4.1 Maximum and Minimum Values

Some of the most important applications of calculus are optimization problems, in which we are required to find the optimal (best) way of doing something, such as in the examples listed on p. 203 of your book. All of these problems involve finding a maximum or minimum value of a function. You can see where the max and min are on the graph in figure 1, and a formal definition follows.

An absolute maximum or minimum can also be called a global max or min. Both of these are called extreme values of $f$. In addition to having an absolute maximum or minimum on a graph, you can also have a local maximum or minimum, as seen in figure 2.

![FIGURE 1](image1.png)

**1 DEFINITION** Let $c$ be a number in the domain $D$ of a function $f$. Then $f(c)$ is the
- **absolute maximum** value of $f$ on $D$ if $f(c) \geq f(x)$ for all $x$ in $D$.
- **absolute minimum** value of $f$ on $D$ if $f(c) \leq f(x)$ for all $x$ in $D$.

![FIGURE 2](image2.png)

**2 DEFINITION** The number $f(c)$ is a
- **local maximum** value of $f$ if $f(c) \geq f(x)$ when $x$ is near $c$.
- **local minimum** value of $f$ if $f(c) \leq f(x)$ when $x$ is near $c$.

The phrase "near $c" means on an open interval containing $c$.

Note that $f(d)$ is both an absolute max and a local max, but $f(a)$ is only an absolute min, NOT a local min. An endpoint of a graph cannot be a local extremum, since there is no open interval around it to compare it with.
Use the graphs below to determine the absolute and local maximum and minimum values of each function.

Sketch the graph of a function \( f \) that is continuous on \([0, 6]\) and has the given properties.

Absolute minimum at 1, absolute maximum at 5, local maximum at 2, local minimum at 4.
Consider the functions

a) \( y = \sin x \)  
\[ \text{Graph of } y = \sin x \]

b) \( y = -x^2 \)  
\[ \text{Graph of } y = -x^2 \]

c) \( y = x^3 \)  
\[ \text{Graph of } y = x^3 \]

d) \( y = 12x^3 - 48x^2 + 36x \)  
\[ \text{Graph of } y = 12x^3 - 48x^2 + 36x \]

Clearly, some functions have extreme values, while others do not. The following theorem gives conditions under which functions are guaranteed to have extreme values:

**THE EXTREME VALUE THEOREM**  
If \( f \) is continuous on a closed interval \([a, b]\), then \( f \) attains an absolute maximum value \( f(c) \) and an absolute minimum value \( f(d) \) at some numbers \( c \) and \( d \) in \([a, b]\).

This theorem does not tell us how to find these extreme values, however, so we use an idea that we have discussed before. Note that at local maximum and minimum points, the slope of the tangent line is 0. This is guaranteed in Fermat's Theorem.

**FERMAT'S THEOREM**  
If \( f \) has a local maximum or minimum at \( c \), and if \( f'(c) \) exists, then \( f'(c) = 0 \).

Note, however, that the converse is not true. Just because \( f'(c) = 0 \) does not guarantee that \( f(c) \) is a local extrema, as we see in the case of \( f(x) = x^3 \). This does give us a starting point, though. Also, if the derivative does not exist at a point, there could be an extreme value, as in the case of \( f(x) = |x| \).
Sketch the graph of $f$ by hand and use your sketch to find any absolute and local maximum and minimum values of $f$.

$f(x) = -2x + 4, \ -1 \leq x \leq 3$  \hspace{1cm}  $f(t) = 2\cos t, \ -3\pi/2 \leq t \leq 3\pi/2$
**DEFINITION** A **critical number** of a function $f$ is a number $c$ in the domain of $f$ such that either $f'(c) = 0$ or $f'(c)$ does not exist.

So, when we're looking for extrema, we need to start by finding these critical numbers, which means that our first step is to take the derivative.

Examples: Find the critical numbers for each of the following functions:

\[ f(x) = 5x^2 - 3x + 7 \quad f(x) = x^2 \ln x \]

\[ f(x) = \frac{x^2 + 8}{x - 1} \quad f(x) = x^{2/5}(4x + 7) \]

In terms of critical numbers, Fermat's Theorem can be rephrased as follows:

If $f$ has a local maximum or minimum at $c$, then $c$ is a critical number.
Examples: Find the absolute maximum and minimum values of each of the following functions on the given intervals:

\[ f(x) = 2x^2 - 3x + 7; \quad [-3, 5] \]
\[ f(x) = 5 + 54x - 2x^3; \quad [0, 4] \]

\[ f(x) = (x^2 - 1)^3; \quad [-1, 2] \]
\[ f(x) = x\sqrt{9 - x^2}; \quad [-3, 3] \]