \( f(x) = ax^2 + bx + c \), \( a \neq 0 \), the vertex of the parabola is \( \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) \).

Graph a Quadratic Function Using the Vertex Formula and I
Given the quadratic function \( f(x) = -2x^2 - 4x + 5 \), address the:

a. Use the vertex formula to determine the vertex.
b. Does the graph "open up" or "open down"?
c. What is the equation of the axis of symmetry?
d. Find any \( x \)-intercepts.
e. Find the \( y \)-intercept.
f. Sketch the graph.
g. State the domain and range in interval notation.

\[
D: (-\infty, \infty), R: (-\infty, 7)
\]
Given the quadratic function \( f(x) = x^2 - 8x \), address the following:

- Use the vertex formula to determine the vertex.
- Does the graph "open up" or "open down"?
- What is the equation of the axis of symmetry?
- Find any \( x \)-intercepts.
- Find the \( y \)-intercept.
- Sketch the graph.
- State the domain and range in interval notation.

\[
f(4) = 4^2 - 8 \cdot 4 = 16 - 32 = -16 \quad \left( -\frac{b}{2a}, f\left(\frac{-b}{2a}\right) \right) = (4, -16)
\]

\[
a = 1, \quad b = -8, \quad c = 0
\]

\[
\frac{b}{2a} = \frac{8}{2} = 4
\]

\[
\text{up (a is pos.)}
\]

\[
x = 4
\]

\[
x = 0, \quad x = 8
\]

\[
0 = x^2 - 8x \quad \Rightarrow \quad 0 = x(x - 8)
\]

\[
d = 0
\]

\[
D: (-\infty, \infty) \quad R: [-16, \infty)
\]

Given the quadratic function \( f(x) = 2x^2 - 5x + 3 \), address the following:

- Use the vertex formula to determine the vertex.
- Does the graph "open up" or "open down"?
- What is the equation of the axis of symmetry?
- Find any \( x \)-intercepts.
- Find the \( y \)-intercept.
- Sketch the graph.
- State the domain and range in interval notation.

\[
f(\frac{5}{4}) = \frac{5^2}{4} - \frac{5 \cdot 5}{4} + 3 = 1.25
\]

\[
2x^2 - 5x + 3
\]

\[
\text{Ans} = \text{Frac} -\frac{1}{8}
\]

\[
\frac{5}{4} \cdot x
\]

\[
2x^2 - 5x + 3
\]

\[
\text{Frac} -\frac{1}{8}
\]

\[
x = \frac{5}{4}
\]

\[
x = 1, \quad x = \frac{3}{2}
\]

\[
0 = 2x^2 - 5x + 3
\]

\[
\frac{2x^2 - 2x - 3x + 3}{2x(x-1) - 3(x-1)} = 0 \quad (x-1)(2x-3)=0
\]

\[
x = 1, \quad x = \frac{3}{2}
\]

\[
D: (-\infty, \infty) \quad R: [-\frac{1}{8}, \infty)
\]
Determine the Equation of a Quadratic Function Given Its Graph

Analyze the graph to address the following about the quadratic function it represents.

a. Is the leading coefficient positive or negative?

b. What is the value of $h$? What is the value of $k$?

c. What is the value of the leading coefficient $a$?

d. Write the equation of the function in standard form $f(x) = a(x - h)^2 + k$.

$$f(x) = -1 \left( x - (-1) \right)^2 + 4$$

$$f(x) = -1 \left( x^2 + 2x + 1 \right) + 4$$

$$f(x) = -x^2 - 2x + 3$$

$$3 = a (0 - (-1))^2 + 4$$

$$3 = a (1)^2 + 4$$

$$-4 = a + 4$$

$$-1 = a$$

e. Write the equation of the function in the form $f(x) = ax^2 + bx + c$.

$$f(x) = -x^2 - 2x + 3$$

$$f(x) = -1 \left( x^2 + 2x + 1 \right) + 4$$

$$f(x) = -x^2 - 2x - 1 + 4$$

$$f(x) = -x^2 - 2x + 3$$

Analyze the graph to address the following about the quadratic function it represents.

a. Is the leading coefficient positive or negative? Graph opens up

b. What is the value of $h$? What is the value of $k$? Vertex: $(h, k)$

c. What is the value of the leading coefficient $a$?

d. Write the equation of the function in standard form $f(x) = a(x - h)^2 + k$.

$$f(x) = 3(x - 2)^2 + 3$$

$$f(x) = 3(x^2 - 4x + 4) + 3$$

$$f(x) = 3x^2 - 12x + 15$$

$$6 = a (1)^2 + 3$$

$$-6 = a + 3$$

$$3 = a$$