5.2 The Definite Integral

We saw in section 5.1 that a limit of the form
\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \lim_{n \to \infty} [f(x_1^*) \Delta x + f(x_2^*) \Delta x + \cdots + f(x_n^*) \Delta x]
\]
arises when we compute an area or when we try to find the distance traveled by an object. It turns out that this can happen even when our function is not positive or even continuous, and the subintervals do not even have to be the same length! We use the notation \(\Delta x_i\) for the length of the \(i^{th}\) subinterval, and we use the notation \(f(x_i^*)\) for the function evaluated at the sample point \(x_i^*\).

A Riemann sum associated with a partition \(P\) and a function \(f\) is constructed by evaluating \(f\) at the sample points, multiplying by the lengths of the corresponding subintervals, and adding:
\[
\sum_{i=1}^{n} f(x_i^*) \Delta x_i = f(x_1^*) \Delta x_1 + f(x_2^*) \Delta x_2 + \cdots + f(x_n^*) \Delta x_n
\]

Note that since the function does not have to be positive, any rectangles below the \(x\)-axis would have negative values for \(f(x_i^*)\), and thus their areas would be subtracted rather than added! The figure below illustrates this concept.
As we discussed in section 5.1, the limit of this sum as $n$ approaches $\infty$ gives us the area under the curve, as long as the largest subinterval approaches 0. The result is called the **definite integral** of $f$ from $a$ to $b$.

2 **DEFINITION OF A DEFINITE INTEGRAL** If $f$ is a function defined on $[a, b]$, the definite integral of $f$ from $a$ to $b$ is the number

$$\int_{a}^{b} f(x) \, dx = \lim_{\max \Delta x \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

provided that this limit exists. If it does exist, we say that $f$ is **integrable** on $[a, b]$.

3 **THEOREM** If $f$ is continuous on $[a, b]$, or if $f$ has only a finite number of jump discontinuities, then $f$ is integrable on $[a, b]$; that is, the definite integral $\int_{a}^{b} f(x) \, dx$ exists.

Now, even though we do not have to use consistent intervals, it is more convenient if we do, so if we use a consistent $\Delta x$ on the interval $[a, b]$, then it is calculated as follows: $\Delta x = (b - a)/n$. If we also choose our sample points consistently, such as right endpoints of the rectangles, then definition 2 becomes much simpler and leads to the following:

2 **THEOREM** If $f$ is integrable on $[a, b]$, then

$$\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$$

where $\Delta x = \frac{b - a}{n}$ and $x_i = a + i \Delta x$. 

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Example: Evaluate the Riemann sum for \( f(x) = x^2 - 6x + 7 \), taking the sample points to be right endpoints and \( a = 0 \), \( b = 8 \), and \( n = 4 \). Note that we are finding an approximation for the area under the curve \( y = x^2 - 6x + 7 \) between \( x = 0 \) and \( x = 8 \), as shown in the graph below.
Example: Find the Riemann sum for \( f(x) = x + x^2, \ -2 \leq x \leq 0 \), if the partition points are -2, -1.5, -1, -0.5, 0, and the sample points are -2, -1.4, -1.2, -0.6.
The graph of a function $f$ is given. Estimate $\int_{0}^{10} f(x) \, dx$ using five subintervals with a) left endpoints, and b) midpoints.
We often choose the sample points to be the right endpoints because it is convenient for computing the limit. But if we are trying to get a better approximation for the area, it is usually preferable to choose $x_i^*$ to be the midpoint of the interval, which we denote by $\bar{x}_i$. If we use midpoints and a regular partition we get the following approximation.

**Midpoint Rule**

$$\int_{a}^{b} f(x) \, dx \approx \sum_{i=1}^{n} f(\bar{x}_i) \Delta x = \Delta x [f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

where $$\Delta x = \frac{b - a}{n}$$

and $$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

Example: Use the Midpoint Rule with $n = 5$ to approximate the following.

$$\int_{0}^{20} \frac{1}{x + 2} \, dx$$
There are several properties of the definite integral which we will use more in the next section, but they are introduced here:

\[
\int_a^b f(x) \, dx = -\int_a^b f(x) \, dx \quad \int_a^b f(x) \, dx = 0
\]

**PROPERTIES OF THE INTEGRAL** Suppose all the following integrals exist.

1. \(\int_a^b c \, dx = c(b - a)\), where \(c\) is any constant

2. \(\int_a^b [f(x) + g(x)] \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx\)

3. \(\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx\), where \(c\) is any constant

4. \(\int_a^b [f(x) - g(x)] \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx\)

5. \(\int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx\)

Example: Use property 5 to evaluate the integral below.

Given \(\int_7^5 f(x) \, dx = 5\) and \(\int_5^7 f(x) \, dx = 3\), find \(\int_5^7 f(x) \, dx\).
Example: The graph of \( f \) is shown. Evaluate each integral below by interpreting it in terms of areas.

\[
\begin{align*}
\int_0^2 f(x)\,dx \\
\int_2^4 f(x)\,dx \\
\int_4^6 f(x)\,dx \\
\int_6^9 f(x)\,dx \\
\int_0^9 f(x)\,dx \\
\int_9^9 f(x)\,dx
\end{align*}
\]
Example: Evaluate the integral below by interpreting it in terms of areas.

\[ \int_{-4}^{5} \left( \frac{1}{2} x + 1 \right) dx \]