5.5 The Substitution Rule

Because of the Fundamental Theorem of Calculus, it is important to be able to find antiderivatives. But our antiderivative formulas don't allow us to handle certain types of integrals (such as those below), so we need to be able to write them in a way that we can handle. The substitution rule allows us to do just that.

The Substitution Rule  If \( u = g(x) \) is a differentiable function whose range is an interval \( I \) and \( f \) is continuous on \( I \), then

\[
\int f(g(x)) g'(x) \, dx = \int f(u) \, du
\]

Examples:

\[
\int 4x^3(6x^4 - 11)^6 \, dx
\]

\[
\int u^6 \, du = \frac{1}{7} u^7 + C = \frac{1}{7} (x^4-11)^7 + C
\]

\[
\int (5t+3)^{30} \, dt = \frac{1}{3} \int (5t+3)^{30} \, dt
\]

\[
\frac{1}{5} \int u^3 \, du = \frac{1}{155} (5t+3)^{31} + C
\]

\[
\int e^{x^2} \, dx
\]

\[
\int e^{u} \, du = \frac{1}{2} e^{u} + C
\]

\[
y = \frac{1}{2} e^{x^2} + C
\]

\[
y' = \frac{1}{2} e^{x^2} \cdot 2x = xe^{x^2}
\]

Recall the property from 5.2 that allows us to take a constant across the integral symbol. This is needed in the next example.
\[
\int \frac{\sin^2 \theta \cos \theta}{du} d\theta = \int u^2 du = \frac{1}{3} u^3 + C = \frac{1}{3} \sin^3 \theta + C
\]
\[
\frac{1}{5} \int \frac{u^{1/2}}{\sqrt{5u-3}} du = \frac{1}{5} \cdot \frac{2}{3} u^{3/2} + C = \frac{2}{15} (5u-3)^{3/2} + C
\]
\[
\int \frac{\sin(\ln x)}{x} dx = \int \sin u du = -\cos u + C = -\cos(\ln x) + C
\]
\[
\frac{1}{2} \int \frac{2x}{(x^2 + 1)^2} dx = \int \frac{1}{u^2} du = \frac{1}{2} \int u^{-2} du = \frac{1}{2} \left(-u^{-1}\right) + C = -\frac{1}{2u} + C
\]
\[
\frac{1}{3} \int 3x^2 \cos(x^3 + 5) dx = \frac{1}{3} \int \cos u du = \frac{1}{3} \sin u + C
\]
\[
\frac{1}{3} \sin (x^3 + 5) + C
\]
When evaluating definite integrals, there are two equally valid methods. One is to calculate the integral as previously shown, replace the $u$ with the function it substituted for, and then evaluate. The other is to use the following:

**THE SUBSTITUTION RULE FOR DEFINITE INTEGRALS**

If $g'$ is continuous on $[a, b]$ and $f$ is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$$

Examples:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(\theta)}{1 + \cos^2(\theta)} \, d\theta$$

$u = \sin(\theta) \rightarrow du = \cos(\theta) \, d\theta$

$g(\theta) = \sin(\theta)$

$g(-\frac{\pi}{2}) = -1$  $g(\frac{\pi}{2}) = 1$

$$\int_{-1}^{1} \frac{1}{1 + u^2} \, du$$

$u = \tan(\theta) \rightarrow du = \sec^2(\theta) \, d\theta$

$g(\theta) = \tan(\theta)$

$g(0) = 0$  $g(\frac{\pi}{4}) = 1$

$$\int_{0}^{\frac{\pi}{4}} \frac{1}{\sqrt{1 + u^2}} \, du$$

$u = \tan(\theta) \rightarrow du = \sec^2(\theta) \, d\theta$

$g(\theta) = \tan(\theta)$

$g(0) = 0$  $g(\frac{\pi}{4}) = 1$

$$\int_{0}^{\frac{\pi}{4}} \frac{1}{\sqrt{1 + u^2}} \, du = \ln(\sqrt{2} + 1)$$
Recall the formula for calculating the average value of a function:

\[ f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x) \, dx \]

Find the average value of each function below on the given interval.

\[ f(x) = \sin(4x); \quad [-\pi, \pi] \]

\[
\begin{align*}
\frac{f_{\text{ave}}}{4} & = \frac{1}{\pi - (-\pi)} \cdot \frac{1}{4} \int_{-\pi}^{\pi} \sin(4x) \, dx \\
& = \frac{1}{2\pi} \cdot \frac{1}{4} \int_{-\pi}^{4\pi} \sin u \, du \\
& = \frac{1}{8\pi} \left[ -\cos u \right]_{-\pi}^{4\pi} \\
& = \frac{1}{8\pi} \left[ -\cos(4\pi) - (-\cos(-4\pi)) \right] \\
& = \frac{1}{8\pi} \left[ -1 + 1 \right] \\
& = 0
\end{align*}
\]

\[ f(x) = xe^{-x^2}; \quad [0,5] \]

\[
\begin{align*}
\frac{f_{\text{ave}}}{5} & = \frac{1}{5} \cdot \frac{1}{2} \int_{-2}^{2} xe^{-x^2} \, dx \\
& = -\frac{1}{10} \int_{0}^{25} e^{-u} \, du = -\frac{1}{10} \int_{0}^{25} e^{-u} \, du \\
& = \frac{1}{10} \left[ e^u \right]_{0}^{25} \\
& = \frac{1}{10} e^0 - \frac{1}{10} e^{-25} \\
& = \frac{1}{10} - \frac{1}{10e^{25}}
\end{align*}
\]