2.6 Implicit Differentiation

So far all of the functions that we have studied have been described by expressing one variable explicitly in terms of another variable, e.g. \( y = 3x^2 \) or \( y = x \sin(x) \). Some functions, however, are defined implicitly by a relation between \( x \) and \( y \), such as the following:

\[
x^2 + y^2 = 25 \\
x^3 - y^3 = 4xy
\]

The first of these could be solved for \( y \) easily into two explicit functions, but the second one would pose some difficulties. So to differentiate an implicitly defined function, we take the derivative of both sides of the equation with respect to \( x \) and then solve the resulting equation for \( y' \) (aka \( \frac{dy}{dx} \)). Keep in mind that \( y \) is a function of \( x \), so when we take the derivative of a function that involves \( y \), we must use the chain rule!

Find \( \frac{dy}{dx} \) by implicit differentiation.

1) \( x^2 + y^2 = 25 \)

\[
\frac{dy}{dx} = \frac{-x}{y}
\]

Find the equation of the tangent to 1) at the point (3, 4).

\[
y - 4 = \frac{-3}{4}(x - 3)
\]

\[
y = \frac{-3}{4}x + \frac{25}{4}
\]
Find $\frac{dy}{dx}$ by implicit differentiation.

2) $x^3 - y^3 = 4xy$

$3x^2 - 3y^2 \frac{dy}{dx} = 4x \cdot \frac{dx}{dx} + y \cdot 4$

$$-4y + \frac{3y^2}{x} \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = -4x$$

$$\frac{3x^2 - 4y^3}{4x + 3y^2} = \frac{4x + 3y^2}{4x + 3y^2}$$

$\sqrt{xy} + x - y = 7$

$$\left(\frac{xy}{2}\right)^{\frac{1}{2}} + x - y = 7$$

$$\frac{-x}{2(xy)^{\frac{1}{2}}} \left( x \cdot \frac{dy}{dx} + y \cdot 1 \right) + 1 - \frac{dy}{dx} = 0$$

$$x \frac{dy}{dx} + \frac{y}{2(xy)^{\frac{1}{2}}} + 1 - \frac{dy}{dx} = 0$$

$$x \frac{dy}{dx} - \frac{dy}{dx} = \frac{-y}{2(xy)^{\frac{1}{2}}} - 1$$

$$\frac{dy}{dx} \left( \frac{x}{2(xy)^{\frac{1}{2}}} - 1 \right) = \frac{-y}{2(xy)^{\frac{1}{2}}} - 1$$

Simplify for 5 pts.

$$\frac{(-y}{2(xy)^{\frac{1}{2}}} - 1) 2(xy)^{\frac{1}{2}} = \frac{-y - 2(xy)^{\frac{1}{2}}}{x - 2(xy)^{\frac{1}{2}}}$$

$$\frac{x}{2(xy)^{\frac{1}{2}}} - 1$$

$$\frac{-y - 2\sqrt{xy}}{x - 2\sqrt{xy}}$$
Find $\frac{dy}{dx}$ (or $y'$) by implicit differentiation for each of the following.

\[
\cos(xy) = 1 + \sin y
\]

\[
-\sin(xy) \left( x \frac{dy}{dx} + y \right) = 0 + \cos y \cdot \frac{dy}{dx}
\]

\[
-\sin(xy) \frac{dy}{dx} - y \sin(xy) = \cos y \cdot \frac{dy}{dx} + x \sin(xy) \frac{dy}{dx}
\]

\[
\frac{-y \sin(xy)}{\cos y + x \sin(xy)} = \frac{dy}{dx} \left( \frac{\cos y + x \sin(xy)}{\cos y + x \sin(xy)} \right)
\]

Remember: $y = f(x)$

\[
y' = f'(x)
\]

\[
\frac{d}{dx} (\cos y) = -\sin y \frac{dy}{dx}
\]

2 \hspace{1ex} x^3 + x^2 y - xy^2 = 2

6x^2 + x^2 \frac{dy}{dx} + y \cdot 2x - (x \cdot 3y^2 \frac{dy}{dx} + y^3 \cdot 1) = 0

6x^2 - x^2 \frac{dy}{dx} + 2xy - 3x y^2 \frac{dy}{dx} - y^3 = 0

\[
\frac{dy}{dx} \left( \frac{x^2 - 3xy^2}{x^2 - 3xy^2} \right) = \frac{-6x^2 - 2xy + y^3}{x^2 - 3xy^2}
\]

\[
\frac{dy}{dx} = \frac{-6x^2 - 2xy + y^3}{x^2 - 3xy^2}
\]
Find the equation of the tangent to the given curve at the point (1, 2).

\[ x^2 + 2xy - y^2 + x = 2 \]

\[ \frac{d}{dx} \left( x^2 + 2xy - y^2 + x \right) = \frac{1}{0} \]

\[ \frac{dy}{dx} \left( \frac{2x - 2y}{2x - 2y} \right) = \frac{-2x - 2y - 1}{2x - 2y} \]

\[ \frac{dy}{dx} \Bigg|_{(1,2)} = \frac{-2(1) - 2(2) - 1}{2(1) - 2(2)} = \frac{-2 - 4 - 1}{2 - 4} = -\frac{7}{2} \]

**Tangent line:** \( y - 2 = -\frac{7}{2} (x - 1) \)

\( y - 2 = -\frac{7}{2} x + \frac{7}{2} \)

\[ y = \frac{7}{2} x - \frac{3}{2} \]
Find the equation of the tangent to the given curve at the point (0, -2).

\[ y^2(y^2 - 4) = x^2(x^2 - 5) \]

\[ y^4 - 4y^2 = x^4 - 5x^2 \]

\[ 4y^3 \frac{dy}{dx} - 8y \frac{dy}{dx} = 4x^3 - 10x \]

\[ m = 0 \]

\[ \left. \frac{dy}{dx} \right|_{(0, -2)} = \frac{0}{-32 + 16} = \frac{0}{-16} = 0 \]

\[ \frac{dy}{dx} \left( \frac{4y^3 - 8y}{4y^3 - 8y} \right) = \frac{4x^3 - 10x}{4y^3 - 8y} = \frac{12x(2x^2 - 5)}{2y - y^2 - 2} \]
Find $y''$ by implicit differentiation.

\[ 16x^4 + y^4 = 16 \]

\[ 64x^3 + 4y^3 \frac{dy}{dx} = 0 \]

\[ -64x^3 = \frac{4y^3}{\frac{dy}{dx}} \]

\[ \frac{dy}{dx} = \frac{-16x^3}{y^3} \]

\[ y'' = y^3 \left( -48x^2 \right) - \left( -16x^3 \right) \cdot 3y^2 \frac{dy}{dx} \]

\[ = \frac{y^3}{y^6} \left( -48x^2 y^3 + 48x^3 y^2 \frac{dy}{dx} \right) \]

\[ = \frac{-48x^2 y^6 - 768x^6 y^2}{y^9} \]

\[ = \frac{-48x^2 y^2 (y^4 + 16x^4)}{y^9} \]

\[ = \frac{-48x^2 (16)}{y^7} \]

\[ = \frac{-768x^2}{y^7} \]

Multiply numerator and denominator by $y^3$ to eliminate the complex fraction.

Simplify.

Factor out the common factor.

From the original problem, $y^4 + 16x^4 = 16$

Final answer!